Randomness and imprecision

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11001010110101010001110101011

111001110010011100111101010111 . . .

WHEN IS A SEQUENCE RANDOM?

Random sequences and random numbers

$$1100101011010101000111010101 1$$

$$111001110010011100111101010111\ldots$$

Random sequences and random numbers

$$0.11001010110101010001110101011$$
$$111001110010011100111101010111\ldots$$

is a real number in $[0, 1]$.

Random sequences and calibrated forecasts

0 1 1 0 0 1 0 1 0 . . .

Random sequences and calibrated forecasts

$\frac{1}{2}0 \quad \frac{1}{2}1 \quad \frac{1}{2}1 \quad \frac{1}{2}0 \quad \frac{1}{2}0 \quad \frac{1}{2}1 \quad \frac{1}{2}0 \quad \frac{1}{2}1 \quad \frac{1}{2}0 \ldots$

Random sequences and calibrated forecasts

$p_1 \, 0 \;\; p_2 \, 1 \;\; p_3 \, 1 \;\; p_4 \, 0 \;\; p_5 \, 0 \;\; p_6 \, 1 \;\; p_7 \, 0 \;\; p_8 \, 1 \;\; p_9 \, 0 \; \ldots$

Random sequences and calibrated forecasts

$I_1$0 $I_2$1 $I_3$1 $I_4$0 $I_5$0 $I_6$1 $I_7$0 $I_8$1 $I_9$0 ...

A BIT OF HISTORY

The classical case of a fair coin

$$\tfrac{1}{2}0 \quad \tfrac{1}{2}1 \quad \tfrac{1}{2}1 \quad \tfrac{1}{2}0 \quad \tfrac{1}{2}0 \quad \tfrac{1}{2}1 \quad \tfrac{1}{2}0 \quad \tfrac{1}{2}1 \quad \tfrac{1}{2}0 \ldots$$

A bit of notation

$$\omega = (x_1, x_2, x_3, \dots, x_n, \dots) \in \Omega$$

with $\Omega = \{0,1\}^{\mathbb{N}} \approx [0,1]$

$$\omega^n = (x_1, x_2, x_3, \dots, x_n) \in \Omega^{\Diamond}$$

with $\Omega^{\Diamond} = \{0,1\}^*$

$$\omega_n = x_n \in \{0,1\}$$

THE APPROACH OF VON MISES, WALD AND CHURCH

The approach of von Mises, Wald and Church

Randomness of ω means:

$$\frac{\sum_{k=1}^{n} x_k}{n} \to \frac{1}{2} \qquad \text{(Law of Large Numbers)}$$

The approach of von Mises, Wald and Church

Randomness of ω means:

$$\frac{\sum_{k=1}^{n} x_k}{n} \to \frac{1}{2} \qquad \text{(Law of Large Numbers)}$$

but also more stringently, for any selection rule $S\colon \{0,1\}^* \to \{0,1\}$ in a countable class $\mathscr{S}$:

$$\frac{\sum_{k=1}^{n} S(x_1,\ldots,x_{k-1})x_k}{\sum_{k=1}^{n} S(x_1,\ldots,x_{k-1})} \to \frac{1}{2}$$

$$\text{whenever } \sum_{k=1}^{n} S(x_1,\ldots,x_{k-1}) \to \infty$$

A selection rule S is a way of selecting subsequences from ω:

$$\begin{cases} S(x_1,\ldots,x_{k-1}) = 1 & \Rightarrow \text{select } x_k \\ S(x_1,\ldots,x_{k-1}) = 0 & \Rightarrow \text{discard } x_k \end{cases}$$

The approach of von Mises, Wald and Church

For von Mises and Wald, $\mathscr{S}$ represented the countable class of selection rules that can be constructed in some given formal system of arithmetic.

For Church, $\mathscr{S}$ represented the countable class of computable selection rules.

$\Rightarrow$ Computable stochasticity

On both approaches, there is an uncountable infinity of 'random' sequences ω associated with $\mathscr{S}$: they have (Lebesgue) measure one on $[0, 1]$.

Criticism

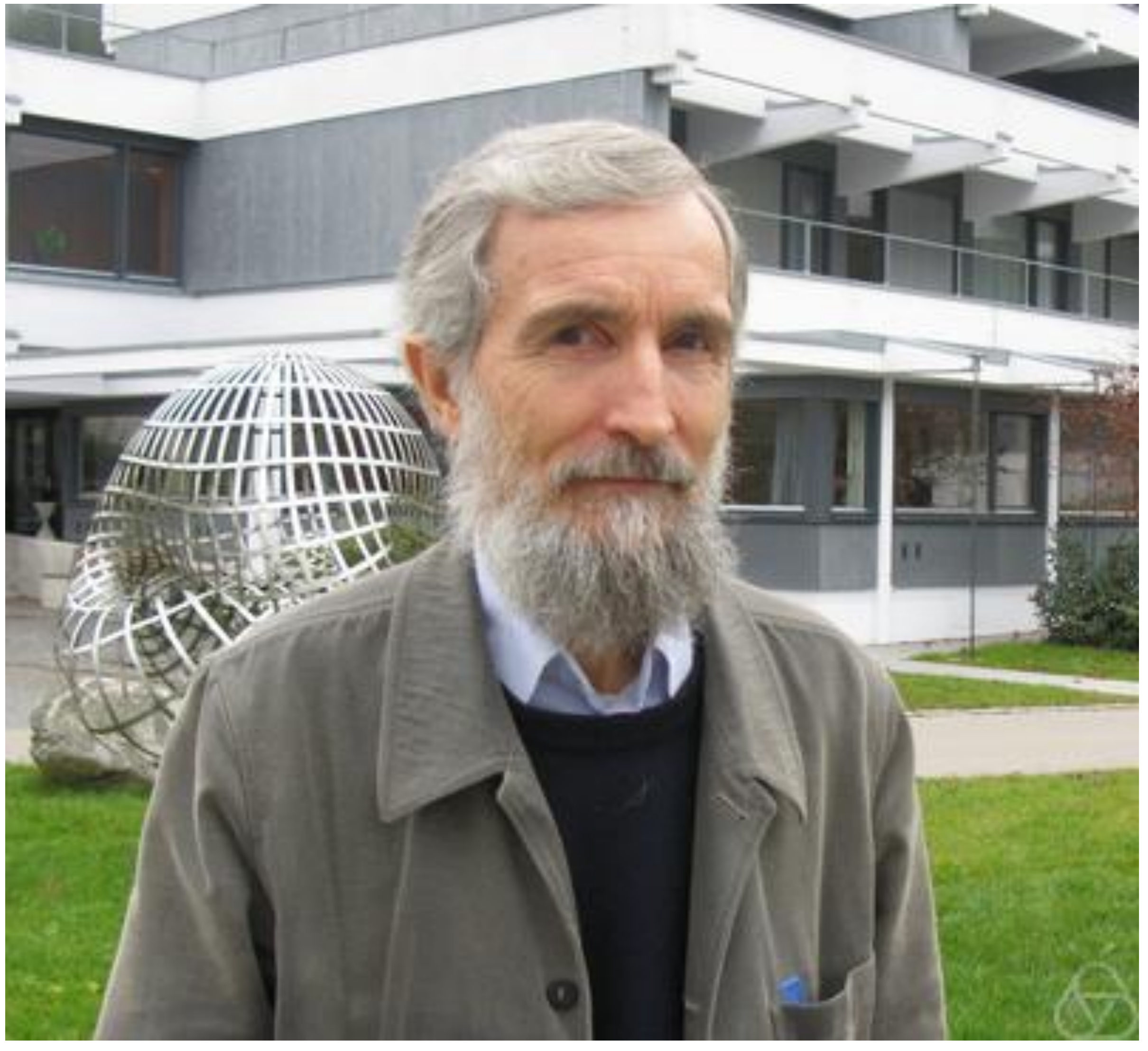
Jean Ville in his *Étude critique de la notion de collectif* (1939):

There are other limit laws than the Law of Large Numbers that are not implied by Computable Stochasticity,

e.g. oscillation around the limit.

Computable stochasticity seems too weak!

THE MARTIN-LÖF APPROACH



Martin-Löf randomness and avoiding null sets

Basic observations:

- randomness is about satisfying limit laws
- randomness is therefore about avoiding null sets
- only countably many null sets can be avoided
- only countably many can be constructed
- a subset A of $[0, 1]$ is null if for all $\varepsilon > 0$ there is a sequence of intervals covering A with total measure at most ε

Effectively null set

A subset A of $[0, 1]$ is effectively null if there is an algorithm that turns any rational $\varepsilon > 0$ into a sequence of intervals covering A with total measure at most ε.

Martin-Löf randomness and avoiding null sets

Conclusions:

- there are only countably many effectively null sets
- their union is null, so its complement has measure one.

Martin-Löf randomness

A sequence ω is Martin-Löf random if it belongs to no effectively null set.

The Martin-Löf random sequences have measure one, and they are computably stochastic.

FORECASTING AND THE MARTINGALE APPROACH

More general precise forecasting

$p_1$0 $p_2$1 $p_3$1 $p_4$0 $p_5$0 $p_6$1 $p_7$0 $p_8$1 $p_9$0 . . .

A single precise forecast r

Forecaster

specifies his expectation r for an unknown outcome X in $\{0,1\}$:
his commitment to adopt r as a fair price for X.

Skeptic

takes Forecaster up on his commitments:

(i) for any $p \leq r$ and $\alpha \geq 0$, Forecaster must accept $\alpha(X-p)$;

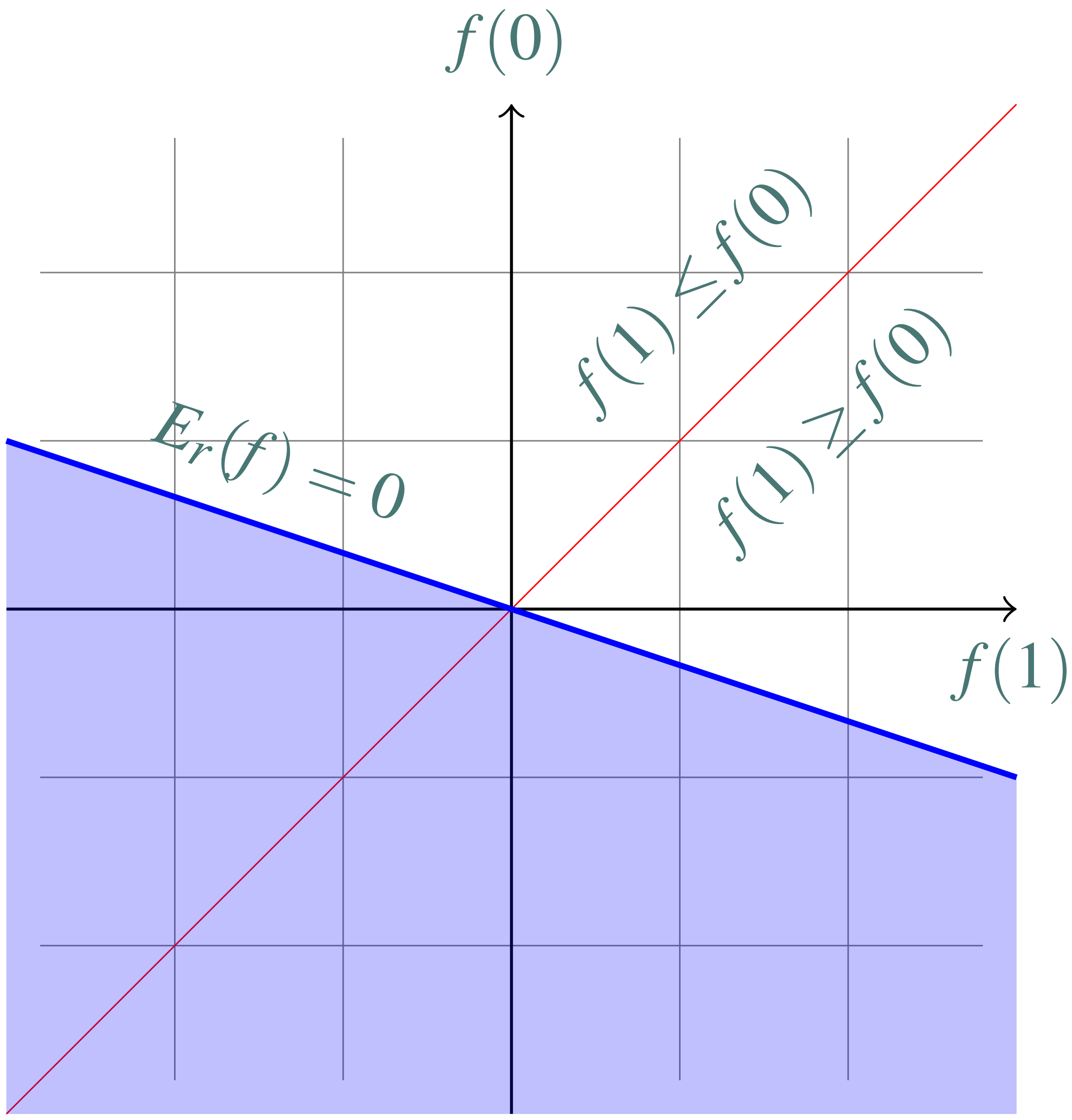
(ii) for any $q \geq r$ and $\beta \geq 0$, Forecaster must accept $\beta(q-X)$.

Reality

determines the value x of X.

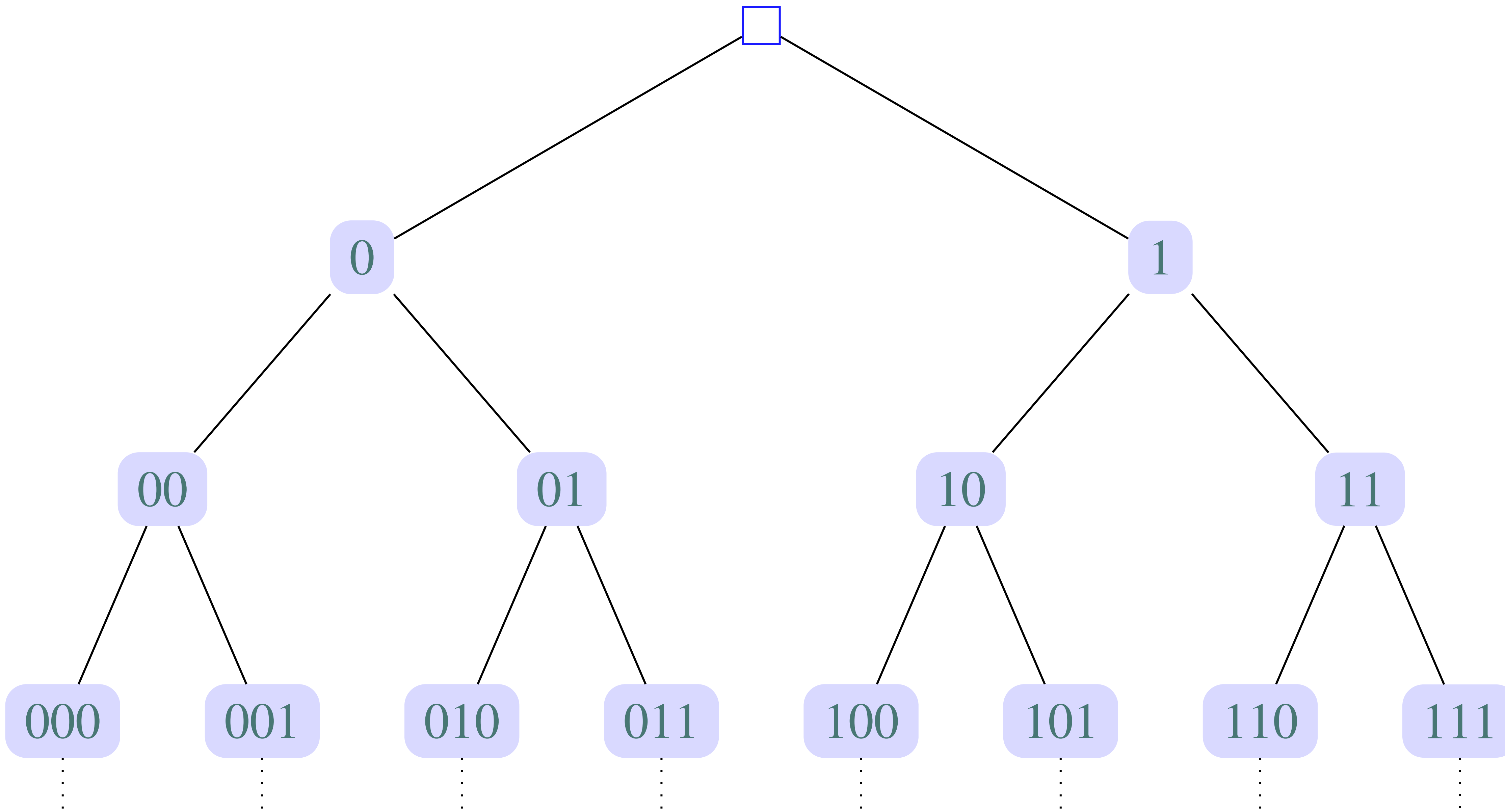
Gambles available to Skeptic: precise forecast r

$$f(X) = -\alpha(X-p) - \beta(q-X) \text{ with } \alpha,\beta \geq 0 \text{ and } 0 \leq p \leq r \leq q \leq 1$$



$$E_r(f) := rf(1) + (1-r)f(0) \leq 0$$

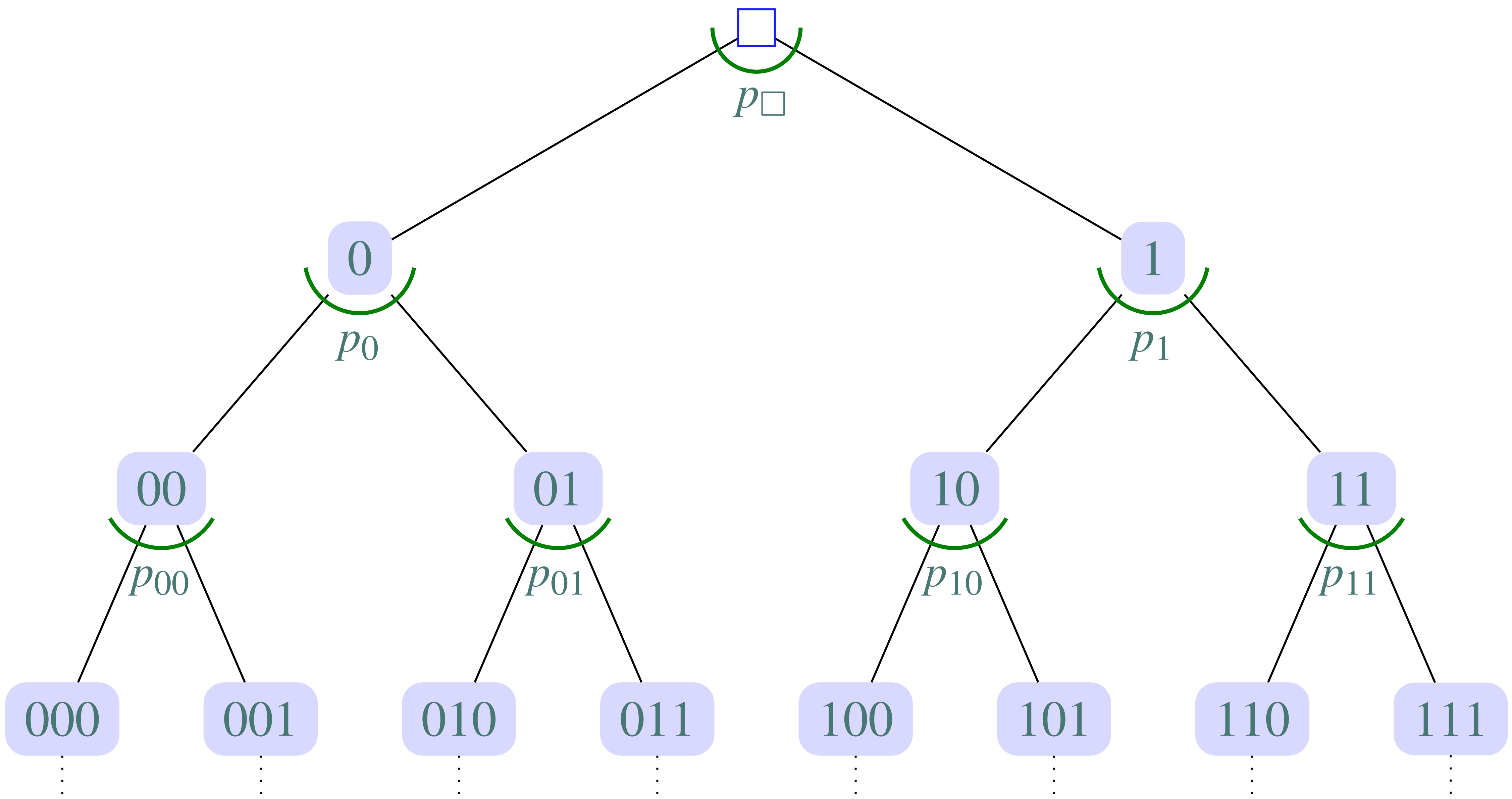
More forecasts: event tree



More forecasts: probability tree

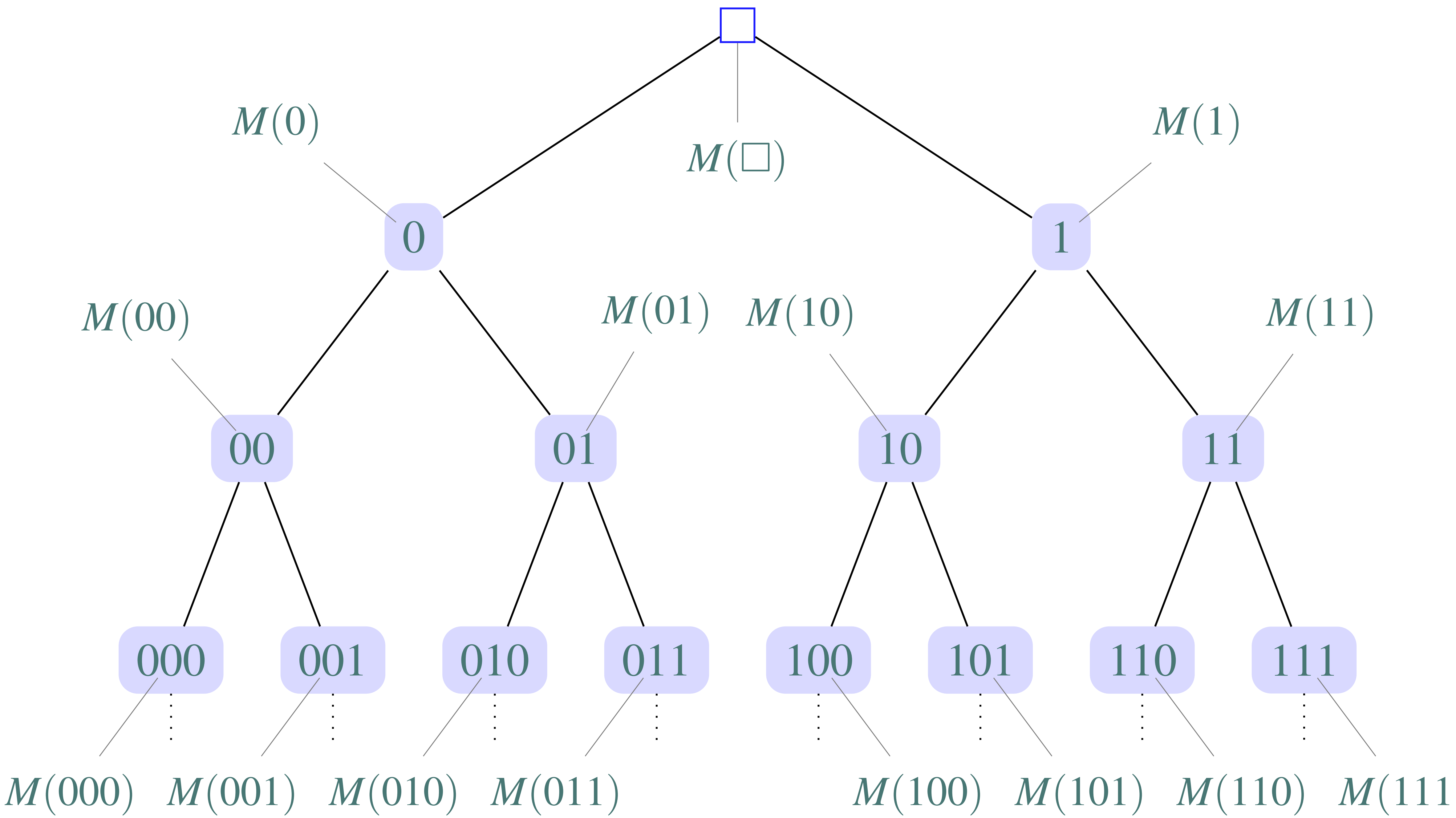
In a probability tree, we associate a precise forecast $\gamma(s) = p_s$ with each situation $s \in \Omega^\lozenge$:

forecasting system $\gamma\colon \Omega^\lozenge \to [0,1]$



Event trees and processes

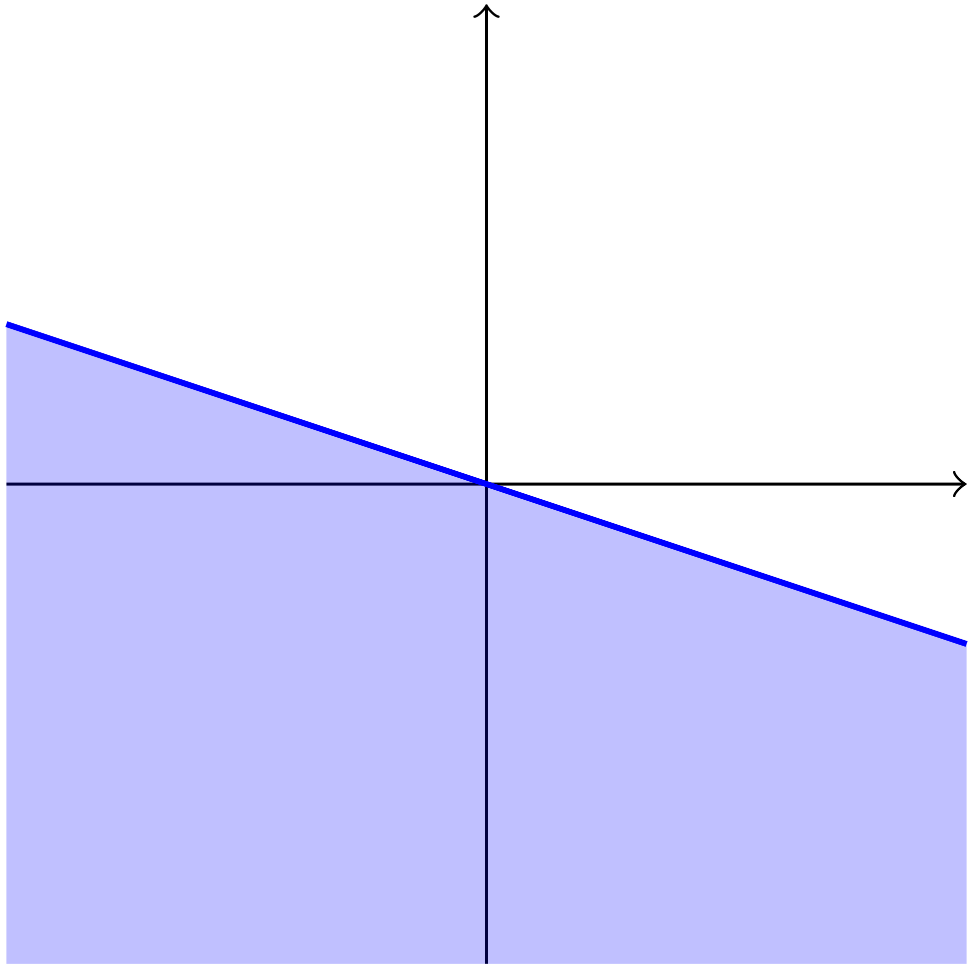
A real process is a map $M\colon \Omega^\lozenge \to \mathbb{R}$, so attaches a real number $M(s)$ to every situation s.



Probability tree and supermartingales

A capital process M for Skeptic is the result of his taking up an available gamble f_s in every possible situation s:

$$\left.\begin{aligned} M(s1) &= M(s) + f_s(1) \\ M(s0) &= M(s) + f_s(0) \end{aligned}\right\} \text{ with } E_s(f_s) \leq 0$$



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Supermartingale

A supermartingale M for a forecasting system γ is a real process whose increments

$$\Delta M(s) := M(s\cdot) - M(s)$$

have non-positive expectation:

$$E_{\gamma(s)}(\Delta M(s)) \leq 0 \text{ in all situations } s.$$

Randomness on the martingale approach

The essential idea idea behind randomness is that there is no system for breaking the bank, for becoming unboundedly rich by betting on the successive outcomes in the sequence.

Randomness

A sequence ω is random for a forecasting system γ if no *non-negative* allowable supermartingale for γ becomes unbounded on ω.

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Martin-Löf randomness

A sequence ω is Martin-Löf random for a forecasting system γ if no *non-negative* lower semicomputable supermartingale for γ becomes unbounded on ω.

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Computable randomness

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Schnorr randomness

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ALLOWING FOR IMPRECISION

More general precise forecasting

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A single interval forecast $I = [\underline{p}, \overline{p}]$

Forecaster

specifies his interval forecast $I = [\underline{p}, \overline{p}]$ for an unknown outcome X in $\{0,1\}$: his commitment to adopt $\underline{p}$ as a highest buying price and $\overline{p}$ as a lowest selling price for X.

Skeptic

takes Forecaster up on his commitments:

(i) for any $p \leq \underline{p}$ and $\alpha \geq 0$, Forecaster must accept $\alpha(X - p)$;

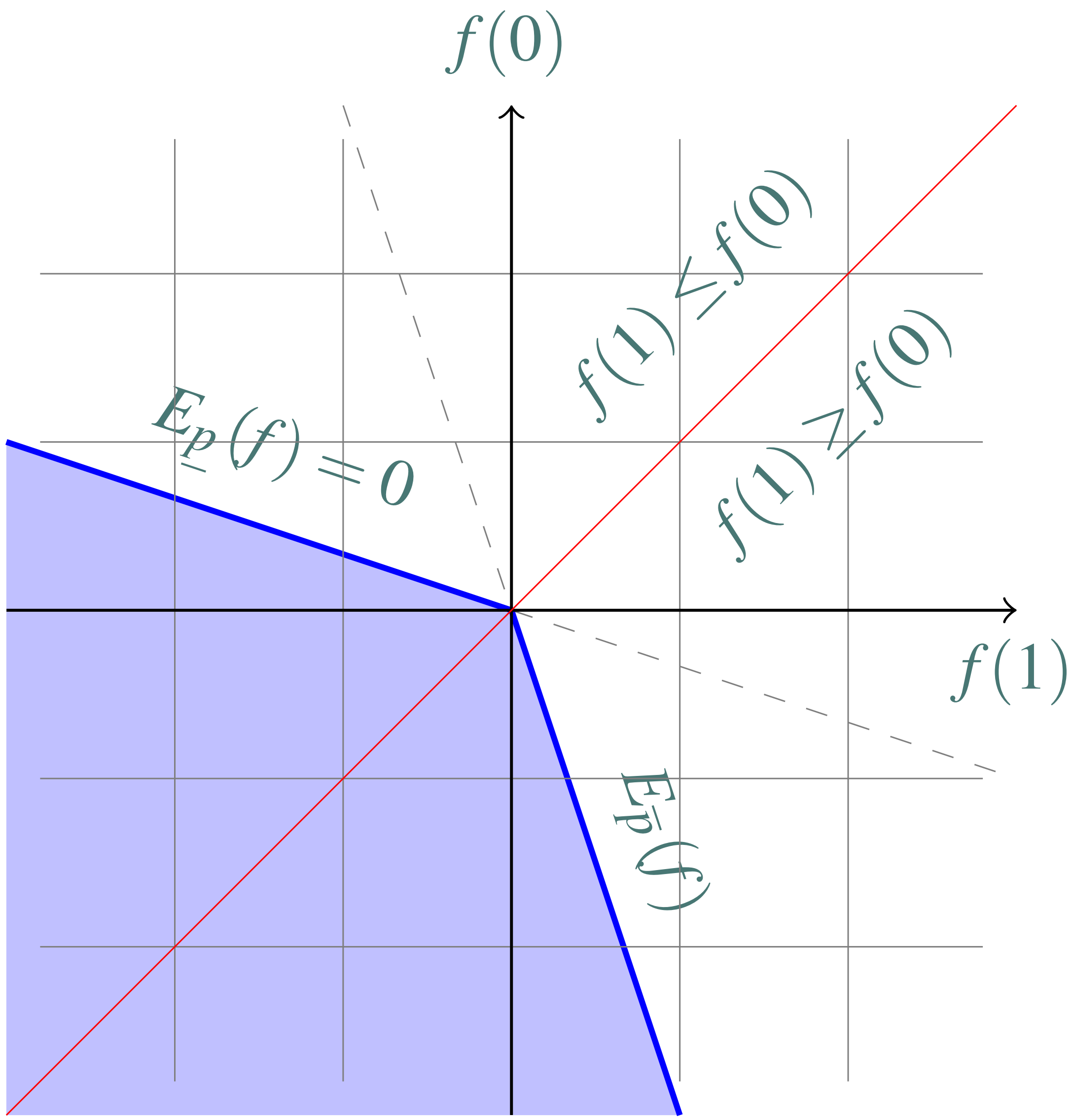
(ii) for any $q \geq \overline{p}$ and $\beta \geq 0$, Forecaster must accept $\beta(q - X)$.

Reality

determines the value x of X.

Gambles available to Skeptic: interval forecast $I = [\underline{p}, \overline{p}]$

$f(X) = -\alpha(X - p) - \beta(q - X)$ with $\alpha, \beta \geq 0$ and $0 \leq p \leq \underline{p} \leq \overline{p} \leq q \leq 1$

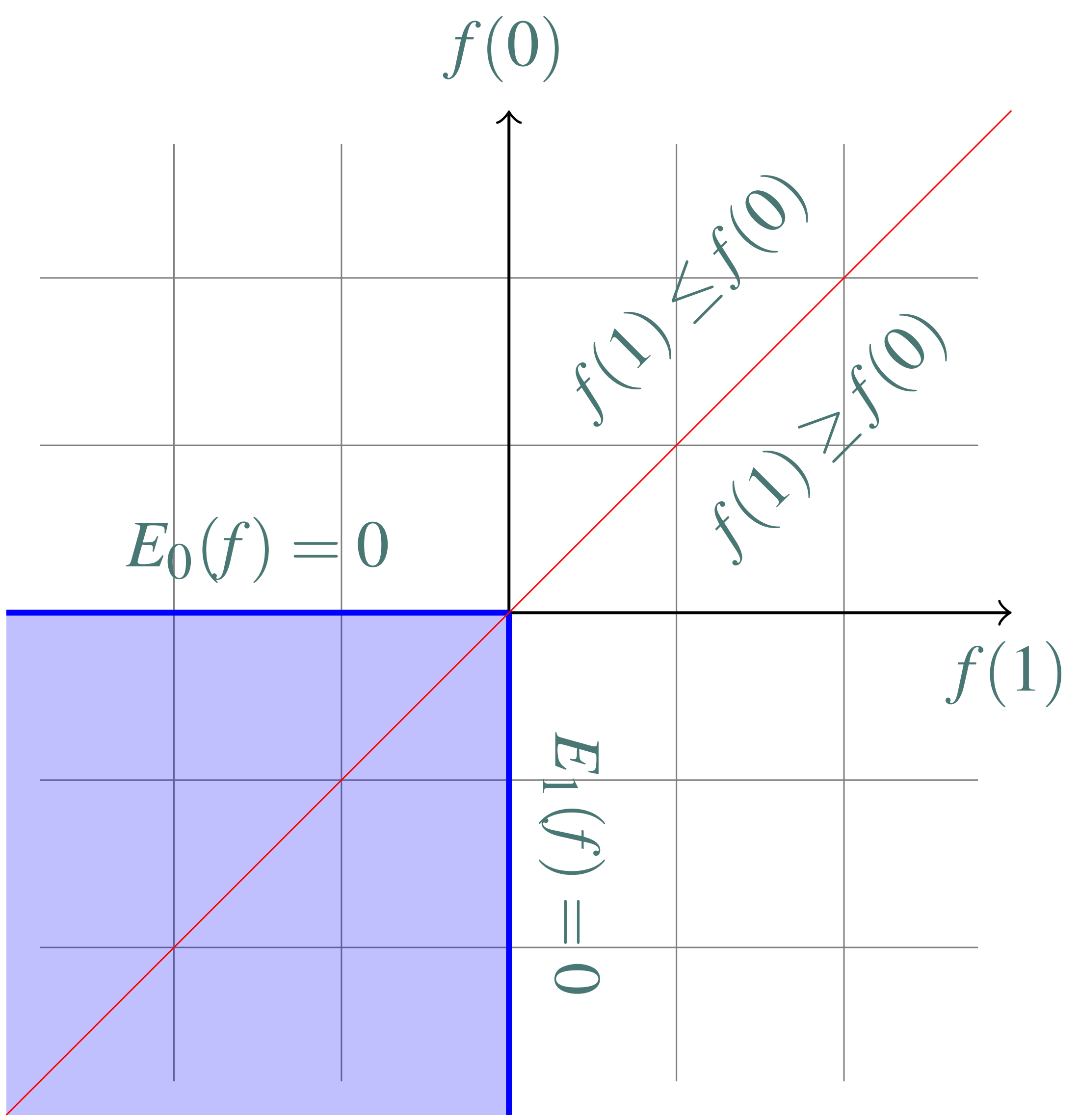


$$\overline{E}_I(f) := \max_{r \in I} E_r(f) \leq 0$$

Gambles available to Skeptic: vacuous forecast

$$I = [0, 1]$$

$f(X) = -\alpha(X - p) - \beta(q - X)$ with $\alpha, \beta \geq 0$ and $0 = p$ and $q = 1$

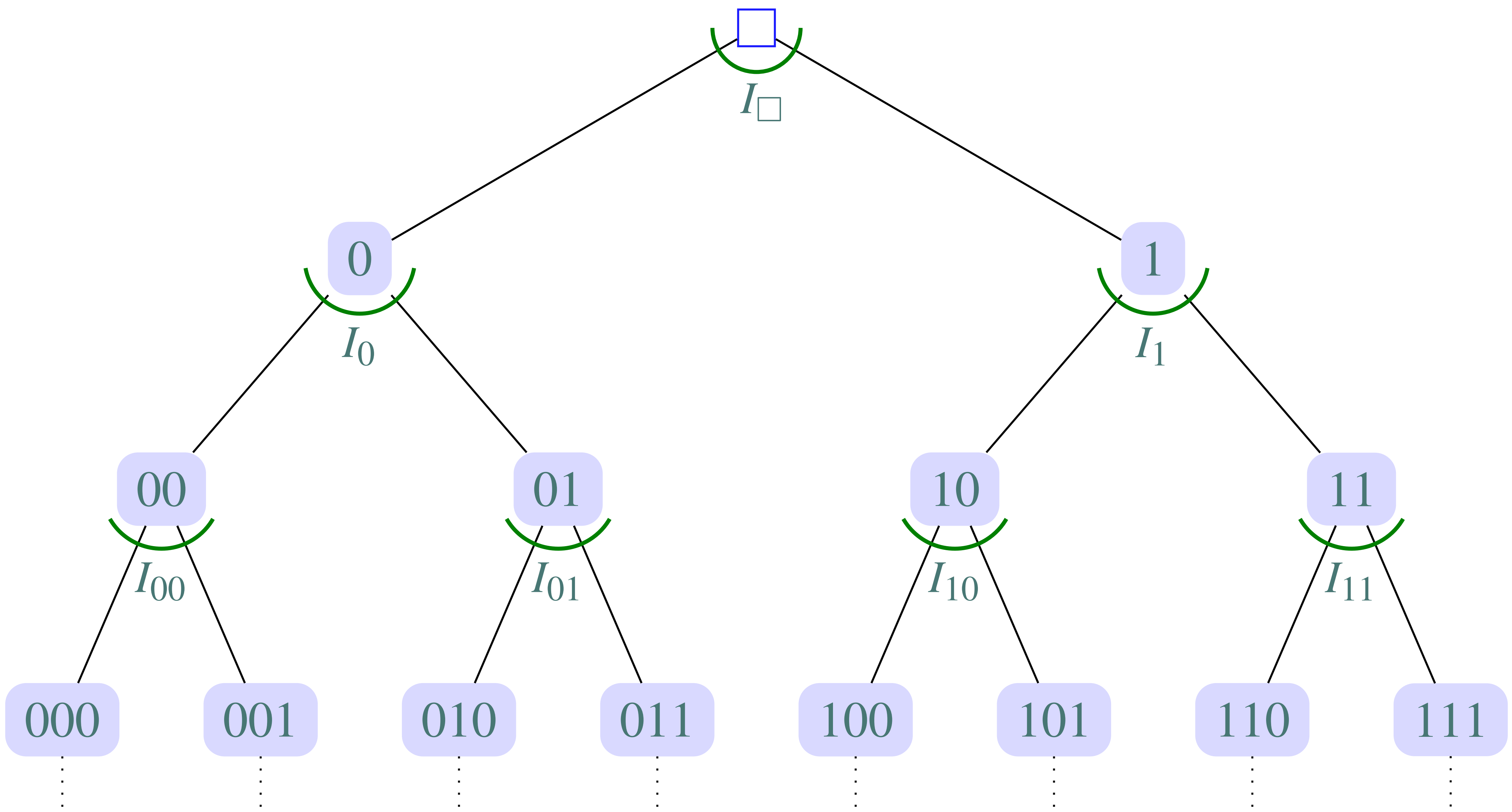


$$\overline{E}_I(f) := \max_{r \in [0,1]} E_r(f) = \max f \leq 0$$

More forecasts: imprecise probability tree

In an imprecise probability tree, we associate an interval forecast $\gamma(s) = I_s = [\underline{p}_s, \overline{p}_s]$ with each situation $s \in \Omega^\lozenge$:

forecasting system $\gamma \colon \Omega^\lozenge \to \mathscr{C}$



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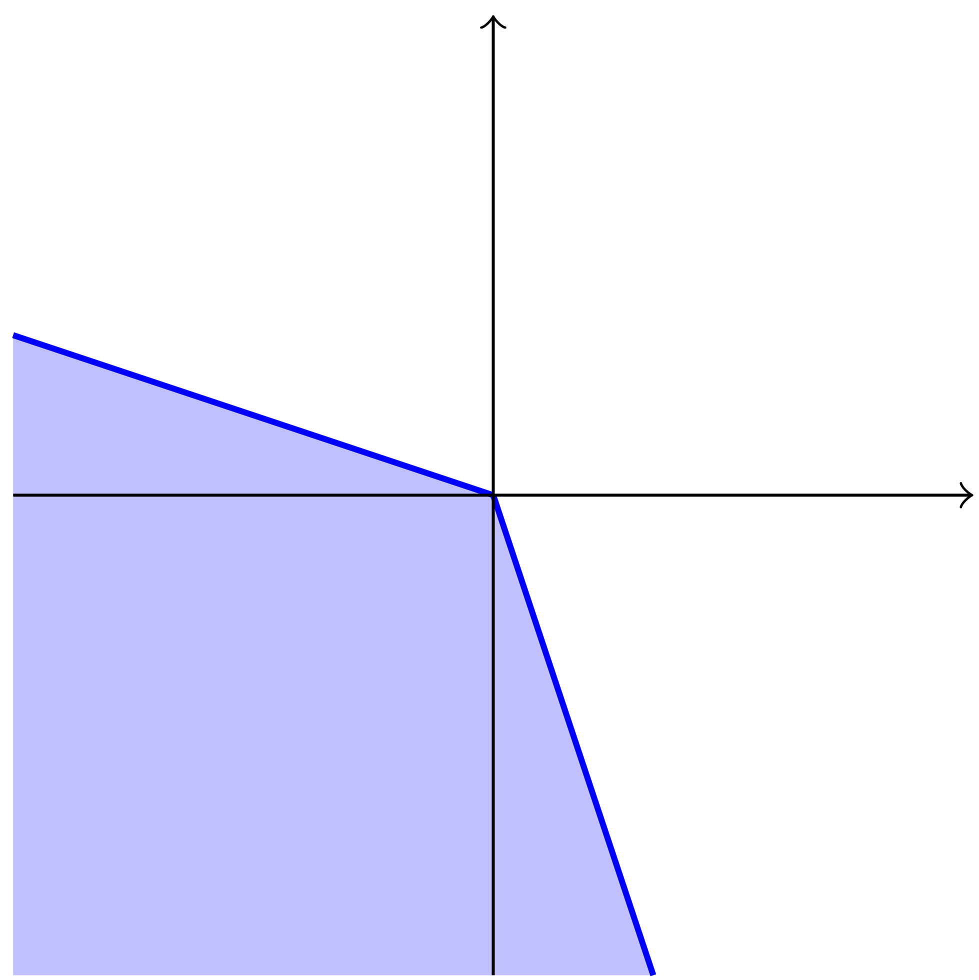

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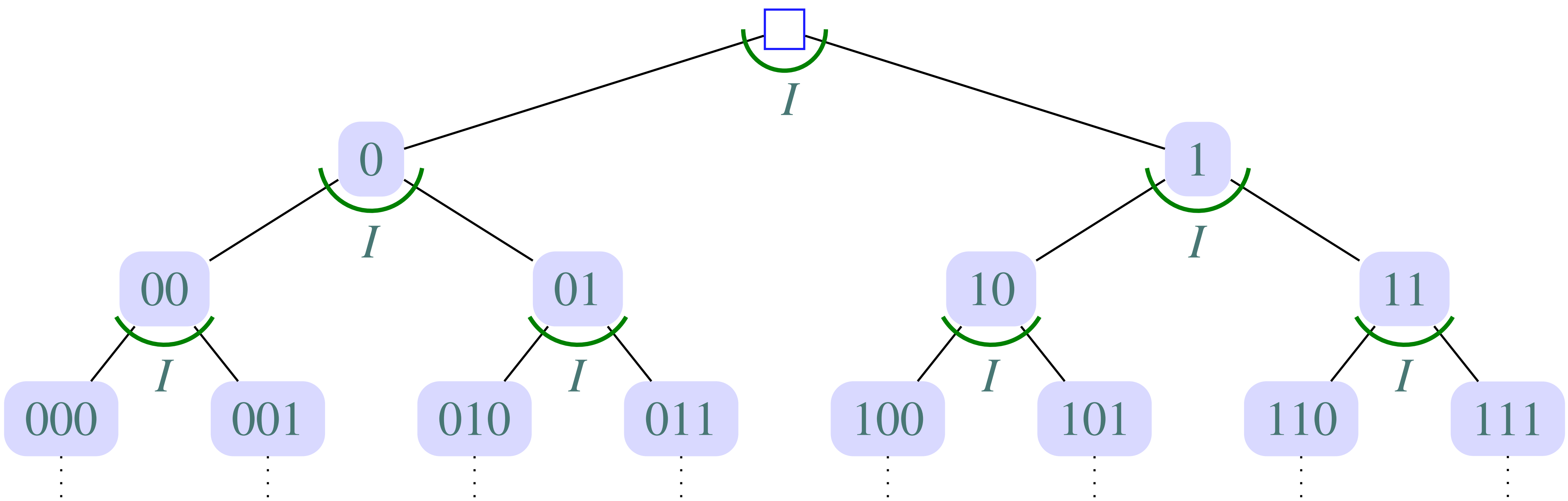
RANDOMNESS IS INHERENTLY IMPRECISE

Constant interval forecasts

$$\gamma_I(s) \coloneqq I \text{ for all } s \in \Omega^\Diamond.$$



$$\mathscr{C}_{\mathrm{C}}(\omega) = \{I \in \mathscr{C} : \gamma_I \text{ makes } \omega \text{ computably random}\}$$

EXAMPLES

A simple example

Consider any p and q in $[0,1]$ with $p \leq q$, and the forecasting system $\gamma_{p,q}$ defined by

$$\gamma_{p,q}(z_1,\ldots,z_n) := \begin{cases} p & \text{if } n \text{ is odd} \\ q & \text{if } n \text{ is even} \end{cases} \quad \text{for all } (z_1,\ldots,z_n) \in \Omega^\lozenge.$$

Theorem

Consider any outcome sequence ω that is computably random for $\gamma_{p,q}$. Then for all $I \in \mathscr{C}$:

$$I \in \mathscr{C}_{\mathrm{C}}(\omega) \Leftrightarrow [p,q] \subseteq I,$$

and therefore

$$\underline{p}_{\mathrm{C}}(\omega) = p \text{ and } \overline{p}_{\mathrm{C}}(\omega) = q.$$

A more complicated example

$$p_n := \frac{1}{2} + (-1)^n \delta_n, \text{ with } \delta_n := e^{-\frac{1}{n+1}} \sqrt{e^{\frac{1}{n+1}} - 1} \text{ for all } n \in \mathbb{N},$$

Consider the precise forecasting system $\gamma_{\sim 1/2}$ defined by

$$\gamma_{\sim 1/2}(z_1, \ldots, z_{n-1}) := p_n \text{ for all } n \in \mathbb{N} \text{ and } (z_1, \ldots, z_{n-1}) \in \Omega^{\Diamond}.$$

Theorem

Consider any outcome sequence ω that is computably random for $\gamma_{\sim 1/2}$. Then for all $I \in \mathscr{C}$:

$$I \in \mathscr{C}_{\mathrm{C}}(\omega) \Leftrightarrow \min I < \frac{1}{2} \text{ and } \max I > \frac{1}{2}$$

and therefore

$$\underline{p}_{\mathrm{C}}(\omega) = \overline{p}_{\mathrm{C}}(\omega) = \frac{1}{2}.$$

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The 20-year anniversary edition of the world's main forum on imprecise probabilities