Minimizar longitud de intervalo

Población gausiana, σ conocida

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- $X \hookrightarrow \mathcal{N}(\mu,\sigma) \implies \bar{X} \hookrightarrow \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \implies$ pivote $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \hookrightarrow \mathcal{N}(0,1) =: \Phi =: \int \varphi$
- $\alpha_1 + \alpha_2 := \alpha \quad \Phi(a) := \alpha_1 \quad \Phi(b) := 1-\alpha_2 = 1-\alpha+\alpha_1$
- $1-\alpha = \Pr\left[a \leqslant \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leqslant b\right] = \Pr\left[\bar{X} - b\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} - a\frac{\sigma}{\sqrt{n}}\right] \implies$ I.C.$(\mu) = \left[\bar{X} - b\frac{\sigma}{\sqrt{n}}, \bar{X} - a\frac{\sigma}{\sqrt{n}}\right]$
- longitud $L = \bar{X} - a\frac{\sigma}{\sqrt{n}} - \left(\bar{X} - b\frac{\sigma}{\sqrt{n}}\right) = \frac{\sigma}{\sqrt{n}}(b-a) = \frac{\sigma}{\sqrt{n}}\left[\Phi^{-1}(1-\alpha+\alpha_1) - \Phi^{-1}(\alpha_1)\right]$
- si en $\alpha_1 \in (0,\alpha)$ hay un extremo, entonces $0 = \frac{\mathrm{d}\,L}{\mathrm{d}\,\alpha_1} = \frac{\sigma}{\sqrt{n}}\left[\frac{\mathrm{d}\,\Phi^{-1}(1-\alpha+\alpha_1)}{\mathrm{d}\,\alpha_1} - \frac{\mathrm{d}\,\Phi^{-1}(\alpha_1)}{\mathrm{d}\,\alpha_1}\right] = \frac{\sigma}{\sqrt{n}}\left[\frac{1}{\varphi(\Phi^{-1}(1-\alpha+\alpha_1))} - \frac{1}{\varphi(\Phi^{-1}(\alpha_1))}\right] \iff \varphi\left(\Phi^{-1}(\alpha_1)\right) = \varphi\left(\Phi^{-1}(1-\alpha+\alpha_1)\right) \iff \Phi^{-1}(\alpha_1) = -\Phi^{-1}(1-\alpha+\alpha_1)$ por simetría de $\mathcal{N}(0,1)$ respecto a $0 \iff \alpha_1 = \alpha_2 = \frac{\alpha}{2}$
- se trata de un mínimo, pues $\frac{\mathrm{d}^2\,L}{\mathrm{d}\,\alpha_1^2} = \frac{\sigma}{\sqrt{n}}\frac{\mathrm{d}}{\mathrm{d}\,\alpha_1}\left[\frac{1}{\varphi(\Phi^{-1}(1-\alpha+\alpha_1))} - \frac{1}{\varphi(\Phi^{-1}(\alpha_1))}\right] =$
$$\frac{\sigma}{\sqrt{n}}\left[\frac{-\overbrace{\varphi'\left(\Phi^{-1}(1-\alpha+\alpha_1)\right)}^{<0 \Leftarrow \Phi^{-1}(1-\alpha+\alpha_1)>0}\overbrace{\frac{1}{\varphi\left(\Phi^{-1}(1-\alpha+\alpha_1)\right)}}^{>0 \Leftarrow \varphi \text{ densidad}}}{\varphi\left(\Phi^{-1}(1-\alpha+\alpha_1)\right)^2} - \frac{-\overbrace{\varphi'\left(\Phi^{-1}(\alpha_1)\right)}^{>0 \Leftarrow \Phi^{-1}(\alpha_1)<0}\frac{1}{\varphi\left(\Phi^{-1}(\alpha_1)\right)}}{\underbrace{\varphi\left(\Phi^{-1}(\alpha_1)\right)^2}_{>0}}\right] > 0$$
- el intervalo más corto se obtiene con $\alpha_1 = \alpha_2 = \alpha/2$, es decir, con $a = -b = -z_{1-\alpha/2} = z_{\alpha/2} := \Phi^{-1}(\alpha/2) = -\Phi^{-1}(1-\alpha/2)$