1. Enunciado

$$X \hookrightarrow N(\mu, \sigma)$$

$$H_0\colon\ \mu = \mu_0 \cap \sigma = \sigma_0 \qquad H_1\colon\ \mu = \mu_1 \cap \sigma = \sigma_1$$

2. Resolución

$$f = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathcal{L} = \frac{1}{\sigma^n \sqrt{2\pi}^n} e^{-\frac{\sum (x_i-\mu)^2}{2\sigma^2}}$$

$$\begin{aligned}
\text{R.C.} &= \left\{\vec{x} \,\middle|\, \frac{\mathcal{L}_1}{\mathcal{L}_0} > k\right\} = \left\{\vec{x} \,\middle|\, \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\left[\frac{\sum (x_i-\mu_1)^2}{2\sigma_1^2} - \frac{\sum (x_i-\mu_0)^2}{2\sigma_0^2}\right]} > k\right\} \\
&= \left\{\vec{x} \,\middle|\, \frac{\sum (x_i-\mu_1)^2}{2\sigma_1^2} - \frac{\sum (x_i-\mu_0)^2}{2\sigma_0^2} < k'\right\} \\
\langle * \rangle &= \left\{\vec{x} \,\middle|\, \frac{\sum (x_i-\mu_1)^2}{\sigma_1^2} - \frac{\sum (x_i-\mu_0)^2}{\sigma_0^2} < k''\right\} \\
&= \left\{\vec{x} \,\middle|\, \sum (x_i-\mu_1)^2 - \frac{\sigma_1^2}{\sigma_0^2}\sum (x_i-\mu_0)^2 < k'''\right\} \\
\left\langle c = \frac{\sigma_1^2}{\sigma_0^2} \right\rangle &= \left\{\vec{x} \,\middle|\, \sum (x_i-\mu_1)^2 - c\sum (x_i-\mu_0)^2 < k'''\right\} \\
&= \left\{\vec{x} \,\middle|\, \sum (x_i^2+\mu_1^2-2\mu_1 x_i) - c\sum (x_i^2+\mu_0-2\mu_0 x_i) < k'''\right\} \\
&= \left\{\vec{x} \,\middle|\, \sum x_i^2 + n\mu_1^2 - 2\mu_1 \sum x_i - \left(c\sum x_i^2 + cn\mu_0 - 2c\mu_0 \sum x_i\right) < k'''\right\} \\
&= \left\{\vec{x} \,\middle|\, \sum x_i^2 + n\mu_1^2 - 2\mu_1 \sum x_i - c\sum x_i^2 - cn\mu_0 + 2c\mu_0 \sum x_i < k'''\right\} \\
&= \left\{\vec{x} \,\middle|\, (1-c)\sum x_i^2 + n\mu_1^2 + 2(c\mu_0-\mu_1)\sum x_i - cn\mu_0 < k'''\right\} \\
&= \left\{\vec{x} \,\middle|\, (1-c)\sum x_i^2 + 2(c\mu_0-\mu_1)\sum x_i < k''''\right\}
\end{aligned}$$

Dados unos parámetros concretos, por ejemplo

$$\mu_0 = 10 \qquad \mu_1 = 12 \qquad \sigma_0 = 1 \qquad \sigma_1 = 2 \qquad n = 25 \qquad \alpha = 0\text{'}05$$

la frontera de la R.C. puede aproximarse mediante montecarlo:

```
> mu0 <- 10
> mu1 <- 12
> sigma0 <- 1
> sigma1 <- 2
> c <- sigma1^2/sigma0^2
> alfa <- 0.05
```

```
> n <- 25
> nr <- 1e6 # número de repeticiones montecarlo
> d1 <- replicate (nr, # distribución montecarlo
+ {
+     x <- rnorm(n,mu0,sigma0)
+     (1-c)*sum(x^2)+2*(c*mu0-mu1)*sum(x)
+ })
> (k.... <- quantile(d1,alfa)) # k''''

      5%
6373.366
```

3. Casos particulares

3.1. $\sigma_0 = \sigma_1 \iff c = 1$

$$
\begin{aligned}
\text{R.C.} &= \left\{ \vec{x} \;\middle|\; (1-c)\sum x_i^2 + 2(c\mu_0 - \mu_1)\sum x_i < k'''' \right\} \\
&= \left\{ \vec{x} \;\middle|\; (1-1)\sum x_i^2 + 2(\mu_0 - \mu_1)\sum x_i < k'''' \right\} \\
&= \left\{ \vec{x} \;\middle|\; 2(\mu_0 - \mu_1)\sum x_i < k'''' \right\} \\
&= \left\{ \vec{x} \;\middle|\; (\mu_0 - \mu_1)\sum x_i < k'''' \right\} \\
\langle \text{supóngase } \mu_1 > \mu_0 \rangle &= \left\{ \vec{x} \;\middle|\; \sum x_i > k^{(5)} \right\} \\
&= \left\{ \vec{x} \;\middle|\; \bar{x} > k^{(6)} \right\}
\end{aligned}
$$

Se obtiene la misma región crítica que intuitivamente.

3.2. $\mu_0 = \mu_1$

Supóngase $\sigma_1 > \sigma_0$, luego $c > 1$. Intuitivamente se podría pensar en una R.C. basada en un estimador de σ:

$$
\left\{ \vec{x} \mid S^2 > k_1 \right\} = \left\{ \vec{x} \;\middle|\; \hat{S}^2 > k_2 \right\}
$$

Aplicando Neyman-Pearson:

$$
\begin{aligned}
\text{R.C.} &= \left\{ \vec{x} \;\middle|\; (1-c)\sum x_i^2 + 2(c\mu_0 - \mu_1)\sum x_i < k'''' \right\} \\
&= \left\{ \vec{x} \;\middle|\; (1-c)\sum x_i^2 + 2(c\mu_0 - \mu_0)\sum x_i < k_4 \right\} \\
&= \left\{ \vec{x} \;\middle|\; (1-c)\sum x_i^2 + 2(c-1)\mu_0\sum x_i < k_4 \right\} \\
&= \left\{ \vec{x} \;\middle|\; (1-c)\sum x_i^2 - 2(1-c)\mu_0\sum x_i < k_4 \right\} \\
\langle 1 - c < 0 \rangle &= \left\{ \vec{x} \;\middle|\; \sum x_i^2 - 2\mu_0\sum x_i > k_5 \right\} \\
&= \left\{ \vec{x} \;\middle|\; \sum x_i^2 - 2\mu_0\sum x_i + n\mu_0^2 > k_6 \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{\vec{x} \;\middle|\; \sum(x_i^2 - 2\mu_0 x_i + \mu_0^2) > k_6\right\} \\
&= \left\{\vec{x} \;\middle|\; \sum(x_i - \mu_0)^2 > k_6\right\} \\
&= \left\{\vec{x} \;\middle|\; \frac{\sum(x_i - \mu_0)^2}{n} > k_7\right\} \\
&= \left\{\vec{x} \mid S_0^2 > k_7\right\}
\end{aligned}
$$

$S_0^2 = \frac{1}{n}\sum(X_i - \mu_0)^2$ sería un estimador de la varianza cuando la esperanza fuese conocida. El estadístico asociado tendría distribución:

$$\frac{nS_0^2}{\sigma^2} \hookrightarrow \chi_n^2 \qquad \Longleftrightarrow \qquad S_0^2 \hookrightarrow \frac{\sigma^2}{n}\chi_n^2$$

Se gana un grado de libertad con respecto a la distribución asociada a la cuasivarianza:

$$\frac{(n-1)\hat{S}^2}{\sigma^2} \hookrightarrow \chi_{n-1}^2 \qquad \Longleftrightarrow \qquad \hat{S}^2 \hookrightarrow \frac{\sigma^2}{n-1}\chi_{n-1}^2$$

Eso supone obtener un contraste con más potencia; de hecho, por Neyman-Pearson, se trata del contraste más potente posible. Por ejemplo:

```
> mu0 <- 0; mu1 <- 0; sigma0 <- 5; sigma1 <- 10
> n <- 10; alfa <- 0.05
> ## RC1 = [cuasivarianza > k1]   RC2 = [S0^2 > k2]
> (k1 <- sigma0^2 * qchisq (1-alfa, n-1) / (n-1))

[1] 46.99716

> (k2 <- sigma0^2 * qchisq (1-alfa, n) / n)

[1] 45.7676

> 1 - pchisq ((n-1) * k1 / sigma1^2, n-1) # potencia RC1

[1] 0.8956509

> 1 - pchisq (n * k2 / sigma1^2, n) # potencia RC2

[1] 0.9176019
```