Computing limit expectations of imprecise continuous-time Markov chains

Alexander Erreygers   Jasper De Bock

WPMSIIP 2018

Ghent University, ELIS, SYSTeMS


Puma

Can we determine $\underline{E}_\infty(f)$

without explicitly evaluating

$$\lim_{t\to+\infty} \underline{E}_{\mathscr{M}_0}\left(\lim_{n\to+\infty}\left(I+\tfrac{t}{n}\underline{Q}\right)^n f\right)?$$

Continuous-time Markov chains

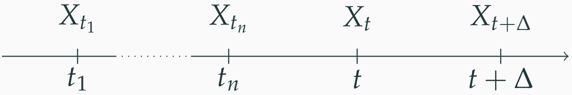
Basic set-up

Objective

making inferences about the *state* X_t of some system

Assumptions

1. state space is finite
2. time parameter is continuous
3. dynamics are non-deterministic, Markovian & homogeneous



$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$

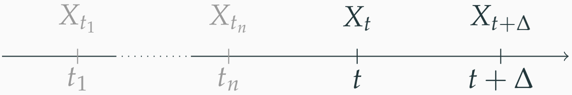
Basic set-up

Objective

making inferences about the *state* X_t of some system

Assumptions

1. state space is finite
2. time parameter is continuous
3. dynamics are non-deterministic, Markovian & homogeneous



$$P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$
$$= P(X_{t+\Delta} = y \mid X_t = x)$$

[Markov property]

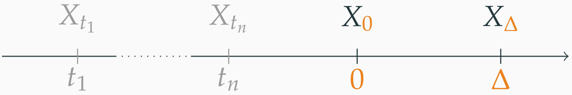
Basic set-up

Objective

making inferences about the *state* X_t of some system

Assumptions

1. state space is finite
2. time parameter is continuous
3. dynamics are non-deterministic, Markovian & homogeneous



$$\begin{aligned} P(X_{t+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\ = P(X_{t+\Delta} = y \mid X_t = x) & \quad \text{[Markov property]} \\ = P(X_\Delta = y \mid X_0 = x) & \quad \text{[homogeneity]} \end{aligned}$$

Characterisation

A *homogeneous CTMC* is fully characterised by

1. a (finite) *state space* $\mathscr{X}$;
2. an *initial distribution* π_0; $[P(X_0 = x) = \pi_0(x)]$
3. a *transition rate matrix* Q.

[nonnegative off-diagonal elements and zero row sums]

Marginal expectations

How do we compute $E(f(X_t))$?

1. solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_\tau f = Q T_\tau f \qquad \text{with initial condition } T_0 f = f$$

[Note: $T_\tau f \colon \mathscr{X} \to \mathbb{R}$]

2. compute

$$E(f(X_t)) = E_{\pi_0}(T_t f)$$

Marginal expectations

How do we compute $E(f(X_t))$?

1. solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_\tau f = Q T_\tau f \qquad \text{with initial condition } T_0 f = f$$

[Note: $T_\tau f \colon \mathscr{X} \to \mathbb{R}$]

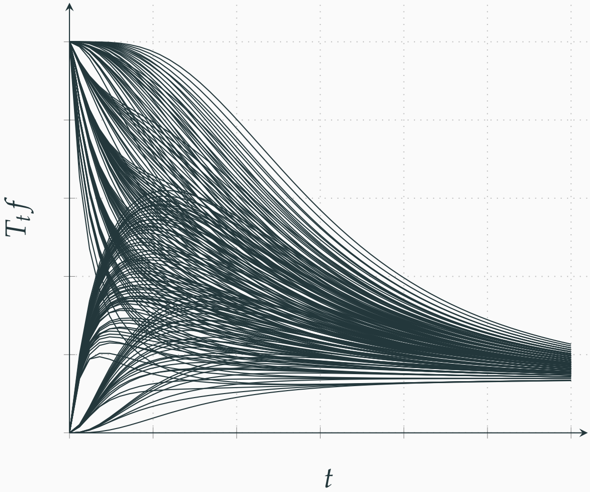
i.e., evaluate

$$T_t f = e^{tQ} f := \lim_{n \to +\infty} \left(I + \frac{t}{n} Q \right)^n f$$

2. compute

$$E(f(X_t)) = E_{\pi_0}(T_t f)$$

Typical temporal behaviour of $T_t f$



Limit expectations

In many applications, one is interested in the limit expectation

$$E_\infty(f) := \lim_{t \to +\infty} E(f(X_t)) = \lim_{t \to +\infty} E_{\pi_0}(T_t f).$$

Definition (Ergodicity)

The transition rate matrix Q is *ergodic* if, for all $f \colon \mathscr{X} \to \mathbb{R}$,

$$E_\infty(f) \text{ does not depend on } \pi_0$$

or equivalently,

$$\lim_{t \to +\infty} T_t f \text{ is a constant function.}$$

Imprecise continuous-time Markov chains

- Thomas Krak, Jasper De Bock, and Arno Siebes. “Imprecise continuous-time Markov chains”. In: *International Journal of Approximate Reasoning* 88 (2017), pp. 452–528
- Jasper De Bock. “The Limit Behaviour of Imprecise Continuous-Time Markov Chains”. In: *Journal of Nonlinear Science* 27.1 (2017), pp. 159–196

Characterisation

An *imprecise* CTMC is characterised by

1. a (finite) state space $\mathscr{X}$;
2. an initial distribution π_0;
3. a transition rate matrix Q.

Characterisation

An *imprecise* CTMC is characterised by

1. a (finite) state space $\mathscr{X}$;
2. a set of initial distributions $\mathscr{M}_0$;
3. a set of transition rate matrices $\mathscr{Q}$.

Characterisation

An *imprecise* CTMC is characterised by

1. a (finite) state space $\mathscr{X}$;
2. a set of initial distributions $\mathscr{M}_0$;
3. a set of transition rate matrices $\mathscr{Q}$.

Problem

These sets do not characterise a single CTMC!

Solution

Consider the sets of stochastic processes that are consistent with $\mathscr{M}_0$ and $\mathscr{Q}$:

$\mathbb{P}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}$ the set of consistent homogeneous CTMCs,

Characterisation

An *imprecise* CTMC is characterised by

1. a (finite) state space $\mathscr{X}$;
2. a set of initial distributions $\mathscr{M}_0$;
3. a set of transition rate matrices $\mathscr{Q}$.

Problem

These sets do not characterise a single CTMC!

Solution

Consider the sets of stochastic processes that are consistent with $\mathscr{M}_0$ and $\mathscr{Q}$:

$\mathbb{P}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}$ the set of consistent homogeneous CTMCs,
$\mathbb{P}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}$ the set of consistent CTMCs,
$\mathbb{P}_{\mathscr{M}_0,\mathscr{Q}}$ the set of consistent stochastic processes.

Lower envelopes

Krak et al. (2017) define the coherent lower expectations

$$\mathbb{P}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}$$

$$\mathbb{P}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}$$

$$\mathbb{P}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}_{\mathscr{M}_0,\mathscr{Q}}$$

Lower envelopes

Krak et al. (2017) define the coherent lower expectations

$$\mathbb{P}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}$$

$$\mathbb{P}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}$$

$$\mathbb{P}_{\mathscr{M}_0,\mathscr{Q}} \xrightarrow{\text{lower envelope}} \underline{E}_{\mathscr{M}_0,\mathscr{Q}}$$

They also define a lower envelope of $\mathscr{Q}$. The *lower transition rate operator* $\underline{Q}\colon \mathscr{L}(\mathscr{X}) \to \mathscr{L}(\mathscr{X})$ is defined by

$$[\underline{Q}f](x) := \inf\{[Qf](x) \colon Q \in \mathscr{Q}\}.$$

[superadditive, nonneg. hom., ~ zero row sums, ~ nonneg. off-diagonal elements]

Marginal lower expectations

Observe that

$$\mathbb{P}_{\mathscr{M}_0,\mathscr{Q}} \supseteq \mathbb{P}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}} \supseteq \mathbb{P}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}.$$

This implies that

$$\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)) \leq \underline{E}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)) \leq \underline{E}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)).$$

Furthermore, Krak et al. (2017) show that [under some conditions on $\mathscr{Q}$]

$$\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)) = \underline{E}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)) \leq \underline{E}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)).$$

Determining $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$

How do we compute $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$?

1. solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_\tau f = Q T_\tau f \qquad \text{with initial condition } T_0 f = f$$

[Note: $T_\tau f \colon \mathscr{X} \to \mathbb{R}$]

i.e., evaluate

$$T_t f = \lim_{n \to +\infty} \left(I + \frac{t}{n} Q \right)^n f$$

2. compute

$$E(f(X_t)) = E_{\pi_0}(T_t f)$$

Determining $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$

How do we compute $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$?

1. solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_\tau f = Q T_\tau f \qquad \text{with initial condition } T_0 f = f$$

[Note: $T_\tau f \colon \mathscr{X} \to \mathbb{R}$]

i.e., evaluate

Underline *all* the operators!

$$T_t f = \lim_{n\to+\infty} \left(I + \frac{t}{n} Q \right)^n f$$

2. compute

$$E(f(X_t)) = E_{\pi_0}(T_t f)$$

Determining $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$

How do we compute $\underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$?

1. solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\underline{T}_\tau f = \underline{Q}\underline{T}_\tau f \qquad \text{with initial condition } \underline{T}_0 f = f$$

[Note: $\underline{T}_\tau f \colon \mathscr{X} \to \mathbb{R}$]

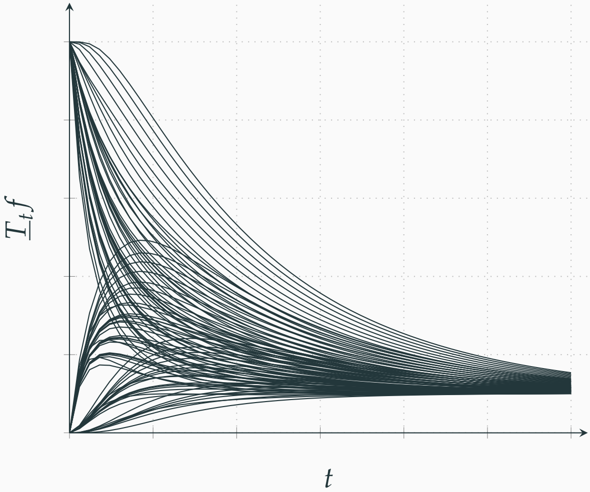
i.e., evaluate

$$\underline{T}_t f = \lim_{n\to+\infty}\left(I + \frac{t}{n}\underline{Q}\right)^n f.$$

2. compute

$$\underline{E}_{\mathcal{M}_0,\mathcal{Q}}(f(X_t)) = \underline{E}_{\mathcal{M}_0}(\underline{T}_t f)$$

Typical temporal behaviour of $\underline{T}_t f$



Limit expectations

We now turn to the *limit lower expectations*

$$\underline{E}_\infty(f) := \lim_{t\to+\infty} \underline{E}_{\mathscr{M}_0}(\underline{T}_t f)$$
$$= \lim_{t\to+\infty} \underline{E}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)) = \lim_{t\to+\infty} \underline{E}^{\mathrm{M}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t))$$

and

$$\underline{E}^{\mathrm{HM}}_\infty(f) := \lim_{t\to+\infty} \underline{E}^{\mathrm{HM}}_{\mathscr{M}_0,\mathscr{Q}}(f(X_t)).$$

Limit expectations

We now turn to the *limit lower expectation*

$$\underline{E}_\infty(f) \coloneqq \lim_{t\to+\infty} \underline{E}_{\mathscr{M}_0}(\underline{T}_t f).$$

Definition (De Bock, 2017)

The lower transition rate operator $\underline{Q}$ is *ergodic* if, for all $f\colon \mathscr{X} \to \mathbb{R}$,

$$\underline{E}_\infty(f) \text{ does not depend on } \mathscr{M}_0$$

or equivalently,

$$\lim_{t\to+\infty} \underline{T}_t f \text{ is a constant function.}$$

Can we determine $\underline{E}_{\infty}(f)$

without explicitly evaluating

$$\lim_{t\to+\infty} \underline{E}_{\mathscr{M}_0}\left(\lim_{n\to+\infty}\left(I+\tfrac{t}{n}\underline{Q}\right)^n f\right)?$$

A well-known result

Theorem

If Q is an ergodic transition rate matrix, then for all $\delta > 0$ with $\delta\|Q\| < 2$, E_∞ is the unique $(I+\delta Q)$-invariant expectation operator:

$$E_\infty((I+\delta Q)f) = E_\infty(f) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X})$$
$$\Leftrightarrow E_\infty(Qf) = 0 \qquad \text{for all } f \in \mathscr{L}(\mathscr{X}).$$

- simply solve the linear system of equations $\pi_\infty Q = 0$

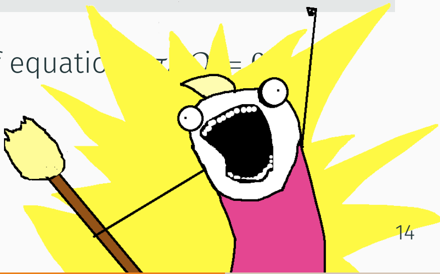
A well-known result

Theorem

If Q is an ergodic transition rate matrix, then for all $\delta > 0$ with $\delta\|Q\| < 2$, E_∞ is the unique $(I+\delta Q)$-invariant expectation operator:

$$E_\infty((I+\delta Q)f) = E_\infty(f) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X})$$
$$\Leftrightarrow E_\infty(Qf) = 0 \qquad \text{for all } f \in \mathscr{L}(\mathscr{X}).$$

- simply solve the linear system of equations $\pi^T Q = 0$



Explicitly determining $\underline{E}_\infty$

Conjecture

If $\underline{Q}$ is an ergodic lower transition rate operator, then

1. for all $\delta > 0$ with $\delta\|\underline{Q}\| < 2$, $\underline{E}_\infty$ is the unique $(I + \delta\underline{Q})$-invariant lower expectation operator:

$$\underline{E}_\infty((I + \delta\underline{Q})f) = \underline{E}_\infty(f) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X});$$

2. $\underline{E}_\infty(\underline{Q}f) = 0$ for all $f \in \mathscr{L}(\mathscr{X})$.

Explicitly determining $\underline{E}_\infty$

Conjecture

If $\underline{Q}$ is an ergodic lower transition rate operator, then

1. for all $\delta > 0$ with $\delta\|\underline{Q}\| < 2$, $\underline{E}_\infty$ is the unique $(I + \delta\underline{Q})$-invariant lower expectation operator:
$$\underline{E}_\infty((I + \delta\underline{Q})f) = \underline{E}_\infty(f) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X});$$
2. ~~$\underline{E}_\infty(\underline{Q}f) = 0$ for all $f \in \mathscr{L}(\mathscr{X})$~~.

Explicitly determining $\underline{E}_\infty$

Conjecture

If $\underline{Q}$ is an ergodic lower transition rate operator, then

1. for all $\delta > 0$ with $\delta <?$, $\underline{E}_\infty$ is the unique $(I + \delta\underline{Q})$-invariant lower expectation operator:

$$\underline{E}_\infty((I + \delta\underline{Q})f) = \underline{E}_\infty(f) \qquad \text{for all } f \in \mathscr{L}(\mathscr{X});$$

2. ~~$\underline{E}_\infty(\underline{Q}f) = 0$ for all $f \in \mathscr{L}(\mathscr{X})$~~.

¿ some alternative general upper bound ?

¿ efficient way to solve this “set of equations” ?

Another well-known result

Theorem

If Q is an ergodic transition rate matrix, then for all $\delta > 0$ with $\delta\|Q\| < 2$,

$$E_\infty(f) = \lim_{m\to+\infty} \min(I + \delta Q)^m f.$$

- works for any (sufficiently small) δ
- non-decreasing in m [Emp.: convergence is faster for larger δ]
- easy to implement

Another well-known result

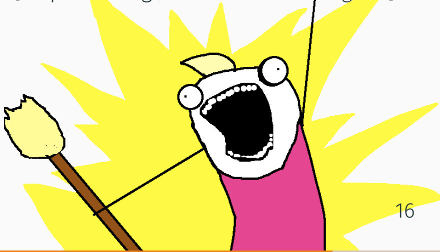
Theorem

If Q is an ergodic transition rate matrix, then for all *$\delta > 0$ with $\delta\|Q\| < 2$,*

$$E_\infty(f) = \lim_{m \to +\infty} \min(I + \delta Q)^m f.$$

Underline *all* the operators?

- works for any (su
- non-decreasing in m [Emp.: convergence is faster for larger δ]
- easy to implement



Iteratively determining $\underline{E}_\infty(f)$

Conjecture

If $\underline{Q}$ is an ergodic lower transition rate operator, then for all $\delta > 0$ with $\delta\|\underline{Q}\| < 2$,

$$\underline{E}_\infty(f) = \lim_{m\to+\infty} \min(I + \delta\underline{Q})^m f.$$

Iteratively determining $\underline{E}_\infty(f)$

Theorem

If $\underline{Q}$ is an ergodic lower transition rate operator, then ~~for all $\delta > 0$ with $\delta\|\underline{Q}\| < 2$,~~

$$\underline{E}_\infty(f) = \lim_{\delta \to 0^+} \lim_{m \to +\infty} \min(I + \delta \underline{Q})^m f.$$

- \- extra limit for δ
- \+ $\min(I + \delta \underline{Q})^m f$ non-decreasing in m
- \+ relatively easy to implement
- ¿ does the value of δ matter ?

Can we determine $\underline{E}_{\infty}^{\mathrm{HM}}(f)$

without explicitly evaluating

$\lim_{t \to +\infty} \inf\{E_P(f(X_t)) \colon P \in \mathbb{P}^{\mathrm{HM}}_{\mathcal{M}_0,\mathcal{Q}}\}$?

Iteratively computing a lower bound on $\underline{E}_\infty^{\mathrm{HM}}(f)$

Theorem

If $\mathcal{Q}$ consists of only ergodic transition rate matrices, then for all n and $\delta > 0$ with $\delta\|\mathcal{Q}\| < 2$,

$$\min(I + \delta \underline{Q})^n f \leq \underline{E}_\infty^{\mathrm{HM}}(f).$$

- $\min(I + \delta \underline{Q})^n f$ converges monotonously for $n \to +\infty$
- ¿ behaviour in function of δ ?