**Ejercicio 1.** Sea $X \equiv N(\mu, \sigma)$ con $\mu$ conocido. Se pretende contrastar

$$\left.\begin{array}{l} H_0 \equiv \sigma^2 = \sigma_0^2 \\ H_1 \equiv \sigma^2 = \sigma_1^2 \end{array}\right\} \qquad \text{con } \sigma_0^2 < \sigma_1^2$$

Usar Neyman-Pearson para contrastar estas hipótesis.

*Resolución.*

$$L(\vec{x}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum(x_i-\mu)^2\right] = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[\frac{-n}{2\sigma^2}S^2\right]$$

$$\begin{aligned} \frac{L_1}{L_0} &= \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} \exp\left[\frac{-n}{2\sigma_1^2}S^2 + \frac{n}{2\sigma_0^2}S^2\right] \\ &= \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} \exp\left[\frac{nS^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right] \end{aligned}$$

$$\begin{aligned} \text{R.C.} &= \left\{\vec{x} \;\middle|\; \frac{L_1(\vec{x})}{L_0(\vec{x})} > k\right\} = \left\{\vec{x} \;\middle|\; \ln\frac{L_1(\vec{x})}{L_0(\vec{x})} > \ln k\right\} \\ &= \left\{\frac{n}{2}\ln\frac{\sigma_0^2}{\sigma_1^2} + n\frac{S^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > \ln k\right\} \\ &= \left\{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)S^2 > \frac{2\ln k}{n} - \ln\frac{\sigma_0^2}{\sigma_1^2}\right\} \\ &= \left\{S^2 > \frac{\frac{2\ln k}{n} - \ln\frac{\sigma_0^2}{\sigma_1^2}}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}\right\} = \{S^2 > k^*\} \end{aligned}$$

ya que $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$. Como

$$\sum_{i=1}^n (X_i - \mu)^2 = \sigma^2 \sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma}\right)^2 = \sigma^2 \cdot \chi_n^2$$

entonces

$$\begin{aligned} P(S^2 > k^* \mid H_0) &= P\left(\frac{\sigma_0^2}{n}\sum\left(\frac{X_i-\mu}{\sigma_0}\right)^2 > k^* \;\middle|\; H_0\right) \\ &= P\left(\frac{\sigma_0^2}{n}\cdot\chi_n^2 > k^*\right) = P\left(\chi_n^2 > \frac{n\cdot k^*}{\sigma_0^2}\right) \\ &\Longrightarrow \frac{n \cdot k^*}{\sigma_0^2} = \texttt{qchisq(1-}\alpha\texttt{,n)} \\ &\Longrightarrow k^* = \sigma_0^2\texttt{/n*qchisq(1-}\alpha\texttt{,n)} \end{aligned}$$

□

**Ejemplo 1.**

$$\left.\begin{array}{l} H_0 \equiv \sigma^2 = 1 \\ H_1 \equiv \sigma^2 = 2 \end{array}\right\} \qquad \mu = 5 \qquad \vec{x} \text{ m.a.s. con } n = 10$$

1. Generar muestra bajo $H_1$.

```
> alfa <- 0.05
> sigma0 <- sqrt(1)
> sigma1 <- sqrt(2)
> mu <- 5
> n <- 10
> options (width = 65)
> (x <- rnorm (n, mu, sigma1))
```

```
 [1] 4.039551 6.884250 4.617631 3.742682 5.560996 6.898465
 [7] 5.304683 7.979297 3.593416 4.083605
```

2. Calcular la R.C.

```
> (k. <- sigma0^2 / n * qchisq (1-alfa, n))
```

```
[1] 1.830704
```

3. Comprobar pertenencia a la R.C.

```
> (S2 <- sum ((x - mu) ^ 2) / n)
```

```
[1] 2.19061
```

```
> S2 > k.
```

```
[1] TRUE
```

4. Calcular P-valor del contraste.

```
> 1 - pchisq (S2 * n / sigma0^2, n)
```

```
[1] 0.01559015
```

**Ejercicio 2.** Sea $X \equiv N(\mu, \sigma)$ con $\mu$ conocida. Se pretende contrastar

$$\begin{array}{l} H_0 \equiv \sigma^2 = \sigma_0^2 \\ H_1 \equiv \sigma^2 > \sigma_0^2 \end{array}$$

a partir de una muestra de tamaño $n$. Aplicar el teorema de Karlin-Rubin.

*Resolución.* Sea un $\sigma_1^2 > \sigma_0^2$. Entonces

$$\begin{aligned} \frac{L(\vec{x},\sigma_1^2)}{L(\vec{x},\sigma_0^2)} &= \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} \exp\left[\frac{-n}{2\sigma_1^2}S^2 + \frac{n}{2\sigma_0^2}S^2\right] \\ &= \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{\frac{n}{2}} \exp\left[\frac{nS^2}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right] \end{aligned}$$

Por ser $\sigma_1^2 > \sigma_0^2 \iff \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$ la razón de verosimilitudes es una función creciente de $S^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2$.

Sea

$$\varphi(\vec{x}) = \begin{cases} 1 & \text{si } S^2 > c \\ \gamma & \text{si } S^2 = c \\ 0 & \text{si } S^2 < c \end{cases}$$

Como

$$S^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2 = \frac{\sigma^2}{n}\sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma}\right)^2 \equiv \frac{\sigma^2}{n} \cdot \chi_n^2$$

es una variable continua, $P(S^2 = c) = 0$ y el test se puede plantear con

$$\text{R.C.} = \left\{\vec{x} \;\middle|\; S^2 > c\right\}$$

□