Exercises: Introduction to imprecise probabilities

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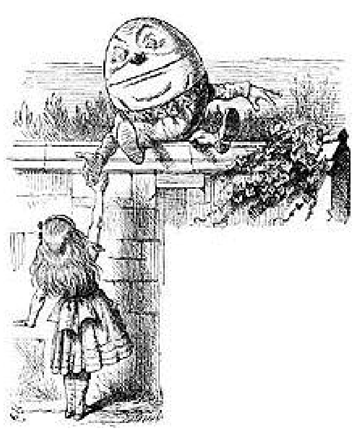
1. **Assessing a set of probabilities.-** Before jumping off the wall, Humpty Dumpty tells Alice the following:

   > "I have a farm with pigs, cows and hens. There are at least as many pigs as cows and hens together and at least as many hens as cows. How many pigs, cows and hens do I have?"



   (a) Determine the set of probabilities compatible with these assessments.

   Solution.- Any probability measure P compatible with these assessments should satisfy $P(\{pigs\}) \geq P(\{hens\}) + P(\{cows\})$, and $P(\{hens\}) \geq P(\{cows\})$. The extreme points of the set of compatible probabilities are

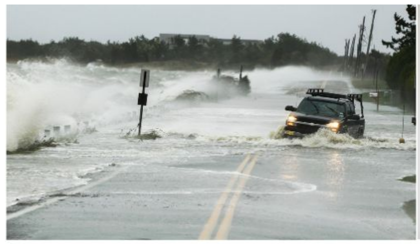
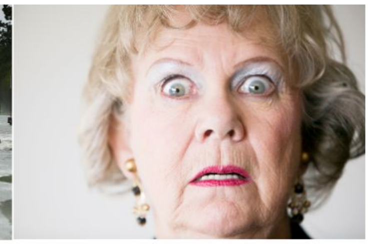
   $$\{(1, 0, 0), (0.5, 0.5, 0), (0.5, 0.25, 0.25)\},$$

   where any vector (p_1, p_2, p_3) above denotes $(P(\{pigs\}), P(\{hens\}), P(\{cows\}))$.

   (b) What is the upper probability of the set {hens, cows}?

   Solution.-Considering the extreme points of the set above, we deduce that the proportion of pigs is at least 0.5, or, using conjugacy, that the upper probability of the set {hens, cows} is 0.5.

2. **Coherence.-** Mr. Play-it-safe is planning his upcoming holidays in the Canary Islands, and he is taking into account three possible disruptions: an unexpected illness (A), severe weather problems (B) and the unannounced visit of his mother in law (C).

He has assessed his lower and upper probabilities for these events:

	A	B	C	D
$\underline{P}$	0.05	0.05	0.2	0.5
$\overline{P}$	0.2	0.1	0.5	0.8

where D denotes the event 'Nothing bad happens'. He also assumes that no two disruptions can happen simultaneously.

Are these assessments coherent?

Solution.-They are not coherent, because $\underline{P}, \overline{P}$ *are not the lower and upper envelopes of the set of probabilities they determine: for instance, the upper envelope* $\overline{E}$ *of* $\mathcal{M}$ *satisfies* $\overline{E}(D) = 0.7 < 0.8 = \overline{P}(D)$.

3. **2-monotonicity.-** Let $\underline{P}$ a 2-monotone capacity defined on a field of sets $\mathcal{A}$, and let us extend it to $\wp(\Omega)$ by

$$\underline{P}_*(A) = \sup\{\underline{P}(B) : B \subseteq A\}.$$

Show that $\underline{P}_*$ is also 2-monotone.

Solution.-Consider events A_1, A_2, *and let us prove that* $\underline{P}_*(A \cup A_2) + \underline{P}_*(A_1 \cap A_2) \geq \underline{P}_*(A_1) + \underline{P}_*(A_2)$. *We have that*

$$\begin{aligned}
&\underline{P}_*(A_1) + \underline{P}_*(A_2) = \sup\{\underline{P}(B_1) : B_1 \in \mathcal{A}, B_1 \subseteq A_1\} + \sup\{\underline{P}(B_2) : B_2 \in \mathcal{A}, B_2 \subseteq A_2\} \\
&= \sup\{\underline{P}(B_1) + \underline{P}(B_2) : B_1 \in \mathcal{A}, B_1 \subseteq A_1, B_2 \in \mathcal{A}, B_2 \subseteq A_2\} \\
&\leq \sup\{\underline{P}(B_1 \cup B_2) + \underline{P}(B_1 \cap B_2) : B_1 \in \mathcal{A}, B_1 \subseteq A_1, B_2 \in \mathcal{A}, B_2 \subseteq A_2\} \\
&\leq \sup\{\underline{P}(C) + \underline{P}(D) : C \in \mathcal{A}, C \subseteq A_1 \cup A_2, D \in \mathcal{A}, D \subseteq A_1 \cap A_2\} \\
&= \sup\{\underline{P}(C) : C \in \mathcal{A}, C \subseteq A_1 \cup A_2\} + \sup\{\underline{P}(D) : D \in \mathcal{A}, D \subseteq A_1 \cap A_2\} \\
&= \underline{P}_*(A_1 \cup A_2) + \underline{P}_*(A_1 \cap A_2),
\end{aligned}$$

where the first inequality follows from the 2-monotonicity of $\underline{P}$ in $\mathcal{A}$.

4. **2-monotonicity.-** Consider $\Omega = \{1,2,3,4\}$, and let $\underline{P}$ be the lower envelope of the probabilities P_1, P_2 given by

$$P_1(\{1\}) = P_1(\{2\}) = 0.5, P_1(\{3\}) = P_1(\{4\}) = 0$$
$$P_2(\{1\}) = P_2(\{2\}) = P_2(\{3\}) = P_2(\{4\}) = 0.25.$$

Show that $\underline{P}$ is not 2-monotone.

Solution-. Consider the events $A = \{1,3\}$ and $B = \{1,4\}$. We obtain

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) = \underline{P}(\{1,3,4\}) + \underline{P}(\{1\}) = 0.5 + 0.25 = 0.75$$

while

$$\underline{P}(A) + \underline{P}(B) = \underline{P}(\{1,3\}) + \underline{P}(\{1,4\}) = 0.5 + 0.5 = 1.$$

Thus, $\underline{P}(A \cup B) + \underline{P}(A \cap B) < \underline{P}(A) + \underline{P}(B)$ and therefore the condition of 2-monotonicity does not hold.

5. **Belief functions.-** Consider $\Omega = \{1,2,3\}$.

(a) Let m be the basic probability assignment given by $m(\{1,2\}) = 0.5, m(\{3\}) = 0.2, m(\{2,3\}) = 3$. Determine the belief function associated with m.

Solution.-Using the formula $Bel(A) = \sum_{B \subseteq A} m(B)$, we obtain the following values:

A	*{1}*	*{2}*	*{3}*	*{1,2}*	*{1,3}*	*{2,3}*	*{1,2,3}*
Bel(A)	*0*	*0*	*0.2*	*0.5*	*0.2*	*0.5*	*1*

(b) Consider the belief function $\underline{P}$ given by $\underline{P}(A) = \frac{|A|}{3}$ for every $A \subseteq \Omega$. Determine its basic probability assignment.

Solution.- Using the formula $m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B)$ we obtain

A	*{1}*	*{2}*	*{3}*	*{1,2}*	*{1,3}*	*{2,3}*	*{1,2,3}*
m(A)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	*0*	*0*	*0*	*0*

Actually, in this case Bel is a probability measure, and all the mass is on the singletons.

6. **Possibility measures.-** Consider $\Omega = \{1,2,3,4\}$.

(a) Let Π be the possibility measure associated with the possibility distribution $\pi(1) = 0.3, \pi(2) = 0.5, \pi(3) = 1, \pi(4) = 0.7$. Determine its focal elements and its basic probability assignment.

Solution.- Using the formula $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ and the conjugacy $N(A) = 1 - \Pi(A^c)$ for every A, we obtain that the belief function that is conjugate to Π satisfies

A	*{1}*	*{2}*	*{3}*	*{4}*	*{1,2}*	*{1,3}*	*{1,4}*
$Bel(A)$	*0*	*0*	*0.3*	*0*	*0*	*0.3*	*0*
A	*{2,3}*	*{2,4}*	*{3,4}*	*{1,2,3}*	*{1,2,4}*	*{1,3,4}*	*{2,3,4}*
$Bel(A)$	*0.3*	*0*	*0.5*	*0.3*	*0*	*0.5*	*0.7*

and $Bel(\{1,2,3,4\}) = 1$. *Using the formula of the Möbius inverse:*

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B)$$

we obtain that the basic probability assignment is

$$m(\{3\}) = 0.3, m(\{3,4\}) = 0.2, m(\{2,3,4\}) = 0.2 \textit{ and } m(\{1,2,3,4\}) = 0.3,$$

and 0 elsewhere. Note that the focal elements are nested as is always the case with possibility measures.

(b) Consider the basic probability assignment given by $m(\{1\}) = 0.2, m(\{1,3\}) = 0.1, m(\{1,2,3\}) = 0.4, m(\{1,2,3,4\}) = 0.3$. Determine the associated possibility measure and its possibility distribution.

Solution.- Using the formula $Bel(A) = \sum_{B \subseteq A} m(B)$, *we obtain that*

$$Bel(\{1,2,3\}) = 0.7, Bel(\{1,2,4\}) = 0.2, Bel(\{1,3,4\}) = 0.3, Bel(\{2,3,4\}) = 0,$$

from which it follows using the conjugacy relation $\Pi(A) = 1 - Bel(A^c)$ *that the possibility distribution is given by:*

$$\pi(\{1\}) = 1, \pi(\{2\}) = 0.7, \pi(\{3\}) = 0.8, \pi(\{4\}) = 0.3.$$

¿From this we can derive value of the possibility measure on any other event using that $\Pi(A) = \max_{\omega \in A} \pi(\omega)$.

7. **Coherent lower previsions.-** Let $\underline{P}$ be the lower prevision on $\mathcal{L}(\{1,2,3\})$ given by

$$\underline{P}(X) = \frac{1}{2}(\min X + \max X)$$

for all X on Ω. Is it coherent?

Solution.-No. A necessary condition for coherence is that it is supper-additive, meaning that $\underline{P}(X+Y) \geq \underline{P}(X) + \underline{P}(Y)$ *for any pair of gambles* X, Y. *To see that this does not hold, consider* $X = (-1,0,-1)$ *and* $Y = (0,-1,-1)$. *Then* $\underline{P}(X) = \underline{P}(Y) = -\frac{1}{2}$, *while* $\underline{P}(X+Y) = -1.5 < \underline{P}(X) + \underline{P}(Y)$.

8. **Natural extension.-** Let $\underline{P}_A$ be the vacuous lower probability relative to a set A, given by the assessment $\underline{P}_A(A) = 1$.

   Prove that the natural extension $\underline{E}$ of $\underline{P}_A$ is equal to the vacuous lower prevision relative to A:
   $$\underline{E}(X) = \inf_{\omega \in A} X(\omega),$$
   for any $X \in \mathcal{L}(\Omega)$.

   Solution.-The credal set associated with $\underline{P}$ is given by
   $$\mathcal{M}(\underline{P}) = \{P : P(B) \geq \underline{P}(B)\ \forall B \subseteq \Omega\} = \{P : P(A) = 1\}.$$
   The extreme points of this set are the probability measures that are degenerate on some element of A. From this it follows that
   $$\underline{E}(X) = \min\{P(X) : P \in \mathcal{M}(\underline{P})\} = \min\{X(\omega) : \omega \in A\}.$$

9. **Sets of desirable gambles.-** Let $\Omega = \{1, 2, 3\}$, and consider the following sets of desirable gambles:
   $$\mathcal{D}_1 := \{X : X(1) + X(2) + X(3) > 0\}.$$
   $$\mathcal{D}_2 := \{X : \max\{X(1), X(2), X(3)\} > 0\}.$$
   Is $\mathcal{D}_1$ coherent? And $\mathcal{D}_2$?

   Solution.- The set $\mathcal{D}_1$ is desirable, because it satisfies all the axioms:

   (D1) $X \leq 0 \Rightarrow X(1) + X(2) + X(3) \leq 0 \Rightarrow X \notin \mathcal{D}_1$.

   (D2) $X \gneq 0 \Rightarrow X(1) + X(2) + X(3) > 0 \Rightarrow X \in \mathcal{D}_1$.

   (D3) $X, Y \in \mathcal{D}_1 \Rightarrow X(1) + X(2) + X(3) > 0$ *and* $Y(1) + Y(2) + Y(3) > 0$, *whence* $(X+Y)(1) + (X+Y)(2) + (X+Y)(3) > 0$ *and therefore* $X + Y \in \mathcal{D}_1$.

   (D4) $X \in \mathcal{D}_1 \Rightarrow X(1) + X(2) + X(3) > 0 \Rightarrow \lambda(X(1) + X(2) + X(3)) = \lambda X(1) + \lambda X(2) + \lambda X(3) > 0 \Rightarrow \lambda X \in \mathcal{D}_1$ *for every* $\lambda > 0$.

   To see that $\mathcal{D}_2$ is not coherent, note that it violates axiom (D3): if we consider the gambles $X = (1, -2, -2)$ and $Y = (-2, 1, -2)$, it follows that $X, Y \in \mathcal{D}_2$ while $X + Y = (-1, -1, -4) \notin \mathcal{D}_2$.

10. **Epistemic irrelevance.-** Suppose that we have three urns. Each of them has 10 balls which are coloured either red or green. We know that the first urn has 5 red (r), 2 green (g), 3 unknown colours. Our knowledge about the other two urns is the same. We know that they have 3 red, 3 green, 4 unknown colours. It is not necessary that Urns 2 and 3 have exactly the same composition. A ball is randomly selected

from the first urn. If it is red then the second ball is selected randomly from the second urn, and if the first ball is green then the second ball is selected randomly from the third urn.

Our uncertainty about the pair of colours is modelled by a set of joint probabilities of the form μ where

- $\mu(\{(r,\omega_2)\}) = \mu_1(\{r\})\mu_{2|r}(\{\omega_2\}), \omega_2 \in \{r,g\}$
- $\mu(\{(g,\omega_2)\}) = \mu_1(\{g\})\mu_{2|g}(\{\omega_2\}), \omega_2 \in \{r,g\}$,

with

- $0.5 \leq \mu_1(\{r\}) \leq 0.8$,
- $0.3 \leq \mu_{2|r}(\{\omega_2\}) \leq 0.7$ and
- $0.3 \leq \mu_{2|g}(\{\omega_2\}) \leq 0.7, \omega_2 \in \{r,g\}$

(a) The above set has eight extreme points, each of them determined by a combination of the extremes of the marginal on Ω_1 and the two conditionals. Determine the collection of eight extreme points of the above set.

Solution.- We will identify each probability measure on $\{r,g\} \times \{r,g\}$ *with the 4-dimensional vector indicating the respective masses on* (r,r), (r,g), (g,r) *and* (g,g). *The eight extremes are:*

$(0.15, 0.35, 0.15, 0.35)$, $(0.15, 0.35, 0.35, 0.15)$, $(0.35, 0.15, 0.15, 0.35)$, $(0.35, 0.15, 0.35, 0.15)$, $(0.24, 0.56, 0.06, 0.14)$, $(0.24, 0.56, 0.14, 0.06)$, $(0.56, 0.24, 0.06, 0.14)$, $(0.56, 0.24, 0.06, 0.14)$.

(b) Calculate the upper probability that the first ball is red, given that the colour of the second ball is green. Does the collection of conditional probabilities $\mathcal{M}_{1|g} = \{\mu_{1|g} : \mu \in \mathcal{M}\}$ coincide with $\mathcal{M}_1$?

Solution.- We must calculate the maximum possible value for the conditional probability $\mu_{1|g}(\{r\}) = \frac{\mu(\{(r,g)\})}{\mu(\{(r,g),(g,g)\})}$. *It is the maximum of the following values:*

$$\max\left\{\frac{0.35}{0.70}, \frac{0.35}{0.50}, \frac{0.15}{0.50}, \frac{0.15}{0.30}, \frac{0.56}{0.70}, \frac{0.56}{0.62}, \frac{0.24}{0.38}, \frac{0.24}{0.30}\right\} = \frac{0.56}{0.62} = \frac{28}{31},$$

which corresponds with the extreme distribution $(0.56, 0.24, 0.14, 0.06)$ *over the set* $\{(r,r),(r,g),(g,r),(g,g)\}$ *and is strictly greater than 0.7. Based on this information, we observe that the set of conditional probabilities* $\mathcal{M}_{1|r} = \{\mu_{1|r} : \mu \in \mathcal{M}\}$ *is not included in* $\mathcal{M}_1$*, and therefore, they do not coincide.*

11. **Independence in the selection and strong independence.-** Assume that we have two urns with the following composition: The first urn has 5 red (r), 2 white (w), 3 unknown colours, while the second one has 3 red, 3 white, 4 unknown colours.

Suppose that the 7 balls in the two urns whose colours are unknown are all the same colour and that the drawings from the two urns are stochastically independent.

Determine the convex hull of the set of probabilities that is compatible with the above information. Does it satisfy independence in the selection? Does it satisfy strong independence?

Solution.- Let μ_1 *and* μ_2 *denote the respective marginal probability distributions, each of them corresponding to one urn. Each of them will be identified with a two-dimensional vector indicating the respective masses on* r *and* w*. There are two possibilities for the pair* (μ_1, μ_2)*:*

- *Option 1:* $\mu_1 = (0.8, 0.2)$ *and* $\mu_2 = (0.7, 0.3)$
- *Option 2:* $\mu_1 = (0.5, 0.5)$ *and* $\mu_2 = (0.3, 0.7)$.

Since both drawings from both urns are stochastically independent, the probability distribution over the Cartesian product must be a product probability. The respective products of the above marginals are $(0.8, 0.2) \otimes (0.7, 0.3) = (0.56, 0.24, 0.14, 0.06)$ *and* $(0.5, 0.5) \otimes (0.3, 0.7) = (0.15, 0.35, 0.15, 0.35)$*. The convex hull is*

$$\mathcal{M} = CH(\{(0.56, 0.24, 0.14, 0.06), (0.15, 0.35, 0.15, 0.35)\})$$

i.e., the collection of linear convex combinations of these two joint probabilities. It satisfies independence in the selection (its extreme points are product probabilities). However it does not satisfy strong independence. In fact, it does not include all the products of the form $\mu_1 \otimes \mu_2$ *with* $\mu_1 \in \mathcal{M}_1$ *and* $\mu_2 \in \mathcal{M}_2$*. In fact, if we pick for instance* $\mu_1 = (0.8, 0.2) \in \mathcal{M}_1$ *and* $\mu_2 = (0.3, 0.7) \in \mathcal{M}_2$*, we can easily observe that the product probability* $\mu = \mu_1 \otimes \mu_2$ *does not belong to* $\mathcal{M}$*.*

12. **Independence in the selection and strong independence (2nd).-** Assume again that we have two urns with the following composition: The first urn has 5 red, 2 white, 3 unknown colours, while the second one has 3 red, 3 white, 4 unknown colours. We take one ball from each urn. The drawings from the two urns are stochastically independent.

Determine the convex hull of the set of probabilities that is compatible with the above information. Does it satisfy independence in the selection? Does it satisfy strong independence?

Solution.- It is

$$CH(\{\mu_1 \otimes \mu_2 \,:\, 0.5 \leq \mu_1(\{r\}) \leq 0.8 \text{ and } 0.3 \leq \mu_2(\{r\}) \leq 0.7\}).$$

It satisfies strong independence (and independence in the selection).

13. **Independence of marginal sets.-** Suppose that we have two urns. Each of the urns has 10 balls which are coloured either red or white. We know that the first urn has 5 red, 2 white, and 3 unknown colours, and the second urn has 3 red, 3 white, and 4 unknown colours. One ball is chosen at random from each of the urns. We have no information about the interaction between the two drawings.

    (a) Determine the marginal credal set on $\Omega_1 = \{r, w\}$ characterizing our incomplete information about the first drawing. Denote it $\mathcal{M}_1$.

    Solution.- The set of probabilities that is compatible with this information is $\{(p, 1-p) \,:\, p \in \{0.5, 0.6, 0.7, 0.8\}$ *The smallest credal set that includes it is* $\mathcal{M}_1 = CH(\{(p, 1-p) \,:\, p \in \{0.5, 0.6, 0.7, 0.8\}\}) =$ $\{(p, 1-p) \,:\, 0.5 \leq p \leq 0.8\}$.

    (b) Determine the marginal credal set on $\Omega_2 = \{r, w\}$ characterizing our incomplete information about the second drawing. Denote it $\mathcal{M}_2$.

    Solution.- $\mathcal{M}_2 = CH(\{(p, 1-p) \,:\, p \in \{0.3, 0.4, 0.5, 0.6, 0.7\}\}) =$ $\{(p, 1-p) \,:\, 0.3 \leq p \leq 0.7\}$.

    (c) Consider the joint possibility space $\Omega = \{rr, rw, wr, ww\}$ and the joint probability $\mu = (0.2, 0.4, 0.3, 0.1)$ defined on it:

    - Check that it belongs to the set $\mathcal{M} = \mathcal{M}_1^* \cap \mathcal{M}_2^*$.

      Solution.- Its marginal on Ω_1 *is* $(0.2 + 0.4, 0.3 + 0.1) = (0.6, 0.4)$, *which belongs to the set* $\mathcal{M}_1$. *Its marginal on* Ω_2 *is* $(0.2+0.3, 0.4+0.1) = (0.5, 0.5)$, *which belongs to the set* $\mathcal{M}_2$.

    - Design a random experiment compatible with the above incomplete information associated with this joint probability.
      Solution.- Suppose that urns respectively contain 6 and 5 red balls (the rest of them are white). Suppose also that the balls are numbered from 1 to 10 in each urn. Suppose that the colours are distributed as follows:

ball	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Urn 1	*r*	*r*	*r*	*r*	*r*	*r*	*w*	*w*	*w*	*w*
Urn 2	*r*	*r*	*w*	*w*	*w*	*w*	*r*	*r*	*r*	*w*

      We take one ball at random from the first urn, and we take the ball with the same number from urn 2.

    (d) Consider the example of two urns described at the beginning of this exercise. What is the credal set on $\Omega = \Omega_1 \times \Omega_2$ that represents our probabilistic information about the joint experiment?
    Solution.- It is $\mathcal{M} = \mathcal{M}_1^* \cap \mathcal{M}_2^*$.

14. **Example of lack of independence of marginal sets.-** Consider the product possibility space $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 = \Omega_2 = \{r, w\}$. Consider the credal set $\mathcal{M} = CH(\{\mu, \mu'\})$ where $\mu = (0.01, 0.09, 0.09, 0.81)$ and $\mu' = (0.81, 0.09, 0.09, 0.01)$. Is independence of the marginal sets satisfied?

    Solution.- No it is not. Let us notice that, in this example, $\mathcal{M}_1 = \{\mu_1, \mu_1'\} = \{(0.1, 0.9), (0.9, 0.1)\}$ *and* $\mathcal{M}_2 = \{\mu_2, \mu_2'\} = \{(0.1, 0.9), (0.9, 0.1)\}$. *Let us take, for instance* $\mu_1 = (0.1, 0.9) \in \mathcal{M}_1$ *and* $\mu_2' = (0.9, 0.1) \in \mathcal{M}_2$. *There is not any probability measure* μ *in* $\mathcal{M}$ *with those two marginals.*

15. **Set valued data.-** Consider the following information:

    - We have two urns. Each of them has 10 balls which are coloured either red or white.
    - Urn 1: 5 red, 2 white, 3 unpainted; Urn 2: 3 red, 3 white, 4 unpainted.
    - One ball is chosen at random from each urn.
    - (If they have no colour, there may be arbitrary correlation between the colours they are finally assigned).

    (a) Check that the focal sets of the joint mass function m are the following nine sets: $\{rr\}$, $\{rw\}$, $\{rr, rw\}$, $\{wr\}$, $\{ww\}$, $\{wr, ww\}$, $\{rr, wr\}$, $\{rw, ww\}$, $\{rr, rw, wr, ww\}$.
    Solution.- The focal sets of each of both marginal mass assignments are, in fact, $\{r\}$, $\{w\}$ *and* $\{r, w\}$. *The focal sets of the joint mass assignment are their respective Cartesian products.*

    (b) Determine the mass values associated with those 9 focal sets.
    Solution.- The respective marginal mass values are:

	$\{r\}$	$\{w\}$	$\{r,w\}$
m_1	*0.5*	*0.2*	*0.3*
m_2	*0.3*	*0.3*	*0.4*

    The joint mass function is the product of them, assigning the following masses to the corresponding focal sets:

focal s.	$\{rr\}$	$\{rw\}$	$\{rr, rw\}$	$\{wr\}$	$\{ww\}$	$\{wr, ww\}$	$\{rr, wr\}$	$\{rw, ww\}$	$\{rr, rw, wr, ww\}$
mass	*0.15*	*0.15*	*0.20*	*0.06*	*0.06*	*0.08*	*0.09*	*0.09*	*0.12*

    (c) Consider the credal set $\mathcal{M}$ associated with m and calculate the minimum possible value for the conditional probability $\mu(\{r, w\} \times \{r\} | \{r\} \times \{r, w\})$: $\min\{\mu(\{r, w\} \times \{r\} | \{r\} \times \{r, w\}) \,|\, \mu \in \mathcal{M}\} = \min\left\{\frac{\mu(\{(r,r)\})}{\mu(\{(r,r),(r,w)\})} \,|\, \mu \in \mathcal{M}\right\}$.
    Solution.- The minimum possible value for $\frac{\mu(\{(r,r)\})}{\mu(\{(r,r),(r,w)\})}$ *in the set* $\mathcal{M}$ *is attained for* $\mu \in \mathcal{M}$ *assigning probability* 0.15 *to* $\{(r, r)\}$ *(the minimum possible probability value) and probability* $0.15 + 0.15 + 0.20 + 0.09 + 0.12 = 0.71$ *to the set*

$\{(r,r),(r,w)\}$. *Therefore, the minimum value for the conditional probability is* $\frac{15}{71}$.

(d) Does the above minimum coincide with 0.3?

Solution.- It does not coincide with 0.3. It is in fact smaller than that.

16. **Random set independence vs independence in the selection.-**

- A light sensor displays numbers between 0 and 255.
- 10 measurements per second.
- If the brightness is higher than a threshold (255), the sensor displays 255 during 3/10s.

Complete the following table, about six consecutive measurements, where the actual values of brightness are independent from each other:

actual values	215	150	200	300	210	280
displayed quantities	215	150	200	255	__	__
set-valued information	$\{215\}$	$\{150\}$	__	__	__	__

Solution.-

actual values	215	150	200	300	210	280
displayed quantities	215	150	200	255	255	255
set-valued information	$\{215\}$	$\{150\}$	200	$[255,\infty)$	$[0,\infty)$	$[0,\infty)$

Let Γ_i denote the random set that represents the (set-valued) information provided by the sensor in the i-th measurement.

What is the value of the following conditional probability?:

$$P(\Gamma_i \supseteq [255,\infty) | \Gamma_{i-1} \supseteq [255,\infty), \Gamma_{i-2} \not\supseteq [255,\infty)).$$

Solution.- It is equal to 1.

17. **Random set independence vs independence in the selection (2nd).-** The random variables X_0 and Y_0 respectively represent the temperature (in °C) of an ill person taken at random in a hospital just before taking an antipyretic (X_0) and 3 hours later (Y_0). The random set Γ_1 represents the information about X_0 using a very crude measure (it reports always the same interval $[37, 39.5]$). The random set Γ_2 represents the information about Y_0 provided by a thermometer with $+/-0.5$ °C of precision.

(a) Are X_0 and Y_0 stochastically independent?

Solution.- No, they are not. According to our (poor) knowledge, they are expected to be positively correlated. The higher the starting point (X_0)*, the higher the temperature after taking the antipyretic* (Y_0) *(in general).*

(b) Are Γ_1 and Γ_2 stochastically independent?

Solution.- Yes, they are, since Γ_1 *is a constant.*

18. **Repetition independence violates convexity (Taken from R. Jeffrey, 1987)**.- Consider the product possibility space $\Omega = \{a, b\} \times \{a, b\}$ and the non-convex set of probabilities $\mathcal{P} = \{\mu \otimes \mu, \mu' \otimes \mu'\}$ with $\mu = (1/3, 2/3)$ and $\mu' = (2/3, 1/3)$. Consider the linear convex combination $\frac{1}{2}(\mu \otimes \mu + \mu' \otimes \mu')$ and check that it is not a product probability (i.e., it cannot be factorised as the product of its marginals).

    Solution.- Let us notice that $\mu \otimes \mu = (1/9, 2/9, 2/9, 4/9)$ *and* $\mu' \otimes \mu' = (4/9, 2/9, 2/9, 1/9)$*. Their convex combination* $\frac{1}{2}(\mu \otimes \mu + \mu' \otimes \mu')$ *can be expressed as* $(5/18, 2/9, 2/9, 5/18)$ *which cannot be written as the product of its marginals, respectively* $(1/2, 1/2)$ *and* $(1/2, 1/2)$*. In fact, the product of its marginals is the distribution* $(1/4, 1/4, 1/4, 1/4)$*.*