ANOVA

análisis de varianza

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Planteamiento

- Sean q poblaciones x_i $(i = 1, \ldots, q)$
  - niveles o modalidades de un “factor”
- $x_i = \mu_i + \epsilon \equiv \mathcal{N}(\mu_i, \sigma) \quad$ con $\epsilon \equiv \mathcal{N}(0, \sigma) = \mathcal{N}_1(0, \sigma^2)$
- De cada x_i se conoce una muestra aleatoria simple de tamaño n_i: $x_{i1}, \ldots, x_{in_i}$
- Tamaño muestral total: $n = \sum_{i=1}^{q} n_i$
- Otra parametrización: $\mu_i = \mu + \alpha_i \quad$ con $\mu = \frac{\sum n_i \mu_i}{n}$
- $\sum_{i=1}^{q} n_i \alpha_i = 0$
- Si $\forall\, i,\ n_i = \frac{n}{q}$ el modelo se dice *equilibrado* o *balanceado*

Planteamiento

Hipótesis previas

$$\forall\, i = 1, \ldots, q \quad \vec{x}_i = (x_{i1}, \ldots, x_{in_i})^t \equiv \mathcal{N}_{n_i}\left(\mu_i \vec{1}_{n_i}, \sigma^2\, I_{n_i}\right)$$

- independencia (muestras aleatorias simples)
- gausianidad
- homoscedasticidad (igualdad de varianzas)

Planteamiento

Hipótesis del análisis de varianza (ANOVA)

$$
\begin{array}{lcccc}
H_0 & \equiv & \forall\, i,j,\;\; \mu_i = \mu_j & \equiv & \forall\, i,\;\; \alpha_i = 0 \\
H_1 & \equiv & \exists\, i,j,\;\; \mu_i \neq \mu_j & \equiv & \exists\, i,\;\; \alpha_i \neq 0
\end{array}
$$

- H_0: el factor no influye en la respuesta
- H_1: el factor sí influye en la respuesta

Repaso de álgebra lineal

- Matriz simétrica $A = [a_{ij}]_{i,j}$ si $a_{ij} = a_{ji} \iff A = A^t$
    - todos sus autovalores $(\lambda_i)_i$ son reales; autovectores: $(\vec{u}_i)_i$
    - $A = U\,\Lambda\,U^t$ con $\Lambda = \operatorname{diag}(\dots, \lambda_i, \dots)$ y $U = [\dots, \vec{u}_i \dots]$
- Matriz idempotente: $A\,A = A$
    - (autovalor A) $\in \{0, 1\}$ pues si $\vec{x} \neq \vec{0}$ es autovector no nulo

$$\lambda\,\vec{x} = A\,\vec{x} = A\,A\,\vec{x} = A\,\lambda\,\vec{x} = \lambda\,A\,\vec{x} = \lambda^2\vec{x}$$

$$\implies \qquad \lambda = 0 \quad \cup \quad \left\{ \vec{x} = \lambda\vec{x} \implies \lambda = 1 \right\}$$

- Traza: $A = [a_{ij}]_{i,j} \implies \operatorname{tr} A = \sum_i a_{ii}$
    - $a = \operatorname{tr}[a]$
    - $\operatorname{tr}(A\,B) = \operatorname{tr}(B\,A)$
    - $\operatorname{tr}(A\,B\,C) = \operatorname{tr}(B\,C\,A) = \operatorname{tr}(C\,A\,B)$
    - $\mathrm{E}\,[\operatorname{tr} A] = \operatorname{tr} \mathrm{E}\,[A]$

Matriz de varianzas y covarianzas

$$\mathrm{Var}(\vec{x}) = \mathrm{Cov}(\vec{x}) = \mathrm{Cov}(\vec{x},\vec{x}) = \mathrm{E}\left[(\vec{x}-\mathrm{E}[\vec{x}])(\vec{x}-\mathrm{E}[\vec{x}])^t\right]$$

$$= \mathrm{E}\left[\begin{pmatrix}(x_1-\mathrm{E}[x_1])\\ \vdots \\ (x_q-\mathrm{E}[x_q])\end{pmatrix}\begin{bmatrix}(x_1-\mathrm{E}[x_1]) & \dots & (x_q-\mathrm{E}[x_q])\end{bmatrix}\right] =$$

$$= \begin{bmatrix}\mathrm{E}\left[(x_1-\mathrm{E}[x_1])^2\right] & \dots & \mathrm{E}\left[(x_1-\mathrm{E}[x_1])(x_q-\mathrm{E}[x_q])\right]\\ \vdots & \ddots & \vdots \\ \mathrm{E}\left[(x_1-\mathrm{E}[x_1])(x_q-\mathrm{E}[x_q])\right] & \dots & \mathrm{E}\left[(x_q-\mathrm{E}[x_q])^2\right]\end{bmatrix}$$

- intradiagonal: $\mathrm{E}\left[(x_i-\mathrm{E}[x_i])^2\right] = \mathrm{Var}(x_i)$
- extradiagonal: $\mathrm{E}\left[(x_i-\mathrm{E}[x_i])(x_j-\mathrm{E}[x_j])\right] = \mathrm{Cov}(x_i,x_j)$

Matriz de varianzas y covarianzas

$$\begin{aligned}
\mathrm{Var}(\vec{x}) = \mathrm{Cov}(\vec{x}) = \mathrm{Cov}(\vec{x},\vec{x}) = \mathrm{E}\left[(\vec{x}-\mathrm{E}\left[\vec{x}\right])(\vec{x}-\mathrm{E}\left[\vec{x}\right])^{t}\right] = \\
= \mathrm{E}\left[\vec{x}\,\vec{x}^{t}\right] - \mathrm{E}\left[\vec{x}\,\mathrm{E}\left[\vec{x}\right]^{t}\right] - \mathrm{E}\left[\mathrm{E}\left[\vec{x}\right]\,\vec{x}^{t}\right] + \mathrm{E}\left[\mathrm{E}\left[\vec{x}\right]\mathrm{E}\left[\vec{x}\right]^{t}\right] = \\
= \mathrm{E}\left[\vec{x}\,\vec{x}^{t}\right] - \mathrm{E}\left[\vec{x}\right]\,\mathrm{E}\left[\vec{x}\right]^{t} - \mathrm{E}\left[\vec{x}\right]\,\mathrm{E}\left[\vec{x}^{t}\right] + \mathrm{E}\left[\vec{x}\right]\,\mathrm{E}\left[\vec{x}\right]^{t} = \\
= \mathrm{E}\left[\vec{x}\,\vec{x}^{t}\right] - \mathrm{E}\left[\vec{x}\right]\,\mathrm{E}\left[\vec{x}\right]^{t}
\end{aligned}$$

Formas cuadráticas

Dados una matriz A (n,n) simétrica e idempotente con rango$(A) = r$ y un vector aleatorio $\vec{x} \equiv \mathcal{N}_n \left(\vec{\mu}, \sigma^2\, I_n\right)$, se verifican las siguientes propiedades:

- $A = U\,\Lambda\,U^t = (U_1 \;\; U_2) \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} \begin{pmatrix} U_1^t \\ U_2^t \end{pmatrix} = U_1\,U_1^t$
  con U_1 vectores propios asociados al valor propio 1
  con U_2 vectores propios asociados al valor propio 0
  y $U^t\,U = I_n$
- $\vec{x}^t\,A\,\vec{x} = \vec{x}^t\,U_1\,U_1^t\,\vec{x}$

Formas cuadráticas

- esperanza

$$
\begin{aligned}
\mathrm{E}\left[\vec{x}^t\, A\, \vec{x}\right] &= \mathrm{E}\left[\vec{x}^t\; U_1\; U_1^t\, \vec{x}\right] = \mathrm{E}\left[\mathrm{tr}\left\{\vec{x}^t\; U_1\; U_1^t\, \vec{x}\right\}\right] \\
&= \mathrm{E}\left[\mathrm{tr}\left\{\, U_1\; U_1^t\, \vec{x}\, \vec{x}^t\right\}\right] = \mathrm{tr}\left\{\, U_1\; U_1^t\, \mathrm{E}\left[\vec{x}\, \vec{x}^t\right]\right\} \\
&= \mathrm{tr}\left\{\, U_1\; U_1^t\; \left(\mathrm{Cov}(\vec{x}) + \mathrm{E}\left[\vec{x}\right]\; \mathrm{E}\left[\vec{x}^t\right]\right)\right\} \\
&= \mathrm{tr}\left\{\, U_1\; U_1^t\; \left(\sigma^2\, I + \mathrm{E}\left[\vec{x}\right]\; \mathrm{E}\left[\vec{x}^t\right]\right)\right\} \\
&= \mathrm{tr}\left\{\sigma^2\; U_1\; U_1^t\right\} + \mathrm{tr}\left\{\, U_1\; U_1^t\, \mathrm{E}\left[\vec{x}\right]\; \mathrm{E}\left[\vec{x}^t\right]\right\} \\
&= \sigma^2\; \mathrm{tr}(U_1\; U_1^t) + \mathrm{tr}\left\{\mathrm{E}\left[\vec{x}^t\right]\;\; U_1\; U_1^t\, \mathrm{E}\left[\vec{x}\right]\right\} \\
&= \sigma^2\; \mathrm{rango}(A) + \mathrm{E}\left[\vec{x}^t\right]\;\; A\, \mathrm{E}\left[\vec{x}\right] \\
&= \sigma^2\, r + \mathrm{E}\left[\vec{x}^t\right]\;\; A\, \mathrm{E}\left[\vec{x}\right]
\end{aligned}
$$

Formas cuadráticas

- Si $A\,\vec{\mu} = \vec{0}$ entonces $\vec{x}^t\,A\,\vec{x} \equiv \sigma^2\,\chi^2_r$. Demostración:

$$\vec{x}^t\,A\,\vec{x} = \vec{x}^t\,U_1\,U_1^t\,\vec{x} = \vec{y}_1^t\,\vec{y}_1, \qquad \text{con } \vec{y}_1 = U_1^t\,\vec{x}$$

$$\vec{\gamma} = \mathrm{E}\left[\vec{y}_1\right] = \mathrm{E}\left[U_1^t\,\vec{x}\right] = U_1^t\,\mathrm{E}\left[\vec{x}\right] = U_1^t\,\vec{\mu}$$

$$U_1\,\vec{\gamma} = U_1\,U_1^t\,\vec{\mu} = A\,\vec{\mu} = \vec{0}$$

$$\vec{\gamma} = I_r\,\vec{\gamma} = U_1^t\,U_1\,\vec{\gamma} = U_1^t\,\vec{0} = \vec{0}$$

$$\mathrm{Cov}(\vec{y}_1) = U_1^t\,\mathrm{Cov}(\vec{x})\,U_1 = \sigma^2\,U_1^t\,U_1 = \sigma^2\,I_r$$

En consecuencia, $\vec{y}_1 \equiv U_1^t\,\mathcal{N}_n\left(\vec{\mu}, \sigma^2\,I\right) = \mathcal{N}_r\left(\vec{0}, \sigma^2\,I_r\right)$

$$\vec{y}_1^t\,\vec{y}_1 = \sum_{j=1}^{r} y_{1j}^2 = \sum_{j=1}^{r} \sigma^2\left(\frac{y_{1j}}{\sigma}\right)^2 \equiv \sigma^2 \sum_{j=1}^{r} \underbrace{\left[\mathcal{N}\left(0,1\right)\right]^2}_{\text{indep.}} = \sigma^2\chi^2_r$$

Análisis de varianza

Sea el vector columna $n \times 1$: $\vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_q \end{pmatrix}$

yuxtaposición de los vectores $n_i \times 1$: $\vec{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in_i} \end{pmatrix}$

para $i = 1, \ldots, q$

Análisis de varianza

$$\vec{1}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1} \qquad J = \frac{1}{n}\,\vec{1}_n\vec{1}_n^t = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}_{n\times n}$$

J es simétrica e idempotente

al multiplicar, genera un vector columna con la media:

$$J\,\vec{x} = \frac{1}{n}\,\vec{1}_n\,\vec{1}_n^t\,\vec{x} = \vec{1}_n\,\frac{1}{n}\,\vec{1}_n^t\,\vec{x} = \bar{x}\,\vec{1}_n$$

Análisis de varianza

$$H = I - J = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \dots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & 1-\frac{1}{n} \end{pmatrix}$$

H es simétrica e idempotente

Análisis de varianza

H es la matriz de centrado

al multiplicar, sustrae la media:

$$\begin{aligned} H\,\vec{x} &= \left(I_n - \frac{1}{n}\,\vec{1}_n\,\vec{1}_n^t\right)\vec{x} = \vec{x} - \vec{1}_n\,\frac{1}{n}\,\vec{1}_n^t\,\vec{x} \\ &= \vec{x} - \vec{1}_n\,\bar{x} = \begin{pmatrix} \vec{x}_1 - \bar{x}\,\vec{1}_{n_1} \\ \vec{x}_2 - \bar{x}\,\vec{1}_{n_2} \\ \vdots \\ \vec{x}_q - \bar{x}\,\vec{1}_{n_q} \end{pmatrix} \end{aligned}$$

Análisis de varianza

$$\begin{aligned}
\vec{x}^t\, H\, \vec{x} &= \vec{x}^t\, H\, H\, \vec{x} = \vec{x}^t\, H^t\, H\, \vec{x} \\
&= \left[(\vec{x}_1 - \bar{x}\,\vec{1}_{n_1})^t, \ldots, (\vec{x}_q - \bar{x}\,\vec{1}_{n_q})^t\right] \begin{pmatrix} \vec{x}_1 - \bar{x}\,\vec{1}_{n_1} \\ \vec{x}_2 - \bar{x}\,\vec{1}_{n_2} \\ \vdots \\ \vec{x}_q - \bar{x}\,\vec{1}_{n_q} \end{pmatrix} \\
&= \sum_{i=1}^{q} (\vec{x}_i - \bar{x}\,\vec{1}_{n_i})^t\, (\vec{x}_i - \bar{x}\,\vec{1}_{n_i}) \\
&= \sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2 = \mathrm{SCT}
\end{aligned}$$

Análisis de varianza

$$H = I - J = I - D + D - J$$

con

$$D = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix}$$

con $J_i = \frac{1}{n_i} \vec{1}_{n_i} \vec{1}_{n_i}^t$

- $I - D$ es simétrica e idempotente
- $\operatorname{tr}(I - D) = \operatorname{tr} I - \operatorname{tr} D = n - q = \operatorname{rango}(I - D)$
- $D - J$ es simétrica e idempotente
- $\operatorname{tr}(D - J) = \operatorname{tr} D - \operatorname{tr} J = q - 1 = \operatorname{rango}(D - J)$

Análisis de varianza

$$D\,\vec{x} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_q \end{pmatrix} = \begin{pmatrix} J_1\,\vec{x}_1 \\ J_2\,\vec{x}_2 \\ \vdots \\ J_q\,\vec{x}_q \end{pmatrix} = \begin{pmatrix} \bar{x}_1\,\vec{1}_{n_1} \\ \bar{x}_2\,\vec{1}_{n_2} \\ \vdots \\ \bar{x}_q\,\vec{1}_{n_q} \end{pmatrix}$$

Análisis de varianza

$$(I - D)\,\vec{x} = \vec{x} - D\,\vec{x} = \begin{pmatrix} \vec{x}_1 - \bar{x}_1\,\vec{1}_{n_1} \\ \vec{x}_2 - \bar{x}_2\,\vec{1}_{n_2} \\ \vdots \\ \vec{x}_q - \bar{x}_q\,\vec{1}_{n_q} \end{pmatrix}$$

Análisis de varianza

$$\vec{x}^t\,(I - D)\,\vec{x} =$$

$$= \vec{x}^t \left[\begin{pmatrix} I_{n_1} & & & \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_q} \end{pmatrix} - \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix} \right] \vec{x}$$

$$= (\vec{x}_1^t, \ldots, \vec{x}_q^t) \begin{pmatrix} I_{n_1} - J_1 & & & \\ & I_{n_2} - J_2 & & \\ & & \ddots & \\ & & & I_{n_q} - J_q \end{pmatrix} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_q \end{pmatrix}$$

$$= \sum_{i=1}^{q} \vec{x}_i^t\,(I_{n_i} - J_i)\,\vec{x}_i = \sum_{i=1}^{q} \sum_{i=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = \text{SCE}$$

Análisis de varianza

$$(D - J)\,\vec{x} = D\,\vec{x} - J\,\vec{x} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_q \end{pmatrix} - J\,\vec{x}$$

$$= \begin{pmatrix} J_1\,\vec{x}_1 \\ J_2\,\vec{x}_2 \\ \vdots \\ J_q\,\vec{x}_q \end{pmatrix} - \bar{x}\,\vec{1}_n = \begin{pmatrix} \bar{x}_1\,\vec{1}_{n_1} \\ \bar{x}_2\,\vec{1}_{n_2} \\ \vdots \\ \bar{x}_q\,\vec{1}_{n_q} \end{pmatrix} - \bar{x}\,\vec{1}_n = \begin{pmatrix} (\bar{x}_1 - \bar{x})\,\vec{1}_{n_1} \\ (\bar{x}_2 - \bar{x})\,\vec{1}_{n_2} \\ \vdots \\ (\bar{x}_q - \bar{x})\,\vec{1}_{n_q} \end{pmatrix}$$

Análisis de varianza

Por ser $D - J$ simétrica e idempotente se tiene que

$$\begin{aligned}
\vec{x}^{t}\,(D-J)\,\vec{x} &= \vec{x}^{t}\,(D-J)^{t}\,(D-J)\,\vec{x} \\
&= \left[(\bar{x}_1-\bar{x})\,\vec{1}^{t}_{n_1},\ldots,(\bar{x}_q-\bar{x})\,\vec{1}^{t}_{n_q}\right]\begin{pmatrix}(\bar{x}_1-\bar{x})\,\vec{1}_{n_1}\\ \vdots \\ (\bar{x}_q-\bar{x})\,\vec{1}_{n_q}\end{pmatrix} \\
&= \sum_{i=1}^{q} n_i\,(\bar{x}_i-\bar{x})^2 = \mathrm{SCF}
\end{aligned}$$

Análisis de varianza

$$\begin{aligned}
\text{SCT} &= \vec{x}^{\,t}\, H\, \vec{x} = \vec{x}^{\,t}\,(I - J)\,\vec{x} = \vec{x}^{\,t}\,(I - D + D - J)\,\vec{x} \\
&= \vec{x}\,(I - D)\,\vec{x} + \vec{x}\,(D - J)\,\vec{x} \\
&= \sum_{i=1}^{q}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2 + \sum_{i=1}^{q} n_i\,(\bar{x}_i - \bar{x})^2 \\
&= \text{SCE} + \text{SCF}
\end{aligned}$$

Análisis de varianza

```
summary(sleep)                       # ejemplo en R

## Vamos (incorrectamente; véase la ayuda "?sleep")
## a suponer que "sleep" contiene una variable
## respuesta que representa el tiempo "extra" de
## sueño en un grupo de control (group=1) y en otro
## grupo en el que se ha administrado cierto
## somnífero (group=1).

X <- sleep$extra
N <- length(X)
G <- sleep$group                     # factor

t.test (X ~ G, var.equal=TRUE)       # =aov <= q=2
```

Análisis de varianza

```
adeva <- aov(X~G)
I <- diag(N)
definirJ <- function (n) matrix(1/n, n, n)
J <- definirJ(N)
SCT <- X %*% (I-J) %*% X
tabla <- summary(adeva)[[1]]
sum(tabla[,"Sum Sq"])                # SCT
stopifnot (identical (G, sort(G)))
Ni <- table(G)
D <- as.matrix(Matrix::bdiag(lapply(Ni, definirJ)))
SCE <- X %*% (I-D) %*% X
tabla["Residuals","Sum Sq"]          # SCE
SCF <- X %*% (D-J) %*% X
tabla[1,"Sum Sq"]                    # SCF
```

Análisis de varianza

A simétrica idempotente de rango r

$\implies \mathrm{E}\left[\vec{x}^{t}\, A\, \vec{x}\right] = \sigma^2\, r + \vec{\mu}\, A\, \vec{\mu}$

$\implies \mathrm{E}\,[\mathrm{SCE}] = \mathrm{E}\left[\vec{x}^{t}\,(I - D)\,\vec{x}\right] = \sigma^2\,(n - q) + \vec{\mu}\,(I - D)\,\vec{\mu}$

En detalle:

$$\begin{aligned}
\mathrm{E}\,[\mathrm{SCE}] &= \mathrm{E}\left[\vec{x}^{t}\,(I - D)\,\vec{x}\right] = \mathrm{E}\left[\mathrm{tr}(\vec{x}^{t}\,(I - D)\vec{x}\right] \\
&= \mathrm{E}\left[\mathrm{tr}([I - D]\,\vec{x}\,\vec{x}^{t})\right] = \mathrm{tr}\left\{\mathrm{E}\left[(I - D)\,\vec{x}\,\vec{x}^{t}\right]\right\} \\
&= \mathrm{tr}\left\{(I - D)\,\mathrm{E}\left[\vec{x}\,\vec{x}^{t}\right]\right\} \\
&= \mathrm{tr}\left\{(I - D)\,\left(\mathrm{Cov}(\vec{x}) + \mathrm{E}\,[\vec{x}]\;\mathrm{E}\left[\vec{x}^{t}\right]\right)\right\} \\
&= \mathrm{tr}\left\{(I - D)\,\left(\sigma^2\, I + \mathrm{E}\,[\vec{x}]\;\mathrm{E}\left[\vec{x}^{t}\right]\right)\right\} \\
&= \sigma^2\;\mathrm{tr}(I - D) + \mathrm{tr}\left\{(I - D)\,\mathrm{E}\,[\vec{x}]\;\mathrm{E}\left[\vec{x}^{t}\right]\right\} \\
&= \sigma^2\,(n - q) + \mathrm{tr}\left\{\mathrm{E}\left[\vec{x}^{t}\right]\;(I - D)\,\mathrm{E}\,[\vec{x}]\right\} \\
&= \sigma^2\,(n - q) + \mathrm{E}\left[\vec{x}^{t}\right]\;(I - D)\,\mathrm{E}\,[\vec{x}]
\end{aligned}$$

Análisis de varianza

$$\mathrm{E}\left[\vec{x}\right] = \begin{pmatrix} \mathrm{E}\left[\vec{x}_1\right] \\ \mathrm{E}\left[\vec{x}_2\right] \\ \vdots \\ \mathrm{E}\left[\vec{x}_q\right] \end{pmatrix} = \begin{pmatrix} \mu_1 \, \vec{1}_{n_1} \\ \mu_2 \, \vec{1}_{n_2} \\ \vdots \\ \mu_q \, \vec{1}_{n_q} \end{pmatrix}$$

$$\begin{aligned} (I-D)\,\mathrm{E}\left[\vec{x}\right] &= \begin{pmatrix} I_{n_1}-J_1 & & & \\ & I_{n_2}-J_2 & & \\ & & \ddots & \\ & & & I_{n_q}-J_q \end{pmatrix} \begin{pmatrix} \mu_1 \, \vec{1}_{n_1} \\ \mu_2 \, \vec{1}_{n_2} \\ \vdots \\ \mu_q \, \vec{1}_{n_q} \end{pmatrix} \\ &= \begin{pmatrix} \vdots \\ (I_{n_i}-J_i)\,\mu_i \, \vec{1}_{n_i} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ H_i \, \mu_i \, \vec{1}_{n_i} \\ \vdots \end{pmatrix} = \vec{0} \end{aligned}$$

$$\Longrightarrow \; \mathrm{E}\left[\vec{x}^{\,t}\right] \, (I-D)\,\mathrm{E}\left[\vec{x}\right] = \mathrm{E}\left[\vec{x}^{\,t}\right] \, \vec{0} = 0 \; \Longrightarrow \; \mathrm{E}\left[\mathrm{SCE}\right] = \sigma^2 \, (n-q)$$

Análisis de varianza

A simétrica idempotente de rango r

$\implies \mathrm{E}\left[\vec{x}^t\, A\, \vec{x}\right] = \sigma^2\, r + \vec{\mu}\, A\, \vec{\mu}$

$\implies \mathrm{E}\,[\mathrm{SCF}] = \mathrm{E}\left[\vec{x}^t\,(D-J)\,\vec{x}\right] = \sigma^2\,(q-1) + \vec{\mu}\,(D-J)\,\vec{\mu}$

En detalle:

$$\begin{aligned}
\mathrm{E}\,[\mathrm{SCF}] &= \mathrm{E}\left[\vec{x}^t\,(D-J)\vec{x}\right] = \mathrm{E}\left[\mathrm{tr}\left\{\vec{x}^t\,(D-J)\,\vec{x}\right\}\right] \\
&= \mathrm{E}\left[\mathrm{tr}\left\{(D-J)\,\vec{x}\,\vec{x}^t\right\}\right] = \mathrm{tr}\left\{(D-J)\,\mathrm{E}\left[\vec{x}\,\vec{x}^t\right]\right\} \\
&= \mathrm{tr}\left\{(D-J)\ \left(\mathrm{Cov}(\vec{x}) + \mathrm{E}\,[\vec{x}]\ \mathrm{E}\left[\vec{x}^t\right]\right)\right\} \\
&= \mathrm{tr}\left\{(D-J)\ \left(\sigma^2\, I + \mathrm{E}\,[\vec{x}]\ \mathrm{E}\left[\vec{x}^t\right]\right)\right\} \\
&= \mathrm{tr}\left\{\sigma^2\,(D-J)\right\} + \mathrm{tr}\left\{(D-J)\,\mathrm{E}\,[\vec{x}]\ \mathrm{E}\left[\vec{x}^t\right]\right\} \\
&= \sigma^2\ \mathrm{tr}(D-J) + \mathrm{tr}\left\{\mathrm{E}\left[\vec{x}^t\right]\ (D-J)\,\mathrm{E}\,[\vec{x}]\right\} \\
&= \sigma^2\,(q-1) + \mathrm{E}\left[\vec{x}^t\right]\ (D-J)\,\mathrm{E}\,[\vec{x}]
\end{aligned}$$

Análisis de varianza

$$\mathrm{E}\left[\vec{x}\right] = \begin{pmatrix} \mathrm{E}\left[\vec{x}_1\right] \\ \vdots \\ \mathrm{E}\left[\vec{x}_q\right] \end{pmatrix} = \begin{pmatrix} \mu\,\vec{1}_{n_1} + \alpha_1\,\vec{1}_{n_1} \\ \vdots \\ \mu\,\vec{1}_{n_q} + \alpha_q\,\vec{1}_{n_q} \end{pmatrix} = \mu\,\vec{1}_n + \begin{pmatrix} \alpha_1\,\vec{1}_{n_1} \\ \vdots \\ \alpha_q\,\vec{1}_{n_q} \end{pmatrix}$$

Análisis de varianza

$$J \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix} = \tfrac{1}{n} \, \vec{1} \, \vec{1}^t \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix} = \vec{1} \, \tfrac{1}{n} \sum n_i \, \alpha_i = \vec{0}$$

$$\Longrightarrow \; (D - J) \, \mathrm{E}\left[\vec{x}\right] = \mu \, (D - J) \, \vec{1} + (D - J) \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix}$$

$$= \vec{0} + D \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix} - J \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix}$$

$$= D \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix} - \vec{0} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix} \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix} = \begin{pmatrix} \alpha_1 \, \vec{1}_{n_1} \\ \vdots \\ \alpha_q \, \vec{1}_{n_q} \end{pmatrix}$$

Análisis de varianza

Por tanto,

$$\begin{aligned}
\mathrm{E}\left[\vec{x}^{t}\right]\,(D-J)\,\mathrm{E}\left[\vec{x}\right] &= \left(\alpha_1\,\vec{1}^{t}_{n_1},\ldots,\alpha_q\,\vec{1}^{t}_{n_q}\right)\begin{pmatrix}\alpha_1\,\vec{1}_{n_1}\\ \vdots\\ \alpha_q\,\vec{1}_{n_q}\end{pmatrix}\\
&= \sum_{i=1}^{q}\alpha_i^2\vec{1}^{t}_{n_i}\,\vec{1}_{n_i} = \sum_{i=1}^{q} n_i\,\alpha_i^2
\end{aligned}$$

y se obtiene

$$\mathrm{E}\left[\mathrm{SCF}\right] = \sigma^2\,(q-1) + \sum n_i\,\alpha_i^2$$

Análisis de varianza

Distribución de la SCE:

Suponiendo que $\vec{x}_i \equiv \mathcal{N}_{n_i}\left(\mu_i\,\vec{1}, \sigma^2\,I_{n_i}\right)$ para $i = 1, \ldots, q$ y que las $\vec{x}_i$ son independientes, se verifica que

$$\mathrm{SCE} = \vec{x}^{\,t}\,(I - D)\,\vec{x} \equiv \sigma^2\,\chi^2_{n-q}$$

Para demostrar esta propiedad, bastará ver que $\mathrm{E}\left[(I - D)\,\vec{x}\right] = \vec{0}$.

Análisis de varianza

$$\begin{aligned}
\mathrm{E}\left[(I-D)\,\vec{x}\right] &= (I-D)\,\mathrm{E}\left[\vec{x}\right] = \mathrm{E}\left[\vec{x}\right] - D\,\mathrm{E}\left[\vec{x}\right] \\
&= \begin{pmatrix} \mu_1\,\vec{1}_{n_1} \\ \vdots \\ \mu_q\,\vec{1}_{n_q} \end{pmatrix} - \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{pmatrix} \begin{pmatrix} \mu_1\,\vec{1}_{n_1} \\ \vdots \\ \mu_q\,\vec{1}_{n_q} \end{pmatrix} \\
&= \begin{pmatrix} \mu_1\,\vec{1}_{n_1} \\ \vdots \\ \mu_q\,\vec{1}_{n_q} \end{pmatrix} - \begin{pmatrix} \mu_1\,J_1\,\vec{1}_{n_1} \\ \vdots \\ \mu_q\,J_q\,\vec{1}_{n_q} \end{pmatrix} \\
&= \vec{0}
\end{aligned}$$

Análisis de varianza

Distribución de la SCF bajo H_0:

$$\mathrm{SCF} = \vec{x}^t \,(D - J)\, \vec{x} \overset{H_0}{\equiv} \sigma^2 \, \chi^2_{q-1}$$

Dado que $D - J$ es idempotente, bastará comprobar que $\mathrm{E}\,[(D - J)\, \vec{x}] = \vec{0}$.

$$\mathrm{E}\,[(D - J)\, \vec{x}] = (D - J)\, \mathrm{E}\,[\vec{x}] = D\, \mathrm{E}\,[\vec{x}] - J\, \mathrm{E}\,[\vec{x}] = \vec{0}$$

ya que bajo $H_0 \equiv \mu_1 = \cdots = \mu_q$ y $\mathrm{E}\,[\vec{x}] = \mu\, \vec{1}$, con lo cual

$$D\, \mu\, \vec{1} = \mu \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{pmatrix} \begin{pmatrix} \vec{1}_{n_1} \\ \vdots \\ \vec{1}_{n_q} \end{pmatrix} = \mu \begin{pmatrix} J_1\, \vec{1}_{n_1} \\ \vdots \\ J_q\, \vec{1}_{n_q} \end{pmatrix} = \mu\, \vec{1}$$

$$J\, \mathrm{E}\,[\vec{x}] = J\, \mu \vec{1} = \mu\, J\, \vec{1} = \mu\, \vec{1}$$

Análisis de varianza

$$\mathrm{CME} = \frac{\mathrm{SCE}}{n-q}$$

Se verifica que $\mathrm{E}\left[\mathrm{CME}\right] = \sigma^2$ y por tanto es un estimador insesgado de la varianza de los residuos. Además,

$$\frac{\mathrm{CME}\left(n-q\right)}{\sigma^2} \equiv \chi^2_{n-q}$$

Análisis de varianza

$$\mathrm{CMF} = \frac{\mathrm{SCF}}{q-1}$$

$$\mathrm{E}\left[\mathrm{CMF}\right] = \sigma^2 + \frac{\sum_{i=1}^{q} n_i\, \alpha_i^2}{q-1}$$

Bajo $H_0 \equiv \alpha_i = 0\,\forall\, i$, se cumple que $\mathrm{E}\left[\mathrm{CMF} \mid H_0\right] = \sigma^2$. Además

$$\frac{\mathrm{CMF}\,(q-1)}{\sigma^2} \overset{H_0}{\equiv} \chi^2_{q-1}$$

Análisis de varianza

CMF y CME son independientes:

- $\vec{x} \equiv \mathcal{N}(\cdot,\cdot) \implies \begin{cases} (I-D)\vec{x} \equiv \mathcal{N}(\cdot,\cdot) \\ (D-J)\vec{x} \equiv \mathcal{N}(\cdot,\cdot) \end{cases}$
- $\mathrm{Cov}[(I-D)\vec{x},(D-J)\vec{x}] = (I-D)\,\mathrm{Var}(\vec{x})(D-J) = (I-D)\sigma^2 I(D-J) = \sigma^2(I-D)(D-J) = \sigma^2(D-J-DD+DJ) = \sigma^2(D-J-D+J) = \mathbf{0}$
- $(I-D)\vec{x}$ independiente de $(D-J)\vec{x}$
- $\vec{x}^t(I-D)\vec{x}$ independiente de $\vec{x}^t(D-J)\vec{x}$
- SCE independiente de SCF

Análisis de varianza

El cociente

$$\frac{\mathrm{CMF}}{\mathrm{CME}} = \frac{\mathrm{CMF}/\sigma^2}{\mathrm{CME}/\sigma^2}$$

tiende a tomar valores cercanos a uno bajo H_0, y más grandes bajo H_1.
Por otra parte, CMF y CME son independientes, luego

$$\frac{\mathrm{CMF}}{\mathrm{CME}} \overset{H_0}{\equiv} \frac{\frac{\chi^2_{q-1}}{q-1}}{\frac{\chi^2_{n-q}}{n-q}} = F_{q-1,n-q}$$

En consecuencia, la región crítica del contraste viene dada por la expresión

$$\mathrm{R.C.} = \left\{ \frac{\mathrm{CMF}}{\mathrm{CME}} > k \right\} \text{ con } P[\mathrm{R.C.} \mid H_0] = \alpha$$

Análisis de varianza

Tabla ANOVA

Fuente de variación	S.C.	g.l.	C.M.	F
Entre / Factor	SCF	$q-1$	CMF	$\frac{\text{CMF}}{\text{CME}}$
Dentro / Error	SCE	$n-q$	CME	
Total	SCT	$n-1$		

Ejemplo

Considérense los datos siguientes incluidos en R:

```
> aves <- data.frame (peso = chickwts$weight,
                       come = factor(chickwts$feed,
                                     labels =
                    c("caseína","fabona","linaza",
                      "har.hueso","soja","girasol")))
> summary (aves)
```

```
      peso                 come   
 Min.    :108.0   caseína   :12  
 1st Qu.:204.5   fabona    :10  
 Median :258.0   linaza    :12  
 Mean   :261.3   har.hueso:11  
 3rd Qu.:323.5   soja      :14  
 Max.    :423.0   girasol   :12  
```

Ejemplo

- Se repartió aleatoriamente en seis grupos una remesa de pollos recién nacidos.
- Cada grupo recibió un complemento alimenticio distinto.
- Se registró el peso en gramos tras seis semanas.

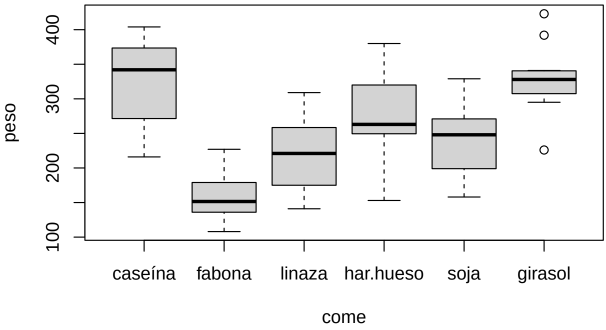
- ¿Influye el complemento en el peso?

Ejemplo

```
> options (digits = 3)
> RcmdrMisc::numSummary (aves$peso, groups=aves$come)
           mean   sd  IQR  0% 25% 50% 75% 100% data:n
caseína     324 64.4 93.5 216 277 342 371  404     12
fabona      160 38.6 39.2 108 137 152 176  227     10
linaza      219 52.2 79.8 141 178 221 258  309     12
har.hueso   277 64.9 70.5 153 250 263 320  380     11
soja        246 54.1 63.2 158 207 248 270  329     14
girasol     329 48.8 27.5 226 313 328 340  423     12
```

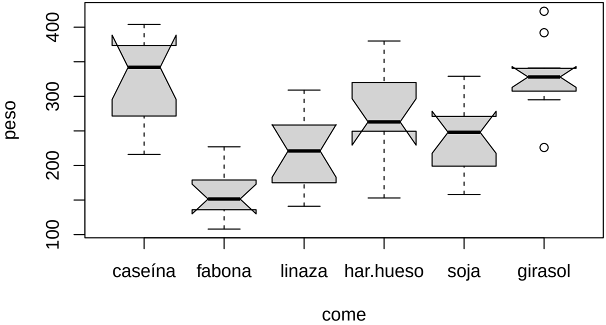
Ejemplo

```
> boxplot (peso ~ come, aves)
```



Ejemplo

```
> boxplot (peso ~ come, aves, notch=TRUE)
```



Ejemplo

```
> bartlett.test (peso ~ come, aves)
        Bartlett test of homogeneity of variances

data:  peso by come
Bartlett's K-squared = 3, df = 5, p-value = 0.7
> car::leveneTest (peso ~ come, aves)
Levene's Test for Homogeneity of Variance (center = median)
      Df F value Pr(>F)
group  5    0.75   0.59
      65
```

Ejemplo

```
> options (width = 60)
> summary (aov (peso ~ come, aves))
            Df Sum Sq Mean Sq F value  Pr(>F)
come         5 231129   46226    15.4 5.9e-10 ***
Residuals   65 195556    3009
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Contrastes a posteriori

$$H_0^{\text{ANOVA}} : \mu_1 = \cdots = \mu_q \qquad \equiv \qquad \bigcap_{1 \leq i < j \leq q} H_0^{ij} : \mu_i = \mu_j$$

¿Qué H_0^{ij} se rechazan?

Contrastes a posteriori

criterio de Bonferroni

- hay $\frac{q\,(q-1)}{2}$ parejas de grupos
- contrastar H_0^{ij} a nivel $\alpha^* = \frac{2\alpha}{q\,(q-1)}$

$$P\left[\text{rechazar alguna } H_0^{ij} \mid H_0^{\text{ANOVA}}\right] \leq \sum_{1\leq i<j\leq q} P\left[\text{rechazar } H_0^{ij} \mid H_0^{ij}\right] = \frac{q\,(q-1)}{2}\,\alpha^* = \alpha$$

- estadístico de contraste

$$\frac{\bar{X}_i - \bar{X}_j}{\sqrt{\left(\frac{1}{n_i}+\frac{1}{n_j}\right)\text{CME}}} \overset{H_0^{ij}}{\equiv} t_{n-q}$$

Contrastes a posteriori

En el ejemplo

```
> q <- length (levels (aves$come))
> q
[1] 6
> q * (q-1) / 2
[1] 15
```

Contrastes a posteriori

```
> pairwise.t.test (aves$peso, aves$come, "none")
        Pairwise comparisons using t tests with pooled SD

data:  aves$peso and aves$come

           caseína fabona linaza har.hueso soja
fabona     2e-09   -      -      -         -
linaza     1e-05   0.02   -      -         -
har.hueso  0.05    7e-06  0.01   -         -
soja       7e-04   3e-04  0.20   0.17      -
girasol    0.81    8e-10  6e-06  0.03      3e-04

P value adjustment method: none
```

Contrastes a posteriori

```
> pairwise.t.test (aves$peso, aves$come, "bonferroni")
        Pairwise comparisons using t tests with pooled SD

data:  aves$peso and aves$come

          caseína fabona linaza har.hueso soja
fabona    3e-08   -      -      -         -
linaza    2e-04   0.228  -      -         -
har.hueso 0.684   1e-04  0.202  -         -
soja      0.010   0.005  1.000  1.000     -
girasol   1.000   1e-08  9e-05  0.397     0.004

P value adjustment method: bonferroni
```

Contrastes a posteriori

criterio de Tukey

- basado en la distribución del *rango estudentizado*
  - $Y_1, \ldots, Y_q \equiv \mathcal{N}(0,1)$ independientes
  - $Z \equiv \chi^2_r$ independiente de las $Y_1, \ldots, Y_q$
  - entonces $\frac{Y_{(q)} - Y_{(1)}}{\sqrt{Z/r}} \equiv Q_{q,r}$ `ptukey(,q,r)` en R
- en ANOVA para contrastar H_0^{ij} se calcula el P-valor

$$P\left[Q_{q,n-q} > \frac{|\bar{X}_i - \bar{X}_j|}{\sqrt{\mathrm{CME}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}\right]$$

Contrastes a posteriori

```
> r <- nrow (aves) - q              # q=6 r=65
> distro <- replicate (1e5,
  {
      medias <- rnorm (q)
      numerador <- diff (range (medias))
      denominador <- sqrt (rchisq (1, r) / r)
      numerador / denominador
  })
> alfas <- c (0.01, 0.025, 0.05, 0.1, 0.5,
              0.9, 0.95, 0.975, 0.99)
> rbind (sim = quantile (distro, alfas),
         num = qtukey (alfas, q, r))
       1% 2.5%    5%   10%   50%   90%   95% 97.5%   99%
sim 0.874 1.06 1.24 1.48 2.48 3.74 4.15  4.52 4.97
num 0.862 1.06 1.24 1.48 2.49 3.75 4.15  4.52 4.97
```

Contrastes a posteriori

```
> a <- aov (peso ~ come, aves)
> TukeyHSD (a)
```

Contrastes a posteriori

```
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = peso ~ come, data = aves)

$come
                     diff     lwr   upr p adj
fabona-caseína    -163.38 -232.35 -94.4 0.000
linaza-caseína    -104.83 -170.59 -39.1 0.000
har.hueso-caseína  -46.67 -113.91  20.6 0.332
soja-caseína       -77.15 -140.52 -13.8 0.008
girasol-caseína      5.33  -60.42  71.1 1.000
linaza-fabona       58.55  -10.41 127.5 0.141
har.hueso-fabona   116.71   46.34 187.1 0.000
soja-fabona         86.23   19.54 152.9 0.004
girasol-fabona     168.72   99.75 237.7 0.000
har.hueso-linaza    58.16   -9.07 125.4 0.128
soja-linaza         27.68  -35.68  91.0 0.793
```

Contrastes a posteriori

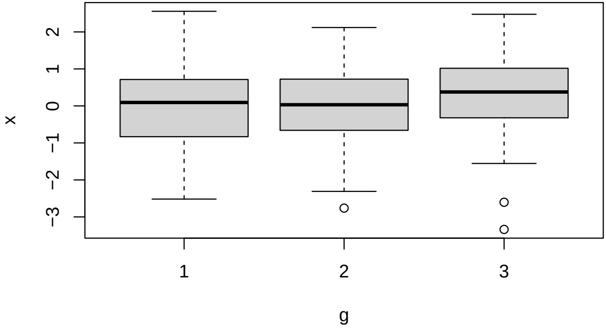
```
> set.seed (122)
> mu <- c (0, 0, 0.2)
> q <- length(mu)
> m <- 100
> g <- factor (rep (1:q, each=m))
> x <- unlist (lapply (mu,
                     function (mui) rnorm (m, mui)))
> a <- aov (x ~ g)
> summary (a)
             Df Sum Sq Mean Sq F value Pr(>F)
g             2      9    4.29     3.8  0.024 *
Residuals   297    336    1.13
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Contrastes a posteriori

```
> options (digits = 7)
> pairwise.t.test (x, g, "bonf")
        Pairwise comparisons using t tests with pooled SD

data:  x and g

  1     2
2 1.000 -
3 0.056 0.050

P value adjustment method: bonferroni
```

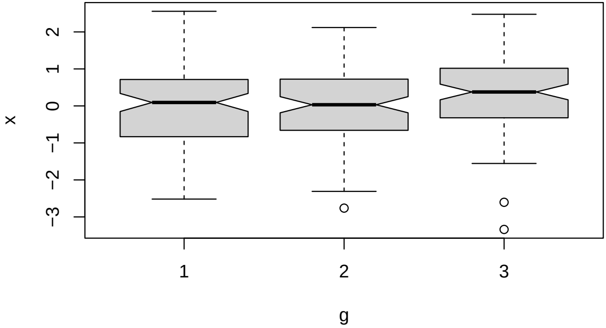
Contrastes a posteriori

```
> boxplot (x ~ g)
```



Contrastes a posteriori

```
> boxplot (x ~ g, notch=TRUE)
```



Contrastes a posteriori

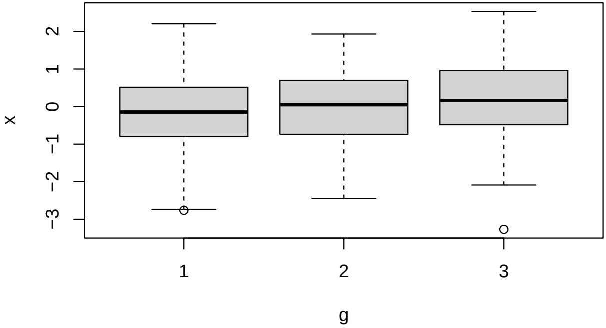
```
> set.seed (614)
> mu <- c (0, 0, 0.2)
> q <- length(mu)
> m <- 100
> g <- factor (rep (1:q, each=m))
> x <- unlist (lapply (mu,
                       function (mui) rnorm (m, mui)))
> a <- aov (x ~ g)
> summary (a)
             Df Sum Sq Mean Sq F value Pr(>F)
g             2   5.99   2.995   2.955 0.0536 .
Residuals   297 301.09   1.014
---
Signif. codes:
0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Contrastes a posteriori

```
> pairwise.t.test (x, g, "bonf")
        Pairwise comparisons using t tests with pooled SD

data:  x and g

  1     2
2 0.546 -
3 0.048 0.831

P value adjustment method: bonferroni
```

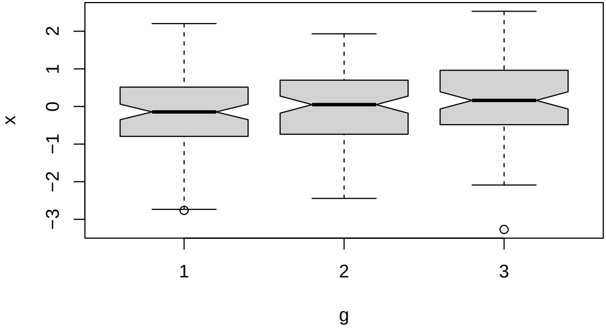
Contrastes a posteriori

```
> boxplot (x ~ g)
```



Contrastes a posteriori

```
> boxplot (x ~ g, notch=TRUE)
```



Contrastes a posteriori

```
> alfa <- 0.05 ; mu <- c (0, 0, 0.2) ; q <- length(mu)
> m <- 100 ; g <- factor (rep (1:q, each=m))
> table (data.frame (t (replicate (10000, {
      x <- unlist (lapply (mu,
              function (mui) rnorm (m, mui)))
      a <- aov (x ~ g)
      H1anova <- summary(a) [[1]] ["g", "Pr(>F)"] < alfa
      H1bonfe <- any (na.omit (c (pairwise.t.test
                                (x,g,"bonf")$p.value))
                     < alfa)
      c (H1anova=H1anova, H1bonfe=H1bonfe)
  }))))
       H1bonfe
H1anova FALSE TRUE
  FALSE  7099   37
  TRUE    308 2556
```

Contrastes a posteriori

```
> rechazos <- sapply (1:1000, function(semilla){
      set.seed (semilla)
      x <- unlist (lapply (mu,
              function (mui) rnorm (m, mui)))
      a <- aov (x ~ g)
      H1anova <- summary(a) [[1]] ["g", "Pr(>F)"] < alfa
      H1bonfe <- any (na.omit (c (pairwise.t.test
                                  (x,g,"bonf")$p.value))
                      < alfa)
      c (H1anova=H1anova, H1bonfe=H1bonfe)
  })
```

Contrastes a posteriori

```
> options (width = 55)
> (semillas <- which (xor (rechazos[1,], rechazos[2,])))
 [1] 122 129 153 218 226 246 267 268 343 369 371 377
[13] 383 397 414 437 457 513 610 614 619 646 654 658
[25] 670 677 706 741 744 762 777 779 793 895 906 930
[37] 973
> rechazos [, semillas]
         [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]
H1anova  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE
H1bonfe FALSE FALSE FALSE FALSE FALSE FALSE FALSE
         [,8]  [,9] [,10] [,11] [,12] [,13] [,14]
H1anova  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE
H1bonfe FALSE FALSE FALSE FALSE FALSE FALSE FALSE
        [,15] [,16] [,17] [,18] [,19] [,20] [,21]
H1anova  TRUE  TRUE  TRUE  TRUE  TRUE FALSE  TRUE
H1bonfe FALSE FALSE FALSE FALSE FALSE  TRUE FALSE
        [,22] [,23] [,24] [,25] [,26] [,27] [,28]
H1anova  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE
```