Imprecise copulas constructed from shock models

Damjan Škulj

University of Ljubljana

11th Workshop on Principles and Methods of Statistical Inference with Interval Probability (WPMSIIP)
31 July 2018

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Outline

1. Copulas allow modelling dependence without the reference to marginal distributions.
2. We study two classes of copulas arising from specific real world models: Marshall and maxmin copulas.
3. We generalize the models to allow imprecision in the probability distributions.
4. The resulting models lead to imprecise copulas.

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Example of shock model

- Consider two components, say Component 1 and Component 2.
- They may be affected by one of the three fatal shocks.
  - The first and the second shock affect one or another component only.
  - The third shock affects both components.
- Two types of components:
  - Without recovery option: fails when affected by any shock.
  - With recovery option: fails when affected by both shocks.
- Let X, Y and Z be the times of the occurrences of the respective shocks.
- The time of failure of a component without recovery option is $U = \min\{X, Z\}$, and the one with recovery option is $V = \max\{Y, Z\}$.

Dependence of lifetimes of components

Our aim is to model the dependence of the lifetimes U and V of the components.

- The times of occurrences of the shocks X, Y, Z are assumed to be independent.
- When both components are of the same type, the dependence can be modelled by the use of Marshall copulas.
- If the components are of different types, a new class of copulas are used, called maxmin copulas.

Contents

1. Introduction
   - Shock models
   - Copulas and Sklar's theorem
2. Imprecise copulas
   - p-boxes
   - Imprecise copulas and a generalization of Sklar's theorem
3. Shock models
   - Marshall copulas
   - Maxmin copulas
4. Imprecise Marshall and maxmin copulas
   - Shock models in the imprecise settings
   - Monotonicity of Marshall and maxmin copulas

Copulas

A function $C\colon [0,1]\times[0,1]\to[0,1]$ is called a copula if it satisfies the following conditions:

(C1) $C(u,0)=C(0,v)=0$ for every $u,v\in[0,1]$;

(C2) $C(u,1)=u$ and $C(1,v)=v$ for every $u,v\in[0,1]$;

(C3) $C(u_2,v_2)-C(u_1,v_2)-C(u_2,v_1)+C(u_1,v_1)\geqslant 0$ for every $0\leqslant u_1\leqslant u_2\leqslant 1$ and $0\leqslant v_1\leqslant v_2\leqslant 1$.

Using copulas it is possible to model dependence between random variables without reference to their marginal distributions.
This is possible thanks to Sklar's theorem.

Sklar's theorem

Let $F\colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0,1]$ be a bivariate distribution function with marginals F_X and F_Y respectively, where $\overline{\mathbb{R}} = [-\infty, \infty]$.
Then there exists a copula C such that

$$F(x,y) = C(F_X(x), F_Y(y)), \tag{1}$$

Conversely, given any copula and a pair of distribution functions F_X and F_Y, (1) is a bivariate distribution function.

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

p-boxes

In the cases where the distribution function of a random variable X is uncertain, it is possible to represent the uncertainty via p-boxes:

- A p-box is a pair of distribution functions $(\underline{F}, \overline{F})$, where $\underline{F}(x) \leqslant \overline{F}(x)$ for every $x \in \mathbb{R}$.
- To every p-box, the closed and convex set of distribution functions $\mathcal{M} = \{F : \underline{F} \leqslant F \leqslant \overline{F}\}$ is assigned.

Bivariate p-boxes

- Given a pair of random variables (X, Y), their joint distribution is described using a bivariate distribution function $F\colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$.
- A bivariate function $F\colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, 1]$ is standardized if it is componentwise increasing and satisfies

$$F(-\infty, y) = F(x, -\infty) = 0 \ \forall x, y \in \overline{\mathbb{R}}, F(\infty, \infty) = 1.$$

- A pair $(\underline{F}, \overline{F})$ of standardized functions such that $\underline{F}(x, y) \leqslant \overline{F}(x, y)$ is a bivariate p-box (Pelessoni et al. [2016], Montes et al. [2015]).
- Note that $\underline{F}$ and $\overline{F}$ are not required to be bivariate distribution functions themselves.

- To each bivariate p-box, the set of distribution functions $\mathcal{M} = \{F$ bivariate distribution function: $\underline{F} \leqslant F \leqslant \overline{F}\}$ is assigned.
- If $\underline{F}(x, y) = \min_{F \in \mathcal{M}} F(x, y)$ and $\overline{F}(x, y) = \max_{F \in \mathcal{M}} F(x, y)$ holds, then the bivariate p-box $(\underline{F}, \overline{F})$ is called coherent.
- Note that in the case where $\underline{F}$ and $\overline{F}$ are themselves distribution functions, the corresponding bivariate p-box is coherent.

Contents

1. Introduction
   - Shock models
   - Copulas and Sklar's theorem
2. Imprecise copulas
   - p-boxes
   - Imprecise copulas and a generalization of Sklar's theorem
3. Shock models
   - Marshall copulas
   - Maxmin copulas
4. Imprecise Marshall and maxmin copulas
   - Shock models in the imprecise settings
   - Monotonicity of Marshall and maxmin copulas

Imprecise copula

A pair of functions $(\underline{C}, \overline{C})$, both mapping $[0,1]^2 \to [0,1]$ is called an imprecise copula (Montes et al. [2015]) if

(IC1) $\underline{C}(0,u) = \underline{C}(u,0) = 0, \underline{C}(1,u) = \underline{C}(u,1) = u \ \forall u \in [0,1]$;

(IC2) $\overline{C}(0,u) = \overline{C}(u,0) = 0, \overline{C}(1,u) = \overline{C}(u,1) = u \ \forall u \in [0,1]$;

(IC3) For every $u_1 \leqslant u_2, v_1 \leqslant v_2$:

- $\underline{C}(u_2,v_2) + \overline{C}(u_1,v_1) - \underline{C}(u_2,v_1) - \underline{C}(u_1,v_2) \geqslant 0$;
- $\overline{C}(u_2,v_2) + \underline{C}(u_1,v_1) - \underline{C}(u_2,v_1) - \underline{C}(u_1,v_2) \geqslant 0$;
- $\overline{C}(u_2,v_2) + \overline{C}(u_1,v_1) - \overline{C}(u_2,v_1) - \underline{C}(u_1,v_2) \geqslant 0$;
- $\overline{C}(u_2,v_2) + \overline{C}(u_1,v_1) - \underline{C}(u_2,v_1) - \overline{C}(u_1,v_2) \geqslant 0$;

Note that $\underline{C}$ and $\overline{C}$ are not necessarily copulas themselves.

Properties of imprecise copulas

The following properties hold:

- If $(\underline{C}, \overline{C})$ is an imprecise copula, then $\underline{C} \leqslant \overline{C}$.
- Given a set of (precise) copulas $\mathcal{C}$, the bounds:

$$\underline{C}(u,v) = \inf_{C \in \mathcal{C}} C(u,v)$$
$$\overline{C}(u,v) = \sup_{C \in \mathcal{C}} C(u,v)$$

  form an imprecise copula $(\underline{C}, \overline{C})$.
- If $\underline{C} \leqslant \overline{C}$ and both are copulas, then $(\underline{C}, \overline{C})$ is an imprecise copula.

Sklar's theorem in the imprecise settings (Montes et al. [2015])

Let X and Y be random variables.

- Their distribution functions are only known up to p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ respectively.
- Let $(\underline{C}, \overline{C})$ be an imprecise copula.
- Define:

$$\underline{F}(x, y) = \underline{C}(\underline{F}_X, \underline{F}_Y)$$
$$\overline{F}(x, y) = \overline{C}(\overline{F}_X, \overline{F}_Y)$$

Then $(\underline{F}, \overline{F})$ is a coherent p-box.

Unlike the precise case, not every coherent p-box can be represented in this way by the means of its marginals.

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Shock model

Let X, Y and Z be independent random variables.

- F_X, F_Y, F_Z are their distribution functions.
- We consider the interdependence of the following pairs of random variables:
  - $U = \max\{X, Z\}$ and $V = \max\{Y, Z\}$ (Marshall [1996])
  - $U = \min\{X, Z\}$ and $V = \min\{Y, Z\}$ (Marshall [1996])
  - $U = \max\{X, Z\}$ and $W = \min\{Y, Z\}$ (Omladič and Ružić [2016])

The first two models behave similarly, while the third one is substantially different.

All of them can be related to modelling lifetimes of certain components that may be affected by fatal shocks.

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Marshall copulas

A Marshall copula is a function $[0,1]^2 \to [0,1]$ defined with

$$C(u,v) = C_{\phi,\psi}(u,v) = uv \min\left\{\frac{\phi(u)}{u}, \frac{\psi(v)}{v}\right\},$$

where

(P1) $\phi(0) = \phi(0) = 0, \psi(1) = \psi(1) = 1;$

(P2) ϕ and ψ are two increasing functions $[0,1] \to [0,1]$;

(P3) $\phi^*(u) = \frac{\phi(u)}{u}$ and $\psi^*(v) = \frac{\psi(v)}{v}$ are decreasing functions.

Modelling shocks with Marshall copulas

- Given independent X, Y, Z and $U = \max\{X, Z\}, V = \max\{Y, Z\}$, the distribution functions for U and V are respectively

$$F = F_X F_Z \qquad \text{and} \qquad G = F_Y F_Z.$$

- Their joint distribution is then

$$H_{U,V}(x, y) = C_{\phi,\psi}(F(x), G(y)),$$

where ϕ and ψ are such that they satisfy $\phi(F) = F_X$ and $\psi(G) = F_Y$.

Contents

1. Introduction
   - Shock models
   - Copulas and Sklar's theorem
2. Imprecise copulas
   - p-boxes
   - Imprecise copulas and a generalization of Sklar's theorem
3. Shock models
   - Marshall copulas
   - Maxmin copulas
4. Imprecise Marshall and maxmin copulas
   - Shock models in the imprecise settings
   - Monotonicity of Marshall and maxmin copulas

Maxmin copulas

A maximin copula is a map $C^*\colon [0,1]^2 \to [0,1]$ defined with

$$C^*(u,v) = C^*_{\phi,\chi}(u,v) = uv + \min\{u(1-v), (\phi(u)-u)(v-\chi(v))\}.$$

where ϕ and χ satisfy

(F1) $\phi(0) = \chi(0) = 0, \chi(1) = \chi(1) = 1;$

(F2) ϕ and χ are non-decreasing;

(F3) $\phi^*(u) = \frac{\phi(u)}{u}$ and $\chi_*(v) = \frac{1-\chi(v)}{v-\chi(v)}$ are non-increasing.

Modelling shocks with maxmin copulas

- Let X, Y and Z be independent
- with respective distribution functions F_X, F_Y and F_Z.
- Take $U = \max\{X, Z\}$ and $W = \min\{Y, Z\}$.
- Let F, K and H respectively be the distribution functions of U, W and the joint distribution of (U, W).
- Then:

$$\begin{aligned} F(x) &= F_X(x)F_Z(x) \\ K(y) &= F_Y(y) + F_Z(y) - F_Y(y)F_Z(y) \\ H(x,y) &= C^*_{\phi,\chi}(F(x), G(y)), \end{aligned}$$

where $F_X = \phi(F)$ and $F_Y = \chi(G)$.

Contents

1 Introduction
- Shock models
- Copulas and Sklar's theorem

2 Imprecise copulas
- p-boxes
- Imprecise copulas and a generalization of Sklar's theorem

3 Shock models
- Marshall copulas
- Maxmin copulas

4 Imprecise Marshall and maxmin copulas
- Shock models in the imprecise settings
- Monotonicity of Marshall and maxmin copulas

Contents

1. Introduction
   - Shock models
   - Copulas and Sklar's theorem
2. Imprecise copulas
   - p-boxes
   - Imprecise copulas and a generalization of Sklar's theorem
3. Shock models
   - Marshall copulas
   - Maxmin copulas
4. Imprecise Marshall and maxmin copulas
   - Shock models in the imprecise settings
   - Monotonicity of Marshall and maxmin copulas

Shock models in the imprecise settings

- Consider random variables X, Y and Z.
- Distribution functions of X and Y are only known up to p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$.
- The distribution function F_Z of Z is given precisely.
- Let $U = \max\{X, Z\}, V = \max\{Y, Z\}, W = \min\{Y, Z\}$.
- Copulas modelling joint distributions of (U, V) and (U, W) are of interest.
- It is natural to consider imprecise copulas for the imprecise case.

Marginal p-boxes

Let $(\underline{F}, \overline{F}), (\underline{G}, \overline{G}), (\underline{K}, \overline{K})$ denote the marginal p-boxes for U, V and W respectively. Then:

$$\underline{F}(x) = \underline{F}_X F_Z \qquad \overline{F}(x) = \overline{F}_X F_Z$$

$$\underline{G}(x) = \underline{F}_Y F_Z \qquad \overline{G}(x) = \overline{F}_Y F_Z$$

$$\underline{K}(x) = \underline{F}_Y + F_Z - \underline{F}_Y F_Z \qquad \overline{K}(x) = \overline{F}_Y + F_Z - \overline{F}_Y F_Z$$

Contents

1. Introduction
   - Shock models
   - Copulas and Sklar's theorem
2. Imprecise copulas
   - p-boxes
   - Imprecise copulas and a generalization of Sklar's theorem
3. Shock models
   - Marshall copulas
   - Maxmin copulas
4. Imprecise Marshall and maxmin copulas
   - Shock models in the imprecise settings
   - Monotonicity of Marshall and maxmin copulas

Monotonicity of Marshall and maxmin copulas

Let (F_X, F_Y, F_Z) and (F'_X, F'_Y, F_Z) be two triples of distribution functions. They give rise to a pair of Marshall and maxmin copulas respectively

$$\begin{aligned} C_{\phi,\psi}, \qquad & C^*_{\phi,\chi} \\ C_{\phi',\psi'}, \qquad & C^*_{\phi',\chi'}, \end{aligned}$$

We now analyse how the order: $F_X \leqslant F'_X$ and $F_Y \leqslant F'_Y$ translates to the order on the corresponding ϕ, ψ and χ.

Explicit derivation

The functions ϕ and ψ have identical properties, where

$$\phi(F) = F_X$$

is the required relation.
The above requirement only defines ϕ on $\mathrm{im} F$:

$$\phi(u) = F_X(F^{-1}(u)) = \frac{u}{F_Z(F^{-1}(u))},$$

where F^{-1} denotes the quasi inverse of F:

$$F^{-1}(u) = \inf\{x \in \mathbb{R} \colon F(x) \geqslant u\}.$$

The explicit derivation for the second parameter function ψ in a Marshall copula is the same.

Explicit expression for χ

The second parameter function χ in a maxmin copula is defined by requiring

$$\chi(K) = F_Y,$$

which is a unique requirement on $\mathrm{im} K$, where it translates to

$$\chi(v) = F_Y(K^{-1}(v)) = \frac{v - F_Z(K^{-1}(v))}{1 - F_Z(K^{-1}(v))}$$

Extension to $[0, 1]$

There are several possible extensions to $[0, 1]$.
One of them uses linear interpolation:

$$\phi(u) = \begin{cases} 0, & \text{if } u = 0, \\ F_X(F^{-1}(u)), & \text{if } u \in \mathrm{im}F \backslash \{0, 1\}, \\ 1, & \text{if } u = 1, \\ \frac{\phi(\overline{u}) - \phi(\underline{u}-)}{\overline{u} - \underline{u}}(u - \underline{u}) + \phi(\underline{u}-), & \text{if } u \notin \mathrm{im}F \cup \{0, 1\}, \end{cases}$$

where $f(x-)$ denotes the left limit and $\overline{u} = F(F^{-1}(u)), \underline{u} = F(F^{-1}(u-))$.

Similarly we can extend χ.

The fact that $F \leqslant F'$ implies $F^{-1} \geqslant F'^{-1}$ implies that

$$\phi(u) = \frac{u}{F_Z(F^{-1}(u))} \leqslant \frac{u}{F_Z(F'^{-1}(u))} = \phi'(u)$$

for every $u \in \mathrm{im}F \cap \mathrm{im}F'$; and similarly

$$\chi(v) = \frac{1 - v\, F_Z(K^{-1}(v))}{1 - F_Z(K^{-1}(v))} \leqslant \frac{1 - v\, F_Z(K'^{-1}(v))}{1 - F_Z(K'^{-1}(v))} = \chi'(v)$$

Unfortunately, these relations are not transferred to every extension to $[0, 1]$.

In particular, they do not hold for the extensions with the linear interpolation.

Constructing extensions satisfying the required order

The following proposition holds:

Proposition

Let $F_X \leqslant F'_X$ and F_Z be given distribution functions, and

$$\phi(u) = \frac{u}{F_Z(F^{-1}(u))}, \qquad \phi'(u) = \frac{u}{F_Z(F'^{-1}(u))}$$

on the corresponding images of $F = F_X F_Z$ and $F' = F'_X F_Z$.
Then if $\mathrm{im} F \subseteq \mathrm{im} F'$ or $\mathrm{im} F \supseteq \mathrm{im} F'$, extensions $\hat{\phi} \leqslant \hat{\phi}'$ of ϕ and ϕ' to the whole interval $[0, 1]$ exist.

Moreover, we prove that for every pair $F \leqslant F'$ of distribution functions, a function $\tilde{F}$ exists, so that $F \leqslant \tilde{F} \leqslant F'$ and $\mathrm{im}F \cup \mathrm{im}F' \subseteq \mathrm{im}\tilde{F}$.
The desired theorem now easily follows:

Theorem

Let $F_X \leqslant F'_Y$ and F_Z be given distribution functions, and the corresponding

$$\phi(u) = \frac{u}{F_Z(F^{-1}(u))}, \qquad \phi'(u) = \frac{u}{F_Z(F'^{-1}(u))}$$

on the corresponding images of F and F'. Then there exist extensions $\hat{\phi}$ and $\hat{\phi}'$ of ϕ and ϕ' respectively to the whole interval $[0, 1]$ such that $\hat{\phi} \leqslant \hat{\phi}'$.

Something similar can be said for the functions ϕ and χ.

Monotonicity of Marshall and maxmin copulas

Theorem

Let

$$(F_X, F_Y, F_Z), \qquad (F'_X, F'_Y, F_Z)$$

be two triples of distribution functions, such that

$$F_X \leqslant F'_X \qquad F_Y \leqslant F'_Y.$$

Then there exist pairs of functions

$$\phi \leqslant \phi' \qquad \psi \leqslant \psi' \qquad \chi \leqslant \chi'$$

so that

$C_{\phi,\psi}$ and $C_{\phi',\psi'}$ model the dependence between U and V

and

$C^*_{\phi,\chi}$ and $C^*_{\phi',\chi'}$ models the dependence between U and W.

assuming the distribution functions F_X, F_Y and F'_X, F'_Y respectively.

Imprecise Marshall and maxmin copulas

The above proposition justifies consideration of the following sets of copulas:

$$\mathcal{C} = \{C_{\phi,\psi} \colon \underline{\phi} \leqslant \phi \leqslant \overline{\phi}, \underline{\psi} \leqslant \psi \leqslant \overline{\psi}\}$$
$$\mathcal{C}^* = \{C^*_{\phi,\chi} \colon \underline{\phi} \leqslant \phi \leqslant \overline{\phi}, \underline{\chi} \leqslant \chi \leqslant \overline{\chi}\}$$

We will call the above sets of copulas imprecise Marshall copula and imprecise maxmin copula respectively.
The following inequalities hold:

$$C_{\underline{\phi},\underline{\psi}} \leqslant C_{\phi,\psi} \leqslant C_{\overline{\phi},\overline{\psi}} \ \forall C_{\phi,\psi} \in \mathcal{C}$$
$$C^*_{\underline{\phi},\overline{\chi}} \leqslant C^*_{\phi,\chi} \leqslant C^*_{\overline{\phi},\underline{\chi}} \ \forall C^*_{\phi,\chi} \in \mathcal{C}^*$$

Joint distributions

Joint distributions between U, V and between U, W are given via the bivariate p-boxes with the following bounds:

$$\underline{H}(x,y) = C_{\underline{\phi},\underline{\psi}}(\underline{F}, \underline{G})$$
$$\overline{H}(x,y) = C_{\overline{\phi},\overline{\psi}}(\overline{F}, \overline{G})$$

for the pair (U, V), and

$$\underline{H}^*(x,y) = C^*_{\underline{\phi},\underline{\chi}}(\underline{F}, \underline{K})$$
$$\overline{H}^*(x,y) = C^*_{\overline{\phi},\overline{\psi}}(\overline{F}, \overline{K})$$

for the pair (U, W).
Notice that the imprecise copula $(\underline{C}^*, \overline{C}^*) = (C^*_{\underline{\phi},\overline{\chi}}, C^*_{\overline{\phi},\underline{\chi}})$ would result in too conservative bounds.

Further work

- Allow the common shocks to have imprecise distributions.
- Analyse and interpret the difference between bounds generated by the imprecise copulas and the 'true' bounds for the bivariate distributions.
- Analyse the effects of different independence assumptions.
- Analyse the dependence properties if the times of shock occurrences are dependent.

References I

Thomas Augustin, Frank PA Coolen, Gert de Cooman, and Matthias CM Troffaes. *Introduction to imprecise probabilities*. John Wiley & Sons, 2014.

Albert W. Marshall. Copulas, marginals, and joint distributions. *Lecture Notes-Monograph Series*, 28:213–222, 1996. ISSN 07492170. URL http://www.jstor.org/stable/4355894.

Ignacio Montes, Enrique Miranda, Renato Pelessoni, and Paolo Vicig. Sklar's theorem in an imprecise setting. *Fuzzy Sets and Systems*, 278:48 – 66, 2015. ISSN 0165-0114. doi: https://doi.org/10.1016/j.fss.2014.10.007. URL http://www.sciencedirect.com/science/article/pii/S0165011414004539. Special Issue on uncertainty and imprecision modelling in decision making (EUROFUSE 2013).

References II

Matjaž Omladič and Nina Ružić. Shock models with recovery option via the maxmin copulas. *Fuzzy Sets and Systems*, 284:113 – 128, 2016. ISSN 0165-0114. doi: https://doi.org/10.1016/j.fss.2014.11.006. URL http://www.sciencedirect.com/science/article/pii/S0165011414004965. Theme: Uncertainty and Copulas.

Renato Pelessoni, Paolo Vicig, Ignacio Montes, and Enrique Miranda. Bivariate p-boxes. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):229–263, 2016. doi: 10.1142/S0218488516500124. URL http://www.worldscientific.com/doi/abs/10.1142/S0218488516500124.