IMPRECISE STOCHASTIC ORDERS AND FUZZY RANKINGS

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Abstract. The aim of this work is to extend the notion of stochastic order to the pairwise comparison of fuzzy random variables. We consider the three most important stochastic orders in the literature: expected utility, stochastic dominance and statistical preference, which are related to the comparisons of the expectations, distribution functions and medians of the underlying variables. We discuss how to generalize these notions to the fuzzy case, when an epistemic interpretation is given to the fuzzy random variables. In passing, we investigate to which extent the earlier extensions of stochastic dominance and expected utility to the comparison of sets of random variables can be useful as fuzzy rankings. Finally, we illustrate our results in a decision making problem.

Keywords: Fuzzy sets, fuzzy random variables, stochastic dominance, statistical preference, expected utility, fuzzy rankings.

1. Introduction

Since they were introduced by Zadeh [1965], fuzzy sets have been widely used as a mathematical model of linguistic information. Their use was boosted further by the introduction by Féron of the notion of fuzzy random variables [Féron, 1976], as generalization of random variables to situations where the images are fuzzy sets (see for example Chaci and Taheri [2011], Couso and Sánchez [2011], González Rodríguez et al. [2009], Thuan et al. [2014] for recent advances).

The widespread use of fuzzy sets has lead naturally to the consideration of decision making problems with fuzzy information (see for example Bellman and Zadeh [1970], Hong and Choi [2000], Mardani et al. [2015], Wu and Chiclana [2014]). In order to solve them, it is necessary to use methods that allow to rank fuzzy sets and fuzzy random variables. The first of these two cases has been widely investigated (see for instance the surveys on fuzzy rankings in Wang and Kerre [2001a], Yuan [1991] or a new approach from an imprecise probability point of view by Destercke and Couso [2015]) but, to the best of our knowledge, the only proposals on rankings for fuzzy random variables have been established in Aiche and Dubois [2010, 2012].

In this paper, we try to remedy this situation somewhat by proposing a few manners in which two fuzzy random variables can be compared. The choice between the very many options that arise can be made by means of several criteria:

- First of all, it is important to take into account the interpretation we give to the fuzziness inherent to our model. In this respect, there are two main interpretations of fuzzy random variables: the *ontic*, where the images of the variable are fuzzy objects, and the *epistemic*, where the fuzzy images are a model for the imprecise knowledge of a crisp value. While the proposals in Aiche and Dubois [2010, 2012] work better under the ontic interpretation, in this paper we will give fuzzy random variables an epistemic interpretation.

- Since then we assume an underlying precise random variable, our proposals will be based in extending orders for pairs of random variables. Specifically, we will focus on stochastic dominance, expected utility and statistical preference. Then one first choice should be which of these models we want to use in the precise case: expected utility is based in the comparison of the expectations, and has been shown to be reasonable for axiomatizing preferences in a decision problem under uncertainty; the stronger notion of stochastic dominance compares the distribution functions and has been deemed useful in economics; while statistical preference can be regarded as a more robust alternative to expected utility, that is also useful when utilities are expressed in a qualitative scale.
- Finally, even for each of these models there will be several possible extensions to the imprecise case. The choice between them can be made by means of the interpretation of the extension, that we shall discuss later, and also by means of their mathematical properties, that we shall also investigate.

The remainder of the paper is organized as follows: we begin the paper in Section 2 by introducing the main notions of fuzzy set theory, stochastic ordering, and imprecise probabilities that are employed later in the paper.

In earlier works [Montes et al., 2014a,b], we already extended these stochastic orders to the comparison of *sets* of random variables, as a first step when modeling imprecise information. We use some of the results from those papers in Section 3, when we compare a number of imprecise probability models related to fuzzy random variables: random sets, or measurable multi-valued mappings, that are determined by the α-cuts of the fuzzy random variable; probability boxes, that correspond to sets of distribution functions and that arise when extending stochastic dominance; and possibility measures, that are an imprecise probability model that is mathematically equivalent to a fuzzy set.

One possibility for comparing fuzzy random variables is to consider a fuzzy set that is representative of the fuzzy random variable (for instance its expectation), and to reduce the problem of comparing the fuzzy random variables to that of comparing their associated fuzzy sets. The latter can be solved by means of fuzzy rankings. The elicitation of a fuzzy ranking among the vast number of proposals in the literature can be done taking into account the desirable properties discussed in Wang and Kerre [2001a,b]. In Section 4 we investigate which of these properties are satisfied by imprecise stochastic dominance and imprecise expected utility.

Next in Section 5 we study how these stochastic orders may be extended towards the comparison of fuzzy random variables: our definitions give rise to the notions of fuzzy expected utility, fuzzy stochastic dominance and fuzzy statistical preference. The first of these possibilities makes the comparison of the fuzzy expectations of the variables by means of a fuzzy ranking, or an imprecise stochastic order; the second compares the values of their fuzzy distribution functions; and the third compares the images of the fuzzy random variables. We also make a fourth proposal where we compare the fuzzy random variables by looking at the random sets that their α-cuts determine, and we investigate the properties of all these notions in the case of fuzzy random variables with trapezoidal values. In addition, we show that our notion of fuzzy expected utility encompasses the orders that can be derived from Walley's upper and lower probability models for a fuzzy random variable.

Section 6 illustrates the proposals in this paper on a decision problem with fuzzy information. We conclude the paper in Section 7 with some additional comments and remarks.

2. Preliminary concepts

2.1. **Fuzzy sets and fuzzy random variables.** From Zadeh [1965], a *fuzzy set* X is a mathematical model for linguistic information, and is determined by a *membership function* $\mu_X : \Omega \to [0,1]$, so that for every $\omega \in \Omega$ the value $\mu_X(\omega)$ represents the acceptability of the statement 'ω satisfies the concept encoded by the fuzzy set X'. The set of elements with strictly positive membership value is called the *support* of X, and we shall denote it supp(X). We shall denote by $\mathcal{F}(\Omega)$ the class of all fuzzy sets over a referential space Ω.

The membership function can be extended to subsets of the possibility space Ω. It has been argued that this extension should be supremum-preserving, so that the acceptability in which the fuzzy concept is satisfied by set A is given by

$$\Pi(A) = \sup_{\omega \in A} \mu(\omega).$$

This function is a *possibility measure* [Dubois and Prade, 1988, Zadeh, 1978], and the membership function μ is its associated *possibility distribution*.

Fuzzy numbers is one prominent family of fuzzy sets that would play an important role in this work. A fuzzy set is a fuzzy number if there exists a closed non-empty interval $[a,b]$ such that:

$$\mu(x) = \begin{cases} 1 & \text{for } x \in [a,b]; \\ l(x) & \text{for } x < a; \\ r(x) & \text{for } x > b; \end{cases}$$

where $l : (-\infty, a) \to [0,1]$ is a non-decreasing and right-continuous function such that $l(x) = 0$ for $x < \omega_1$, and $r : (b, \infty) \to [0,1]$ is a non-increasing and left-continuous function such that $r(x) = 0$ for $x > \omega_2$, for some ω_1, ω_2 in the real line $\mathbb{R}$. From this definition, it follows that the α-cuts of a fuzzy number are closed intervals (see for example Figure 1).

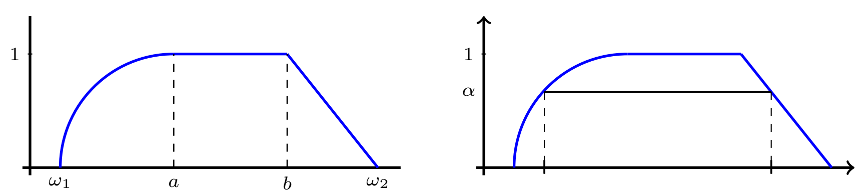

Figure 1. Example of a fuzzy number (left picture) and one of its α-cuts (right picture).

In this paper we focus on fuzzy random variables, which extend the notion of random variable to the case where the images are fuzzy sets.

Definition 1. Kruse and Meyer [1987] Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy random variable is a map $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ such that the α-cuts $\widetilde{X}_\alpha : \Omega \to \mathcal{P}(\mathbb{R})$ given by

$$\widetilde{X}_\alpha(\omega) = \{r \in \mathbb{R} : \widetilde{X}(\omega)(r) \geq \alpha\}$$

are *random sets*, meaning that

$$\{\omega \in \Omega : \widetilde{X}_\alpha(\omega) \cap A \neq \emptyset\} \in \mathcal{A} \ \forall A \in \beta_{\mathbb{R}}, \tag{1}$$

where $\beta_{\mathbb{R}}$ denotes the class of Borel subsets of $\mathbb{R}$.

Fuzzy random variables were introduced by Féron [1976] as a generalization of random variables to the case where the images are fuzzy subsets. They have been given two main interpretations: the *ontic* one, considered by Puri and Ralescu [1986], where they can be regarded as random variables whose values are (vague) linguistic assessments, and the *epistemic* one, developed by Kruse and Meyer [1987], which has its roots in the work by Kwakernaak [1978], in which fuzzy random variables are a model for the imprecise knowledge of a random variable. We refer to Couso and Sánchez [2008], Couso et al. [2014] for a review of the different interpretations. In this paper we align with the works in Couso and Sánchez [2008], de Cooman [2005], de Cooman and Walley [2002] and follow the epistemic interpretation. According to it, fuzzy random variables model the imprecise knowledge about a random variable U_0: for any $\omega' \in \mathbb{R}$, $\widetilde{X}(\omega)(\omega')$ is interpreted as the acceptability degree of the proposition "$U_0(\omega) = \omega'$". Following these lines, it is possible to define a fuzzy set on the class of measurable functions from Ω to $\mathbb{R}$, $\mu_{\widetilde{X}}$, such that it associates the value:

$$\mu_{\widetilde{X}}(U) = \inf\{\widetilde{X}(\omega)(U(\omega)) : \omega \in \Omega\} \tag{2}$$

to any measurable function $U : \Omega \to \mathbb{R}$. This value can then be understood as the acceptability degree of the proposition "$U = U_0$".

Using this interpretation, Couso [1999] defined the probabilistic envelope of a fuzzy random variable.

Definition 2. [Couso, 1999, Definition 5.1.1] Let $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ be a fuzzy random variable. The *probabilistic envelope* of $\widetilde{X}$ is the map $P_{\widetilde{X}} : \mathcal{A} \to \mathcal{F}([0,1])$ with the membership function

$$P_{\widetilde{X}}(A)(p) = \sup\{\mu_{\widetilde{X}}(U) \mid U : \Omega \to \mathbb{R} \text{ r.v.}, P_U(A) = p\}$$

for any $A \in \mathcal{A}$ and $p \in [0,1]$.

With a similar reasoning, $P_{\widetilde{X}}(A)(p)$ can be interpreted as the acceptability degree of the proposition "$P_{U_0}(A) = p$". In fact, using the probabilistic envelope it is possible to define the envelope of the cumulative distribution function of $\widetilde{X}$ as the map $F_{\widetilde{X}} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ such that

$$F_{\widetilde{X}}(x)(p) = \sup\{\mu_{\widetilde{X}}(U) : F_U(x) = p\}. \tag{3}$$

Then, $F_{\widetilde{X}}(x) = P_{\widetilde{X}}((-\infty, x])$ for any $x \in \mathbb{R}$, and $F_{\widetilde{X}}(x)(p)$ can be interpreted as the acceptability degree of the proposition "$F_{U_0}(x) = p$".

Following Couso [1999], Grzegorzewski [1998], Kruse and Meyer [1987], the fuzzy version of any parameter can be defined for fuzzy random variables. Formally, if the parameter belongs to the parametric space Θ, the fuzzy version of the parameter is defined by:

$$\theta_{\widetilde{X}} \in \mathcal{F}(\Theta), \quad \theta_{\widetilde{X}}(\theta') = \sup\{\mu_{\widetilde{X}}(U) : \theta(P_U) = \theta'\}. \tag{4}$$

$\theta_{\widetilde{X}}(\theta')$ represents the acceptability degree of the proposition $\theta(P_{U_0}) = \theta'$. In particular, the expectation of a fuzzy random variable can be defined using Eq. (4):

$$E(\widetilde{X})(r) = \sup\{\mu_{\widetilde{X}}(U) : E(U) = r\}, \tag{5}$$

and $E(\widetilde{X})(r)$ can be interpreted as the acceptability degree of the proposition "$E(U_0) = r$". Under some regularity conditions, and in particular when the images of the fuzzy random variable are fuzzy numbers, this coincides [Couso, 1999, Kruse and Meyer, 1987, Puri and Ralescu, 1986] with the fuzzy set whose α-cuts are given by

$$(E(\widetilde{X}))_\alpha = (A)\int \widetilde{X}_\alpha dP, \tag{6}$$

where the integral above is the *Aumann integral* [Aumann, 1965] of the random set determined by the α-cut of $\widetilde{X}$, whose definition was given in Eq. (1).

2.2. **Stochastic orders.** Stochastic orders are methods that compare random variables by means of their probabilistic information [Müller and Stoyan, 2002, Shaked and Shantikumar, 2006]. We shall denote a stochastic order by $\succeq$, and shall use $\succ$ and $\equiv$ to denote its associated strict preference and indifference relations.

The most important stochastic order is *expected utility*: given two random variables X, Y defined on a probability space $(\Omega, \mathcal{A}, P)$, we define

$$X \succeq_{\mathrm{E}} Y \Leftrightarrow E(X) \geq E(Y),$$

whenever both expectations exist.

In this paper, we shall also consider two other stochastic orders. The first one is called stochastic dominance.

Definition 3. Let X and Y be two random variables and let F_{X} and F_{Y} be their respective cumulative distribution functions. X is said to *stochastically dominate* Y if

$$F_{\mathrm{X}}(t) \leq F_{\mathrm{Y}}(t) \text{ for every } t \in \mathbb{R},$$

and it is denoted by $X \succeq_{\mathrm{SD}} Y$.

Stochastic dominance can be characterized by means of the comparison of the expectations of adequate transformations of the random variables.

Theorem 1. *[Müller and Stoyan, 2002] Let X and Y be two random variables, and define*

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R} \textit{ increasing}\}.$$

Then, $X \succeq_{\mathrm{SD}} Y$ if and only if $E(u(X)) \geq E(u(Y))$ for all $u \in \mathcal{U}$. In particular, $X \succeq_{\mathrm{SD}} Y$ implies $X \succeq_{\mathrm{E}} Y$.

Another alternative for comparing random variables is statistical preference [David, 1963, De Schuymer et al., 2003], which is based on the notion of probabilistic relation [Bezdek et al., 1978]. Given a set of alternatives $\mathcal{A}$, a map $Q : \mathcal{A}\times\mathcal{A} \to [0,1]$ is called a *probabilistic relation* if $Q(a,b) + Q(b,a) = 1$ for any $a, b \in \mathcal{A}$.

Definition 4. Consider two random variables X, Y, and let Q be the probabilistic relation given by

$$Q(X,Y) = P(X > Y) + \frac{1}{2}P(X = Y).$$

Then X is said to be *statistically preferred* to Y, and it is denoted $X \succeq_{\mathrm{SP}} Y$, when $Q(X,Y) \geq Q(Y,X)$, or equivalently, when $P(X \geq Y) \geq P(Y \geq X)$.

Unlike expected utility and stochastic dominance, statistical preference takes into account the joint behavior of the two variables, and it can be used to establish degrees of preference between the two variables. It has been proved useful in decision making problems with qualitative utilities by Dubois et al. [2003].

2.3. **Lower and upper previsions.** In this paper we will consider earlier works extending stochastic orders to the comparisons of sets of random variables. For this, we shall consider different models that can be used to summarize a set of probability measures. One of these will be the possibility measures induced by fuzzy sets we saw in Section 2.1; in this section we shall introduce other possible models.

A prominent model within imprecise probability theory are *lower and upper previsions* [Walley, 1991]. Given a possibility space Ω, a gamble is a bounded random variable $f : \Omega \to \mathbb{R}$. The set of all gambles is denoted $\mathcal{L}(\Omega)$. A lower prevision is a mapping $\underline{P} : \mathcal{K} \to \mathbb{R}$ where $\mathcal{K}$ is a subset of $\mathcal{L}(\Omega)$. It can be given a behavioral interpretation in terms of buying and selling prices. A lower prevision defined on indicators of events only is called a *lower probability.*

A lower prevision $\underline{P}$ defines an upper prevision $\overline{P}$ from $-\mathcal{K} = \{-f : f \in \mathcal{K}\}$ to $\mathbb{R}$ by $\overline{P}(f) = -\underline{P}(-f)$.

When $\mathcal{K} = \mathcal{L}(\Omega)$ and $\underline{P}(f) = \overline{P}(f)$ for every f, then $\underline{P}$ is called a *linear prevision*, and its restriction to events is a finitely additive probability. Then if we consider the set

$$\mathcal{M}(\underline{P}) = \{P \text{ linear prevision} : \underline{P}(f) \leq P(f) \leq \overline{P}(f)\ \forall f \in \mathcal{K}\},$$

it is said that $\underline{P}$ is *coherent* when it is the lower envelope of $\mathcal{M}(\underline{P})$.

Completely monotone and minimum-preserving lower previsions are particular instances of coherent lower previsions:

Definition 5. A lower prevision $\underline{P}$ defined on a lattice of gambles $\mathcal{K}$ is n-monotone when:

$$\sum_{I \subseteq \{1,\dots,p\}} (-1)^{|I|} \underline{P}\left(f \wedge \bigwedge_{i \in I} f_i\right) \geq 0$$

for any $p \leq n$, $f, f_1, \dots, f_p \in \mathcal{K}$. It is called *completely monotone* when it is n-monotone for any n, and it is *minimum preserving* when $\underline{P}(f \wedge g) = \underline{P}(f) \wedge \underline{P}(g)$ for any $f, g \in \mathcal{K}$.

It can be proven that a minimum preserving lower prevision is in particular completely monotone. Although coherent lower and upper previsions are more informative in general than their associated lower and upper probabilities, in the sense that two different coherent lower previsions may have the same restriction to events, under 2-monotonicity this is not the case, in the sense that a 2-monotone lower probability $\underline{P}$ defined on $\mathcal{P}(\Omega)$ has only one 2-monotone extension to $\mathcal{L}(\Omega)$: its *Choquet integral* [Choquet, 1953, Denneberg, 1994], given by

$$\underline{P}(f) = (C)\int f \mathrm{d}\underline{P} = \inf f + \int_{-\infty}^{\infty} \underline{P}(f \geq t)\mathrm{d}t \text{ and}$$
$$\overline{P}(f) = (C)\int f \mathrm{d}\overline{P} = \inf f + \int_{-\infty}^{\infty} \overline{P}(f \geq t)\mathrm{d}t.$$

for every gamble f.

For a complete review on lower and upper previsions, we refer to Miranda [2008].

3. Imprecise stochastic orders

In Montes et al. [2011, 2012], we extended the stochastic orders introduced in Section 2.2 towards the comparison of *sets* of random variables. We refer to Montes et al. [2014a,b] for a unified approach, to Denoeux [2009] for a study of stochastic dominance for belief functions and to Couso and Dubois [2012] for a posterior alternative study in terms of lower and upper previsions. The following definitions were considered:

Definition 6. [Montes et al., 2014a, Definition 5] Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of random variables, and let $\succeq$ be a stochastic order. It is said that:

(1) $\mathcal{X} \succeq_1 \mathcal{Y}$ if and only if for every $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ it holds that $X \succeq Y$.
(2) $\mathcal{X} \succeq_2 \mathcal{Y}$ if and only if there is some $X \in \mathcal{X}$ such that $X \succeq Y$ for every $Y \in \mathcal{Y}$.
(3) $\mathcal{X} \succeq_3 \mathcal{Y}$ if and only if for every $Y \in \mathcal{Y}$ there is some $X \in \mathcal{X}$ such that $X \succeq Y$.
(4) $\mathcal{X} \succeq_4 \mathcal{Y}$ if and only if there are $X \in \mathcal{X}, Y \in \mathcal{Y}$ such that $X \succeq Y$.
(5) $\mathcal{X} \succeq_5 \mathcal{Y}$ if and only if there is some $Y \in \mathcal{Y}$ such that $X \succeq Y$ for every $X \in \mathcal{X}$.
(6) $\mathcal{X} \succeq_6 \mathcal{Y}$ if and only if for every $X \in \mathcal{X}$ there is $Y \in \mathcal{Y}$ such that $X \succeq Y$.

The first of these extensions is the most restrictive one, and it requires that each element of $\mathcal{X}$ dominates each element of $\mathcal{Y}$; as such, it produces many instances of incomparability. The second and third definitions have an underlying interpretation of risk-seeking, as they focus on the best alternative within each set: they require that each alternative in $\mathcal{Y}$ is dominated by some alternative in $\mathcal{X}$ (the same in $\succeq_2$, not necessarily in $\succeq_3$). The risk-aversion counterpart is given by the fifth and sixth definitions, where we require that each alternative in $\mathcal{X}$ dominates some alternative in $\mathcal{Y}$ (the same in the case of $\succeq_5$, not necessarily in the case of $\succeq_6$). Finally, the fourth extension is the weakest, as it only requires the existence of a pair $X \in \mathcal{X}, Y \in \mathcal{Y}$ such that X dominates Y.

The relationships between these conditions are summarized in the following figure:

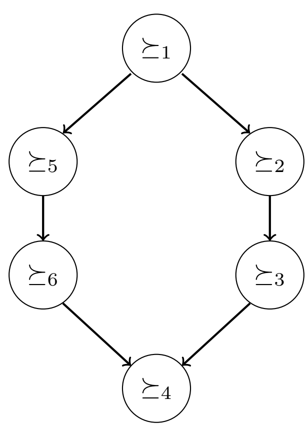

FIGURE 2. Relationships between the extensions of the stochastic orders.

It can be checked that no additional implication holds in general.

When $\succeq$ is given by stochastic dominance, statistical preference or expected utility, we shall refer to the extensions $\succeq_i$ for $i = 1, \ldots, 6$ as *imprecise stochastic* *dominance, imprecise statistical preference* or *imprecise expected utility*, and we shall denote them by $\succeq_{\mathrm{SD_i}}$, $\succeq_{\mathrm{SP_i}}$ or $\succeq_{\mathrm{E_i}}$, respectively.

In particular, in Montes et al. [2014b] we showed that imprecise expected utility is related to some other generalizations of expected utility to the imprecise framework, such as interval dominance [Zaffalon et al., 2003], maximax [Satia and Lave, 1973], maximin [Gilboa and Schmeidler, 1989] or E-admissibility [Levi, 1980]. See Couso and Dubois [2012], Troffaes [2007] for a review on these methods.

Next, we are going to show how the definition above can be applied in a number of particular cases: the comparison of random sets, possibility measures and sets of distribution functions. These three models are related to fuzzy random variables and, as a consequence, our results in this section shall be useful when comparing fuzzy random variables in Section 5.

3.1. **Random sets.** As we mentioned in Section 2.1, the α-cuts of a fuzzy random variable are random sets. As we will discuss in Section 5.5, one possible way of comparing two fuzzy random variables is to compare their α-cuts for all, or some, α in $[0,1]$. In this section we investigate how this comparison of the α-cuts can be done. Recall that in this paper we are considering an *epistemic* interpretation [Montes et al., 2014a], and regard random sets as a model for the imprecise knowledge of a random variable. We refer to Cascos and Molchanov [2003] to a study of stochastic orders for random sets under an ontic interpretation.

Our focus in this paper shall be on *random closed intervals* Miranda et al. [2005]. These are random sets $\Gamma = [L, R]$, where L, R are random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and such that $L \leq R$. Under the epistemic interpretation mentioned above, the information about the unknown random variable is given by the set of *measurable selections* of Γ, given by

$$S(\Gamma) = \{U \text{ random variable} : U(\omega) \in \Gamma(\omega)\ \forall \omega\}.$$

Thus, the comparison between two random closed intervals $\Gamma_{\mathrm{X}}, \Gamma_{\mathrm{Y}}$ shall be done by means of their associated sets of measurable selections $S(\Gamma_{\mathrm{X}})$ and $S(\Gamma_{\mathrm{Y}})$, using Definition 6. We shall denote $\Gamma_{\mathrm{X}} \succeq_i \Gamma_{\mathrm{Y}}$ to refer to $S(\Gamma_{\mathrm{X}}) \succeq_i S(\Gamma_{\mathrm{Y}})$.

Proposition 1. *Let $\Gamma_{\mathrm{X}} = [L_{\mathrm{X}}, R_{\mathrm{X}}]$ and $\Gamma_{\mathrm{Y}} = [L_{\mathrm{Y}}, R_{\mathrm{Y}}]$ be two random closed intervals, and let $\succeq$ be a stochastic order that is compatible with monotonicity, in the sense that*

$$\text{if } W_1 \geq V_1 \text{ and } V_2 \geq W_2, \text{ then } V_1 \succeq V_2 \Rightarrow W_1 \succeq W_2 \tag{7}$$

for any random variables W_1, W_2, V_1, V_2 defined on the same probability space. Then the following conditions hold:

(1) $\Gamma_{\mathrm{X}} \succeq_1 \Gamma_{\mathrm{Y}} \Leftrightarrow L_{\mathrm{X}} \succeq R_{\mathrm{Y}}$.
(2) $\Gamma_{\mathrm{X}} \succeq_2 \Gamma_{\mathrm{Y}} \Leftrightarrow \Gamma_{\mathrm{X}} \succeq_3 \Gamma_{\mathrm{Y}} \Leftrightarrow R_{\mathrm{X}} \succeq R_{\mathrm{Y}}$.
(3) $\Gamma_{\mathrm{X}} \succeq_4 \Gamma_{\mathrm{Y}} \Leftrightarrow R_{\mathrm{X}} \succeq L_{\mathrm{Y}}$.
(4) $\Gamma_{\mathrm{X}} \succeq_5 \Gamma_{\mathrm{Y}} \Leftrightarrow \Gamma_{\mathrm{X}} \succeq_6 \Gamma_{\mathrm{Y}} \Leftrightarrow L_{\mathrm{X}} \succeq L_{\mathrm{Y}}$.

Proof. Since $L_{\mathrm{X}}, R_{\mathrm{X}} \in S(\Gamma_{\mathrm{X}})$ and $L_{\mathrm{Y}}, R_{\mathrm{Y}} \in S(\Gamma_{\mathrm{Y}})$, it follows from Eq. (7) that

$$R_{\mathrm{X}} \succeq V \succeq L_{\mathrm{X}}\ \forall V \in S(\Gamma_{\mathrm{X}}) \text{ and } R_{\mathrm{Y}} \succeq V \succeq L_{\mathrm{Y}}\ \forall V \in S(\Gamma_{\mathrm{Y}}).$$

The result then follows immediately from Definition 6. □

Note that Eq. (7) is satisfied by the three stochastic orders we have introduced in Section 2.2: expected utility, stochastic dominance and statistical preference.

3.2. **Sets of distribution functions.** Another possible manner in which we can compare two fuzzy random variables is to consider a fuzzy set that is representative of each fuzzy random variable (for instance its expectation). This fuzzy set is mathematically equivalent to a possibility measure, and also to a set of distribution functions. In our next two sections we consider these two models and study how imprecise stochastic orderings apply to them.

Definition 7. Let $\underline{F}, \overline{F} : \mathbb{R} \to [0,1]$ be two monotone functions satisfying $\underline{F} \leq \overline{F}$ and such that $\lim_{x\to-\infty} \underline{F}(x) = \lim_{x\to-\infty} \overline{F}(x) = 0$ and $\lim_{x\to+\infty} \underline{F}(x) = \lim_{x\to+\infty} \overline{F}(x) = 1$. Then the *p-box* $(\underline{F}, \overline{F})$ is the set of distribution functions bounded between $\underline{F}$ and $\overline{F}$.

An instance is represented in the following figure:

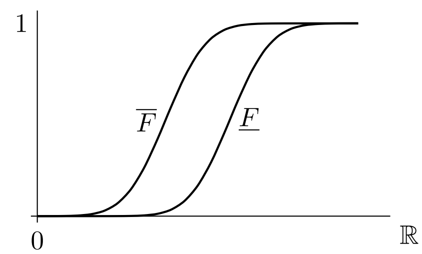

FIGURE 3. Example of a p-box.

Note that $\underline{F}, \overline{F}$ need not be distribution functions, because we are not requiring them to be right-continuous; they will only be distribution functions associated with a finitely additive probability. In this paper, we shall follow the work in Montes et al. [2014b] and regard p-boxes as sets of σ-additive distribution functions; for other works in the literature where p-boxes are considered sets of *finitely additive* distribution functions we refer to Troffaes and Destercke [2011].

When the imprecise probability models to be compared are given by two p-boxes, $(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}})$ and $(\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}})$, the comparison between them can be made by means of the extensions of stochastic dominance and expected utility (we do not consider statistical preference here because it requires the comparison of the images of the random variables inducing these distributions, which we are not known in general).

Consider a p-box $(\underline{F}, \overline{F})$ with bounded support and such that both $\underline{F}, \overline{F}$ are distribution functions. They induce probabilities $P_{\underline{F}}, P_{\overline{F}}$ on the field generated by

$$\{(-\infty, x], (x, +\infty) : x \in \mathbb{R}\}.$$

We shall denote by $\mu_{\underline{F}}, \mu_{\overline{F}}$ the inner extensions of $P_{\underline{F}}, P_{\overline{F}}$ to $\mathcal{P}(\mathbb{R})$, and by

$$(C)\int u d\underline{F} \quad \text{and} \quad (C)\int u d\overline{F}$$

the Choquet integrals of a function u with respect to $\mu_{\underline{F}}, \mu_{\overline{F}}$, respectively. It follows from monotonicity that for any $u \in \mathcal{U}$

$$(C)\int u d\underline{F} \leq \inf_{F\in(\underline{F},\overline{F})} \int u dF \quad \text{and} \quad (C)\int u d\overline{F} \geq \sup_{F\in(\underline{F},\overline{F})} \int u dF; \qquad (8)$$

note that since $\underline{F}, \overline{F}$ have bounded support, we can assume without loss of generality that any $u \in \mathcal{U}$ is bounded. In addition, when $\underline{F}, \overline{F}$ are distribution functions, then it holds that $(C)\int ud\underline{F} = \int ud\underline{F}$ and $(C)\int ud\overline{F} = \int ud\overline{F}$.

With respect to imprecise expected utility, it is then easy to establish the following:

Proposition 2. *Consider two p-boxes $(\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $(\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$, with bounded support and including their respective lower and upper distribution functions.*

(1) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_1} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\overline{F}_{\rm X} \geq \int id \ {\rm d}\underline{F}_{\rm Y}$.

(2) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_2} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_3} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\underline{F}_{\rm X} \geq \int id \ {\rm d}\underline{F}_{\rm Y}$.

(3) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_4} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\underline{F}_{\rm X} \geq \int id \ {\rm d}\overline{F}_{\rm Y}$.

(4) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_5} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm E_6} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int id \ {\rm d}\overline{F}_{\rm X} \geq \int id \ {\rm d}\overline{F}_{\rm Y}$.

Proof. The result follows immediately from Eq. (8), taking also into account that by assumption $\underline{F}_{\rm X}, \overline{F}_{\rm X} \in (\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $\underline{F}_{\rm Y}, \overline{F}_{\rm Y} \in (\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$. □

With respect to imprecise stochastic dominance, some results were already established in [Montes et al., 2014a, Theorem 8] and [Montes et al., 2014b, Proposition 3] for the comparison of arbitrary sets of distribution functions. Next we show that the converses of this second result hold when the sets of distribution functions to be compared are determined by respective p-boxes.

Proposition 3. *Consider two p-boxes $(\underline{F}_{\rm X}, \overline{F}_{\rm X})$ and $(\underline{F}_{\rm Y}, \overline{F}_{\rm Y})$ with bounded support and including their respective lower and upper distribution functions.*

(1) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_1} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\overline{F}_{\rm X} \geq \int u{\rm d}\underline{F}_{\rm Y}$ *for every* $u \in \mathcal{U}$.

(2)
$$(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_2} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_3} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\underline{F}_{\rm X} \geq \int u{\rm d}\underline{F}_{\rm Y} \text{ for every } u \in \mathcal{U}.$$

(3) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_4} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\underline{F}_{\rm X} \geq \int u{\rm d}\overline{F}_{\rm Y}$ *for every* $u \in \mathcal{U}$.

(4)
$$(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_5} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_6} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \int u{\rm d}\overline{F}_{\rm X} \geq \int u{\rm d}\overline{F}_{\rm Y} \text{ for every } u \in \mathcal{U}.$$

Proof. The result follows taking into account that, by [Montes et al., 2014b, Corollary 1], the following equivalences hold:

(SD1) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_1} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \overline{F}_{\rm X} \leq \underline{F}_{\rm Y}$;

(SD2-3) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_2} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_3} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \underline{F}_{\rm X} \leq \underline{F}_{\rm Y}$;

(SD4) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_4} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \underline{F}_{\rm X} \leq \overline{F}_{\rm Y}$;

(SD5-6) $(\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_5} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow (\underline{F}_{\rm X}, \overline{F}_{\rm X}) \succeq_{\rm SD_6} (\underline{F}_{\rm Y}, \overline{F}_{\rm Y}) \Leftrightarrow \overline{F}_{\rm X} \leq \overline{F}_{\rm Y}$, and also that, given two distribution functions F_1, F_2, it follows from Theorem 1 that $F_1 \leq F_2$ if and only if $\int u dF_1 \geq \int u dF_2$ for every $u \in \mathcal{U}$. □

It follows from these two results that $\succeq_{\mathrm{SD_i}}$ implies $\succeq_{\mathrm{E_i}}$ for every $i = 1, \ldots, 6$ in this context, and this agrees with Theorem 1.

3.3. **Possibility measures.** Another imprecise probability model closely related to fuzzy set theory are possibility measures [Zadeh, 1978]: recall that the membership function of a fuzzy set can be interpreted as a possibility distribution, and as a consequence its associated possibility measure extends the membership function towards subsets of the referential space.

A possibility measure Π on $\mathcal{P}(\mathbb{R})$ determines a set of probability measures by means of

$$\mathcal{M}(\Pi) = \{P \text{ prob.} : P(A) \leq \Pi(A) \ \forall A \in \beta_{\mathbb{R}}\}. \tag{9}$$

It coincides with the set of probability measures that dominate the conjugate *necessity measure* of Π, given by $N(A) = 1 - \Pi(A^c) \ \forall A \subseteq \Omega$.

In this paragraph we shall consider possibility measures induced by fuzzy numbers. For them, we can establish a simple characterization of stochastic dominance and of expected utility. The case of stochastic dominance was already considered in [Montes et al., 2014b, Section 4.4], for possibility measures on the unit interval. What follows is a straightforward extension of [Montes et al., 2014b, Lemma 2].

Consider thus Π_1, Π_2 two possibility measures defined on the power set of $\mathbb{R}$ whose respective possibility distributions are fuzzy numbers, and let $\mathcal{M}(\Pi_1), \mathcal{M}(\Pi_2)$ be the sets they induce by Eq. (9). We shall also denote by $\mathcal{F}_1, \mathcal{F}_2$ the corresponding sets of cumulative distribution functions. Our following lemma shows that these sets are indeed determined by the possibility measures and their conjugate necessity measures.

Lemma 1. *Let Π be a possibility measure on $\mathbb{R}$ associated with a fuzzy number, and let $\mathcal{F}$ denote the set of cumulative distribution functions associated with $\mathcal{M}(\Pi)$. Then the lower and upper envelopes of $\mathcal{F}$ belong to $\mathcal{F}$ and they coincide with the ones determined by N and Π.*

Proof. Denote by F_{N} and F_{Π} the cumulative distribution functions associated with N and Π, respectively. Then it holds that:

$$\begin{aligned} \overline{F}(x) &= \sup_{P \leq \Pi} P((-\infty, x]) = \Pi((-\infty, x]) = F_{\Pi}(x). \\ \underline{F}(x) &= \inf_{P \leq \Pi} P((-\infty, x]) = 1 - \Pi((x, +\infty)) \\ &= 1 - \sup_{y > x} \pi(y) = N((-\infty, x]) = F_N(x). \end{aligned}$$

To see that these bounds are indeed attained, note that by Goodman [1982] the random set $\Gamma : [0,1] \to \mathcal{P}(\mathbb{R})$ given by $\Gamma(\alpha) = \{x : \pi(x) \geq \alpha\}$ satisfies $P^*_{\Gamma} = \Pi$. Moreover, it follows that when Π is associated with a fuzzy number, this random set is compact-valued. Applying [Castaldo et al., 2004, Lemma 3.3] to the identity map $id : \mathbb{R} \to \mathbb{R}$ we deduce that there exists a measurable selection $U \in S(\Gamma)$ such that $F_U = F_{\Pi}$. Since the probability measure of any measurable selection is always dominated by the upper probability, it follows that $F_{\Pi} = \max\{F_P : P \leq \Pi\}$. The result for F_N follows by duality. □

Using this lemma, we can establish the following characterization of imprecise expected utility for possibility measures:

Corollary 1. *Let $\Pi_{\rm X}$ and $\Pi_{\rm Y}$ be two possibility measures on $\mathbb{R}$ associated with fuzzy numbers, and let $\mathcal{F}_{\rm X}, \mathcal{F}_{\rm Y}$ denote their associated sets of distribution functions.*

(1) $\mathcal{F}_{\rm X} \succeq_{\rm E_1} \mathcal{F}_{\rm Y} \Leftrightarrow \int id \; {\rm d}F_{\Pi_{\rm X}} \geq \int id \; {\rm d}F_{N_{\rm Y}}$.

(2) $\mathcal{F}_{\rm X} \succeq_{\rm E_2} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm E_3} \mathcal{F}_{\rm Y} \Leftrightarrow \int id \; {\rm d}F_{N_{\rm X}} \geq \int id \; {\rm d}F_{N_{\rm Y}}$.

(3) $\mathcal{F}_{\rm X} \succeq_{\rm E_4} \mathcal{F}_{\rm Y} \Leftrightarrow \int id \; {\rm d}F_{N_{\rm X}} \geq \int id \; {\rm d}F_{\Pi_{\rm Y}}$.

(4) $\mathcal{F}_{\rm X} \succeq_{\rm E_5} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm E_6} \mathcal{F}_{\rm Y} \Leftrightarrow \int id \; {\rm d}F_{\Pi_{\rm X}} \geq \int id \; {\rm d}F_{\Pi_{\rm Y}}$.

Proof. This is a consequence of Proposition 2 and Lemma 1. □

A similar result can be established for imprecise stochastic dominance:

Proposition 4. *Consider two possibility measures $\Pi_{\rm X}$ and $\Pi_{\rm Y}$ on $\mathbb{R}$ associated with fuzzy numbers, and denote by $\mathcal{F}_{\rm X}, \mathcal{F}_{\rm Y}$ their associated sets of distribution functions. The following statements hold:*

(1) $\mathcal{F}_{\rm X} \succeq_{\rm SD_1} \mathcal{F}_{\rm Y} \Leftrightarrow \Pi_{\rm X}((-\infty, t]) \leq N_{\rm Y}((-\infty, t]) \; \forall t$.
(2) $\mathcal{F}_{\rm X} \succeq_{\rm SD_2} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm SD_3} \mathcal{F}_{\rm Y} \Leftrightarrow N_{\rm X}((-\infty, t]) \leq N_{\rm Y}((-\infty, t]) \; \forall t$.
(3) $\mathcal{F}_{\rm X} \succeq_{\rm SD_4} \mathcal{F}_{\rm Y} \Leftrightarrow N_{\rm X}((-\infty, t]) \leq \Pi_{\rm Y}((-\infty, t]) \; \forall t$.
(4) $\mathcal{F}_{\rm X} \succeq_{\rm SD_5} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm SD_6} \mathcal{F}_{\rm Y} \Leftrightarrow \Pi_{\rm X}((-\infty, t]) \leq \Pi_{\rm Y}((-\infty, t]) \; \forall t$.

Proof. From the proof of Lemma 1, we see that

$$\overline{F}_{\rm X}(t) = \Pi_{\rm X}((-\infty, t]), \quad \underline{F}_{\rm X}(t) = N_{\rm X}((-\infty, t])$$
$$\overline{F}_{\rm Y}(t) = \Pi_{\rm Y}((-\infty, t]), \quad \underline{F}_{\rm Y}(t) = N_{\rm Y}((-\infty, t])$$

for every $t \in \mathbb{R}$. The result follows then from Eqs. (SD1)– (SD5-6). □

Proposition 3 established a connection between the comparison of p-boxes by imprecise stochastic dominance and the comparison of integrals. Next we establish an analogous result for particular case of p-boxes induced by possibility measures.

Corollary 2. *Let $\Pi_{\rm X}$ and $\Pi_{\rm Y}$ be two possibility measures on $\mathbb{R}$ associated with fuzzy numbers, and let $\mathcal{F}_{\rm X}, \mathcal{F}_{\rm Y}$ denote their associated sets of distribution functions. The following statements hold:*

(1) $\mathcal{F}_{\rm X} \succeq_{\rm SD_1} \mathcal{F}_{\rm Y} \Leftrightarrow \int u {\rm d}F_{\Pi_{\rm X}} \geq \int u {\rm d}F_{N_{\rm Y}}$ *for every* $u \in \mathcal{U}$.

(2) $\mathcal{F}_{\rm X} \succeq_{\rm SD_2} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm SD_3} \mathcal{F}_{\rm Y} \Leftrightarrow \int u {\rm d}F_{N_{\rm X}} \geq \int u {\rm d}F_{N_{\rm Y}}$ *for every* $u \in \mathcal{U}$.

(3) $\mathcal{F}_{\rm X} \succeq_{\rm SD_4} \mathcal{F}_{\rm Y} \Leftrightarrow (C) \int u {\rm d}F_{N_{\rm X}} \geq \int u {\rm d}F_{\Pi_{\rm Y}}$ *for every* $u \in \mathcal{U}$.

(4) $\mathcal{F}_{\rm X} \succeq_{\rm SD_5} \mathcal{F}_{\rm Y} \Leftrightarrow \mathcal{F}_{\rm X} \succeq_{\rm SD_6} \mathcal{F}_{\rm Y} \Leftrightarrow \int u {\rm d}F_{\Pi_{\rm X}} \geq \int u {\rm d}F_{\Pi_{\rm Y}}$ *for every* $u \in \mathcal{U}$.

Proof. The result follows immediately from Proposition 3 and Lemma 1. □

Taking these results into account, whenever we have to compare two possibility measures $\Pi_{\rm X}, \Pi_{\rm Y}$ whose possibility distributions are fuzzy numbers, we shall consider only definitions $\succeq_1, \succeq_2, \succeq_4$ and $\succeq_5$ whenever $\succeq$ refers to stochastic dominance or expected utility, since in both cases we have $\succeq_2 \Leftrightarrow \succeq_3$ and $\succeq_5 \Leftrightarrow \succeq_6$. Moreover, we shall use the notation $\Pi_{\rm X} \succeq_i \Pi_{\rm Y}$ to refer to $\mathcal{F}_{\rm X} \succeq_i \mathcal{F}_{\rm Y}$ for simplicity.

Also, it follows that imprecise stochastic dominance implies imprecise expected utility when applied to fuzzy numbers. This is consistent with the relation between stochastic dominance and expected utility in Theorem 1.

4. Imprecise stochastic orders as fuzzy rankings

A *fuzzy ranking* is a method that allows to establish comparisons between fuzzy sets. Several fuzzy rankings have been proposed in the literature [Abbasbandy and Asady, 2006, Chen, 1985, de Campos and González Muñoz, 1989, Deng, 2014, Ezzati et al., 2012, Fortemps and Roubens, 1996, Kim and Park, 1990, Lee et al., 2004, Wang, 2015]; we refer to Bortolan and Degani [1985], Wang and Kerre [2001a,b], Yuan [1991] for critical reviews. They can be classified in three types: those that transform the fuzzy sets into a real number and compare these with the usual order in $\mathbb{R}$; those that measure the distance from the fuzzy sets to a reference set (usually the maximum of the fuzzy sets); and those that provide a fuzzy relationship in order to compare the fuzzy sets.

Since a fuzzy set is formally equivalent to a possibility measure, we can use the ideas from Section 3.3 and regard the methods we have considered in the previous section as fuzzy rankings. There is, however, one fundamental difference with the ones considered above: we are allowing for incomparability between the fuzzy sets, which in our view is natural under the epistemic interpretation we are giving to fuzziness in this paper. In this sense, our proposal aligns with the one by Dubois and Prade [1983]: they propose several indices for the comparison between two fuzzy sets but in case of discrepancy leave the final choice to the decision maker, under the light of the information provided. See Zhang et al. [2014] for a somewhat related idea, where the image of the fuzzy ranking is a fuzzy set, and Bronevich and Rozenberg [2014] for an approach based on inclusion indices.

The use of stochastic orders as fuzzy rankings has already been investigated by Destercke and Couso [2015]. A key difference with our approach is that the authors of that paper make assumptions on the possible dependence between the fuzzy sets, according to which they express stochastic dominance, statistical preference and the comparison of expectations in terms of the comparison of lower/upper expectations with respect to adequate functions. On the other hand, in this paper we are not making any assumption about dependence between the fuzzy sets to be compared. As a consequence, we shall not consider statistical preference in this section, because it requires the knowledge of a joint model for the two fuzzy sets, and we are considering only marginal information.

We shall study the properties of imprecise stochastic dominance and imprecise expected utility as fuzzy rankings. We shall focus on the application of these orders to *trapezoidal fuzzy numbers*, for which the orders shall take a simple expression. Recall that a trapezoidal fuzzy number [Cheng, 1998] is a fuzzy set determined by four parameters (t_1, t_2, t_3, t_4); its membership function is given by:

$$\mu(x) = \begin{cases} 0 & \text{if } x < t_1. \\ \frac{x-t_1}{t_2-t_1} & \text{if } t_1 \leq x < t_2. \\ 1 & \text{if } t_2 \leq x \leq t_3. \\ \frac{t_4-x}{t_4-t_3} & \text{if } t_3 < x \leq t_4. \\ 0 & \text{if } x > t_4. \end{cases} \tag{10}$$

In Figure 4 we can see an example of the membership function of a triangular fuzzy number.

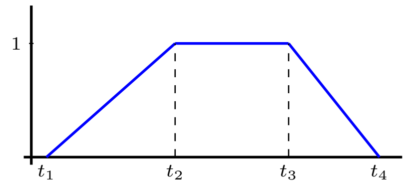

FIGURE 4. Example of a trapezoidal fuzzy number.

The α-cuts of a trapezoidal fuzzy number are given by the interval $[T_L(\alpha), T_U(\alpha)]$, where:

$$[T_L(\alpha), T_U(\alpha)] = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha]. \tag{11}$$

One particular family of trapezoidal fuzzy sets are triangular fuzzy sets. A triangular fuzzy set is denoted by (z_1, z_2, z_3), and it represents the trapezoidal fuzzy set (z_1, z_2, z_2, z_3). This means that the maximum membership degree is only attained in one point: z_2.

4.1. **Imprecise expected utility as a fuzzy ranking.** Given a possibility measure induced by a trapezoidal fuzzy number, we also establish a simple characterization of imprecise expected utility. To see how this comes about, note that if Π is determined by the membership function of a trapezoidal fuzzy number (t_1, t_2, t_3, t_4), then

$$\int id\ dF_{\Pi} = \frac{t_1 + t_2}{2} \quad \text{and} \quad \int id\ dF_N = \frac{t_3 + t_4}{2}.$$

Applying Corollary 1, we deduce the following:

Proposition 5. *Let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) be two trapezoidal fuzzy numbers, and denote by Π_X and Π_Y the possibility measures they determine.*

(1) $\Pi_X \succeq_{E_1} \Pi_Y \Leftrightarrow \frac{x_1+x_2}{2} \geq \frac{y_3+y_4}{2}$.
(2) $\Pi_X \succeq_{E_2} \Pi_Y \Leftrightarrow \frac{x_3+x_4}{2} \geq \frac{y_3+y_4}{2}$.
(3) $\Pi_X \succeq_{E_4} \Pi_Y \Leftrightarrow \frac{x_3+x_4}{2} \geq \frac{y_1+y_2}{2}$.
(4) $\Pi_X \succeq_{E_5} \Pi_Y \Leftrightarrow \frac{x_1+x_2}{2} \geq \frac{y_1+y_2}{2}$.

In Wang and Kerre [2001a,b], a number of desirable properties for fuzzy rankings $\succeq$ are discussed. Here we shall consider the following[1]:

(A0) For any pair of fuzzy numbers A, B, either $A \succeq B$ or $B \succeq A$. [Completeness]
(A1) $A \succeq A$ for any fuzzy number A. [Reflexivity]
(A2) $A \succeq B, B \succeq A \Rightarrow A = B$. [Antisymmetry]
(A3) $A \succeq B$ and $B \succeq C \Rightarrow A \succeq C$. [Transitivity]
(A4) $\inf \operatorname{supp}(A) > \sup \operatorname{supp}(B) \Rightarrow A \succ B$.
(A5) $A \succeq B \Rightarrow A + C \succeq B + C$ for any fuzzy number C.
(A6) $\operatorname{supp}(C) \subseteq [0, +\infty), A \succeq B \Rightarrow AC \succeq BC$.

[1]Wang and Kerre assume that the fuzzy ranking produces a complete order, which is not always the case for our imprecise stochastic orders; this is why we have included (A0) in our discussion. In addition, our axiom (A4),(A5) and (A6) correspond to (A4'),(A6) and (A7) in that paper, their (A5) not being too interesting in our context.

Using Proposition 5, we can establish the following:

Proposition 6. *Let $\succeq_{\mathrm{E_i}}$ denote the extension of expected utility to the imprecise case by means of Definition 6. They satisfy the following properties as fuzzy rankings of trapezoidal fuzzy numbers:*

	(A0)	(A1)	(A2)	(A3)	(A4)	(A5)	(A6)
$\succeq_{\mathrm{E_1}}$			•	•	•		
$\succeq_{\mathrm{E_2}}$	•	•		•	•	•	
$\succeq_{\mathrm{E_4}}$	•	•			•	•	•
$\succeq_{\mathrm{E_5}}$	•	•		•	•	•	

Proof. The properties that are satisfied follow immediately from Proposition 5. Let us give counterexamples for those that do not hold:

(A0,A1) Take $A = (1, 2, 3, 4)$. Then A is incomparable to itself with respect to $\succeq_{\mathrm{E_1}}$, and as a consequence $\succeq_{\mathrm{E_1}}$ does not induce a complete relationship.

(A2) Note that $(1, 2, 3, 4) \sim_{\mathrm{E_i}} (0, 1, 3, 4)$, $i = 2, 4$, and $(1, 2, 3, 4) \sim_{\mathrm{E_5}} (1, 2, 4, 5)$.

(A3) $(0, 1, 1, 2) \succeq_{\mathrm{E_4}} (1, 2, 2, 3) \succeq_{\mathrm{E_4}} (2, 3, 3, 4)$, but $(0, 1, 1, 2) \not\succeq_{\mathrm{E_4}} (2, 3, 3, 4)$.

(A5) Take $A = (2, 3, 3, 4), B = (1, 2, 2, 3)$ and $C = (0, 1, 1, 2)$. Then $A \succeq_{\mathrm{E_1}} B = B + C$. However, $A + C = (2, 4, 4, 6) \not\succeq_{\mathrm{E_1}} (1, 3, 3, 5)$.

(A6) The same A, B, C in the previous item satisfy also $AC = (0, 3, 3, 8) \not\succeq_{\mathrm{E_1}} (0, 2, 2, 6) = BC$.

On the other hand, if we consider $A' = (-2, -1, 1, 2), B' = (-3, 0, 0, 3)$ and $C' = (0, 1, 2, 3)$, it holds that $A' \sim_{\mathrm{E_2,E_5}} B'$ but $A'C' = (-6, -2, 2, 6)$, $B'C' = (-9, 0, 0, 9)$, whence $A'C' \succ_{\mathrm{E_2}} B'C' \succ_{\mathrm{E_5}} A'C'$. □

Unsurprisingly, the antisymmetry property is only satisfied by the most stringent extension ($\succeq_{\mathrm{E_1}}$), while transitivity is not satisfied by the weakest one ($\succeq_{\mathrm{E_4}}$). It is also interesting to remark that the first extension may produce instances of incomparability, unlike the others. Note also that, although all our versions of imprecise expected utility satisfy axiom (A4), this is not the case for all fuzzy rankings: as shown by Wang and Kerre [2001a], it is violated for instance by Kerre's fuzzy ranking [Kerre, 1982].

Although it is an open problem at this stage, we think that the properties in Proposition 6 also hold when we compare arbitrary fuzzy sets by means of imprecise expected utility.

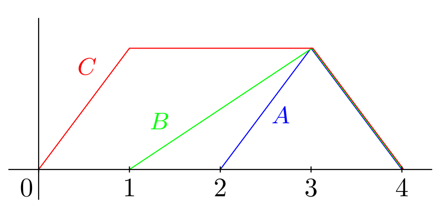

FIGURE 5. Graphical representation of the trapezoidal fuzzy sets.

It is also interesting to discuss the behavior of these orders in the controversial case discussed in [Wang and Kerre, 2001a, Section 1.2], that we depict in Figure 5: we consider the trapezoidal fuzzy sets $A = (2, 3, 3, 4), B = (1, 3, 3, 4)$ and $C =$

$(0,1,3,4)$. We deduce from Proposition 5 that

$$A \succ_{\mathrm{E}_5} B \succ_{\mathrm{E}_5} C, \quad A \equiv_{\mathrm{E}_2} B \equiv_{\mathrm{E}_2} C, \quad A \equiv_{\mathrm{E}_4} B \equiv_{\mathrm{E}_4} C, \tag{12}$$

and that they are incomparable with respect to $\succeq_{\mathrm{E}_1}$. This is because the order $\succeq_{\mathrm{E}_5}$ is looking at the lower limits of the fuzzy sets, for which we can establish a strict order, while $\succeq_{\mathrm{E}_2}$ is looking at the upper limits, where the three fuzzy sets coincide.

4.2. **Imprecise stochastic dominance as a fuzzy ranking.** When the possibility measures to be compared are induced by trapezoidal fuzzy numbers, imprecise stochastic dominance takes a simple expression:

Proposition 7. *Let X and Y be two trapezoidal fuzzy numbers with parameters (x_1,x_2,x_3,x_4) and (y_1,y_2,y_3,y_4), respectively, and let Π_{X} and Π_{Y} be the possibility measures they determine.*

(1) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_1} \Pi_{\mathrm{Y}} \Leftrightarrow y_3 \leq x_1$ *and* $y_4 \leq x_2$.
(2) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_2} \Pi_{\mathrm{Y}} \Leftrightarrow y_3 \leq x_3$ *and* $y_4 \leq x_4$.
(3) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_4} \Pi_{\mathrm{Y}} \Leftrightarrow y_1 \leq x_3$ *and* $y_2 \leq x_4$.
(4) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_5} \Pi_{\mathrm{Y}} \Leftrightarrow y_1 \leq x_1$ *and* $y_2 \leq x_2$.

Proof. From Eq. (10), we can deduce that the lower and upper distributions induced by a trapezoidal fuzzy number with parameters (t_1,t_2,t_3,t_4) are given by:

$$\underline{F}(x) = \begin{cases} 0 & \text{if } x \leq t_3. \\ 1 - \frac{t_4 - x}{t_4 - t_3} & \text{if } t_3 < x \leq t_4. \\ 1 & \text{if } x > t_4. \end{cases} \qquad \overline{F}(x) = \begin{cases} 0 & \text{if } x < t_1. \\ \frac{x - t_1}{t_2 - t_1} & \text{if } t_1 \leq x < t_2. \\ 1 & \text{if } x \geq t_2. \end{cases}$$

Taking these formulas into account, the result follows from Proposition 4. □

We shall use this result to study the properties of imprecise stochastic dominance as a fuzzy ranking. They are summarized in the following proposition:

Proposition 8. *Let $\succeq_{\mathrm{SD_i}}$ denote the extensions of stochastic dominance to the imprecise case by means of Definition 6. They satisfy the following properties as fuzzy rankings of trapezoidal fuzzy numbers:*

	(A0)	(A1)	(A2)	(A3)	(A4)	(A5)	(A6)
$\succeq_{\mathrm{SD}_1}$			•	•	•	•	
$\succeq_{\mathrm{SD}_2}$		•		•	•	•	•
$\succeq_{\mathrm{SD}_4}$	•	•			•	•	•
$\succeq_{\mathrm{SD}_5}$		•		•	•	•	•

Proof. Let us study each of the axioms:

(A0) Given $A=(1,2,3,4)$ and $B=(0,2.5,2.5,4.5)$, it follows from Proposition 7 that A,B are incomparable with respect to $\succeq_{\mathrm{SD_i}}$ for $i=1,2,5$. On the other hand, given $X=(x_1,x_2,x_3,x_4)$ and $Y=(y_1,y_2,y_3,y_4)$, if $y_1 > x_3$ or $y_2 > x_4$ it follows that $x_1 \leq y_3$ and $x_2 \leq y_4$. Thus, $\succeq_{\mathrm{SD}_4}$ induces a complete relationship.

(A1) For $\succeq_{\mathrm{SD_i}}$, $i=2,4,5$, this follows immediately from Proposition 7. To see that it does not hold for $\succeq_{\mathrm{SD}_1}$, use the same counterexample of Proposition 6.

(A2) $A \sim_{\mathrm{SD}_1} B$ implies that $b_3 \leq a_1 \leq a_3 \leq b_1$ and that $b_4 \leq a_2 \leq a_4 \leq b_2$, whence $a_1 = \cdots = a_4 = b_1 = \cdots = b_4$ and in particular $A = B$. To see that (A2) does not hold for the other relationships, use the same examples as in Proposition 6.

(A3) For $\succeq_{\mathrm{SD_i}}$, $i = 1, 2, 5$, this follows immediately from Proposition 7. To see that it does not hold for $\succeq_{\mathrm{SD}_4}$, use the same example as in Proposition 6.

(A4) This is a consequence of Proposition 7.

(A5) Given trapezoidal fuzzy numbers $A = (a_1, a_2, a_3, a_4), B = (b_1, b_2, b_3, b_4)$ and $C = (c_1, c_2, c_3, c_4)$,it holds that $A + C = (a_1 + c_1, a_2 + c_2, a_3 + c_3, a_4 + c_4)$ and $B + C = (b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4)$. From here the result follows by Proposition 7.

(A6) Given trapezoidal fuzzy numbers $A = (a_1, a_2, a_3, a_4), B = (b_1, b_2, b_3, b_4)$ and $C = (c_1, c_2, c_3, c_4)$ with $c_1 \geq 0$, it follows that

$$\begin{aligned} AC = (\,&\min\{a_1c_1, a_1c_4, a_4c_1, a_4c_4\}, \min\{a_2c_2, a_2c_3, a_3c_2, a_3c_3\},\\ &\max\{a_2c_2, a_2c_3, a_3c_2, a_3c_3\}, \max\{a_1c_1, a_1c_4, a_4c_1, a_4c_4\}) \end{aligned}$$

and, similarly

$$\begin{aligned} BC = (\,&\min\{b_1c_1, b_1c_4, b_4c_1, b_4c_4\}, \min\{b_2c_2, b_2c_3, b_3c_2, b_3c_3\},\\ &\max\{b_2c_2, b_2c_3, b_3c_2, b_3c_3\}, \max\{b_1c_1, b_1c_4, b_4c_1, b_4c_4\}). \end{aligned}$$

Using these expressions and Proposition 7, we can show that $\succeq_{\mathrm{SD_i}}$, $i = 2, 4, 5$, satisfy (A6). To see that it is not the case for $\succeq_{\mathrm{SD}_1}$, take for instance $A = (2, 3, 3, 4), B = (1, 2, 2, 3)$ and $C = (0, 1, 1, 2)$; then $A \succeq_{\mathrm{SD}_1} B$ while $AC = (0, 3, 3, 8) \not\succeq_{\mathrm{SD}_1} (0, 2, 2, 4) = BC$. $\square$

It is also interesting to discuss again the behavior of these orders in the case discussed in Figure 5. From Proposition 7, we immediately see that

$$A \succ_{\mathrm{SD}_5} B \succ_{\mathrm{SD}_5} C, \quad A \equiv_{\mathrm{SD}_2} B \equiv_{\mathrm{SD}_2} C, \quad A \equiv_{\mathrm{SD}_4} B \equiv_{\mathrm{SD}_4} C,$$

and that they are incomparable with respect to $\succeq_{\mathrm{SD}_1}$. In fact, taking into account that $\succeq_{\mathrm{SD_i}}$ implies $\succeq_{\mathrm{E_i}}$ for $i = 1, \ldots, 6$ when comparing trapezoidal fuzzy numbers, this also allows us to obtain some of the relationships in Eq. (12).

We conclude this section by giving necessary and sufficient conditions for imprecise stochastic dominance when applied to fuzzy numbers (not necessarily trapezoidal). They shall be useful when considering fuzzy stochastic dominance in Section 5.6.

It can be shown that the comparison of the possibility measures induced by fuzzy numbers by means of imprecise stochastic dominance is related to the comparison of the α-cuts. In order to do this, let us introduce the following orders between real intervals $[a, b]$ and $[a', b']$ Dubois et al. [2000], Fisburn [1985]:

$$[a, b] \lesssim [a', b'] \text{ when } a \leq b'. \tag{13}$$

$$[a, b] \leq [a', b'] \text{ when } a \leq a'. \tag{14}$$

$$[a, b] \leqslant [a', b'] \text{ when } b \leq b'. \tag{15}$$

$$[a, b] \ll [a', b'] \text{ when } b \leq a'. \tag{16}$$

These orders are at the basis of Denoeux's work on stochastic dominance for belief functions [Denoeux, 2009]. We have established the following:

Proposition 9. *Let X, Y be two fuzzy numbers and denote by $\Pi_{\mathrm{X}}, \Pi_{\mathrm{Y}}$ the possibility measures they induce. Then:*

(1) $\pi_{\mathrm{Y}}^{\alpha} \ll \pi_{\mathrm{X}}^{\alpha}$ *for any* $\alpha \in (0,1] \Rightarrow \Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_1} \Pi_{\mathrm{Y}} \Rightarrow \pi_{\mathrm{Y}}^{\alpha} \ll \pi_{\mathrm{X}}^{\alpha}$ *for any* $\alpha \in (0.5,1]$;
(2) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_2} \Pi_{\mathrm{Y}} \Leftrightarrow \pi_{\mathrm{Y}}^{\alpha} \leqslant \pi_{\mathrm{X}}^{\alpha}$ *for any* $\alpha \in (0,1]$;
(3) $\pi_{\mathrm{Y}}^{\alpha} \lesssim \pi_{\mathrm{X}}^{\alpha}$ *for* $\alpha = 1 \Rightarrow \Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_4} \Pi_{\mathrm{Y}}$;
(4) $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_5} \Pi_{\mathrm{Y}} \Leftrightarrow \pi_{\mathrm{Y}}^{\alpha} \leq \pi_{\mathrm{X}}^{\alpha}$ *for any* $\alpha \in (0,1]$.

Proof. It follows from its definition that the α-cut of the fuzzy number determined by the numbers a, b and the maps l, r is given by

$$[\inf\{x : l(x) \geq \alpha\}, \sup\{x : r(x) \geq \alpha\}]$$

for $\alpha \in (0,1]$.

On the other hand, if we consider the possibility measure Π it induces and its corresponding p-box $(\underline{F}, \overline{F})$, the latter is given by:

$$\overline{F}(t) = \begin{cases} 0 & \text{if } t < \omega_1. \\ l(t) & \text{if } t \in [\omega_1, a). \\ 1 & \text{if } t \geq a. \end{cases} \qquad \underline{F}(t) = \begin{cases} 0 & \text{if } t \leq b. \\ 1 - r(t) & \text{if } t \in (b, \omega_2]. \\ 1 & \text{if } t > \omega_2. \end{cases} \tag{17}$$

Now, let π_{x} and π_{Y} be the fuzzy numbers defined by l_{X}, r_{X}, a, b, ω_1, ω_2 and l_{Y}, r_{Y}, a', b', δ_1, δ_2, respectively. We shall apply the characterisation of imprecise stochastic dominance in Proposition 4.

(1) Assume that $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_1} \Pi_{\mathrm{Y}}$, whence $\overline{F}_{\mathrm{X}} \leq \underline{F}_{\mathrm{Y}}$. From Eq. (17), we deduce that:
   - $b' \leq \omega_1$;
   - $\delta_2 \leq a$; and
   - $l_{\mathrm{X}}(t) \leq 1 - r_{\mathrm{Y}}(t)$ for any $t \in [\omega_1, \delta_2]$.

   Now, assume ex-absurdo that there is $\alpha > \frac{1}{2}$ such that $\pi_{\mathrm{Y}}^{\alpha} \not\ll \pi_{\mathrm{X}}^{\alpha}$. This implies that:
   $$\inf\{x : l_{\mathrm{X}}(x) \geq \alpha\} < \sup\{x : r_{\mathrm{Y}}(x) \geq \alpha\}.$$
   Thus, there are x_1, x_2 such that:
   $$x_1 < x_2,\ l_{\mathrm{X}}(x_1) \geq \alpha,\ r_{\mathrm{Y}}(x_2) \geq \alpha.$$
   Furthermore, since $x_1 < x_2$, $l_{\mathrm{X}}(x_2) \geq l_{\mathrm{X}}(x_1) \geq \alpha$, and therefore:
   $$l_{\mathrm{X}}(x_2) + r_{\mathrm{X}}(x_2) \geq 2\alpha > 1,$$
   a contradiction. We conclude that $\pi_{\mathrm{Y}}^{\alpha} \ll \pi_{\mathrm{X}}^{\alpha}$ for any $\alpha \in (0.5, 1]$.

   On the other hand, if $\pi_{\mathrm{Y}}^{\alpha} \ll \pi_{\mathrm{X}}^{\alpha}$ for every $\alpha > 0$, then it must be $\delta_2 \leq \omega_1$, since
   $$\begin{aligned} \omega_1 = \inf\{t : l_{\mathrm{Y}}(t) > 0\} &= \lim_{\alpha \downarrow 0} \inf\{t : l_{\mathrm{Y}}(t) \geq \alpha\} \\ &\geq \lim_{\alpha \downarrow 0} \sup\{t : r_{\mathrm{X}}(t) \geq \alpha\} = \sup\{t : r_{\mathrm{X}}(t) > 0\} = \delta_2, \end{aligned}$$
   where the second and third equalities follow from the right-continuity of l_{Y} and the left-continuity of r_{X}. As a consequence, $r_{\mathrm{Y}}(t) + l_{\mathrm{X}}(t) \leq 1$ for every $t \in [b', a]$, because only one of the terms is non-zero for each t.
(2) Assume now that $\Pi_{\mathrm{X}} \succeq_{\mathrm{SD}_2} \Pi_{\mathrm{Y}}$, whence $\underline{F}_{\mathrm{X}} \leq \underline{F}_{\mathrm{Y}}$. From Eq. (17), we deduce that:
   - $\delta_2 \leq \omega_2$;

- $b' \leq b$, and
- $1 - r_{\text{X}}(t) \leq 1 - r_{\text{Y}}(t)$ for any $t \geq b'$.

Then, $r_{\text{X}}(t) \geq r_{\text{Y}}(t)$ for $t \geq b'$. As a consequence, if $r_{\text{Y}}(t) \geq \alpha$, also $r_{\text{X}}(t) \geq \alpha$, whence

$$\sup\{x : r_{\text{Y}}(x) \geq \alpha\} \leq \sup\{x : r_{\text{X}}(x) \geq \alpha\}.$$

We conclude from Eq. (15) that $\pi_{\text{Y}}^{\alpha} \leqslant \pi_{\text{X}}^{\alpha}$ for $\alpha \in (0, 1]$.

Conversely, assume that $\pi_{\text{Y}}^{\alpha} \leqslant \pi_{\text{X}}^{\alpha}$ for $\alpha \in (0, 1]$. Taking $\alpha = 1$ we obtain that $b' \leq b$. Ex-absurdo, assume that $\omega_2 < \delta_2$. Then:

$$r_{\text{X}}\left(\frac{\omega_2 + \delta_2}{2}\right) = 0 < r_{\text{Y}}\left(\frac{\omega_2 + \delta_2}{2}\right).$$

Taking $\alpha = r_{\text{Y}}\left(\frac{\omega_2+\delta_2}{2}\right)$, it holds that

$$r_{\text{Y}}\left(\frac{\omega_2 + \delta_2}{2}\right) \geq \alpha > r_{\text{X}}\left(\frac{\omega_2 + \delta_2}{2}\right),$$

but this contradicts the hypothesis $\pi_{\text{Y}}^{\alpha} \leqslant \pi_{\text{X}}^{\alpha}$. Finally, it only remains to see that $1 - r_{\text{X}}(t) \leq 1 - r_{\text{Y}}(t)$ for any $t \geq b$. Ex-absurdo, assume that $1 - r_{\text{Y}}(t) < 1 - r_{\text{X}}(t)$, and then $r_{\text{X}}(t) < r_{\text{Y}}(t)$ for a given t. Take $\alpha = r_{\text{Y}}(t)$. Then:

$$r_{\text{Y}}(t) = \alpha > r_{\text{X}}(t),$$

and this implies that $\pi_{\text{X}}^{\alpha} \not\leqslant \pi_{\text{Y}}^{\alpha}$, a contradiction.

(3) Since $[a', b'] = \pi_{\text{Y}}^{1} \lesssim \pi_{\text{X}}^{1} = [a, b]$, it follows from Eq. (13) that $a' \leq b$. Thus, $\underline{F}_{\text{X}} \leq \overline{F}_{\text{Y}}$, and applying (SD4) we conclude that $\Pi_{\text{X}} \succeq_{\text{SD}_4} \Pi_{\text{Y}}$.

(4) Finally, assume that $\Pi_{\text{X}} \succeq_{\text{SD}_5} \Pi_{\text{Y}}$. By (SD5-6), this is equivalent to $\overline{F}_{\text{X}} \leq \overline{F}_{\text{Y}}$. This implies that:
- $\delta_1 \leq \omega_1$;
- $a' \leq a$; and
- $l_{\text{X}}(t) \leq l_{\text{Y}}(t)$ for any $t \leq a'$.

Thus $l_{\text{X}}^{-1]}(\alpha) \geq l_{\text{Y}}^{-1]}(\alpha)$, and therefore $\pi_{\text{Y}}^{\alpha} \leq \pi_{\text{X}}^{\alpha}$ for any $\alpha \in (0, 1]$.

Conversely, assume that $\pi_{\text{Y}}^{\alpha} \leq \pi_{\text{X}}^{\alpha}$ for any $\alpha \in (0, 1]$. By Eq. (14), if we consider $\alpha = 1$, $\pi_{\text{Y}}^{1} = [a', b'] \leq [a, b] = \pi_{\text{X}}^{1}$, then $a' \leq a$. Ex-absurdo, assume that $\omega_1 < \delta_1$. Then $l_{\text{X}}\left(\frac{\omega_1+\delta_1}{2}\right) > 0 = l_{\text{Y}}\left(\frac{\omega_1+\delta_1}{2}\right)$. Taking $\alpha = l_{\text{X}}\left(\frac{\omega_1+\delta_1}{2}\right)$, we conclude that $l_{\text{X}}^{-1]}(\alpha) < l_{\text{Y}}^{-1]}(\alpha)$, but this implies that $\pi_{\text{Y}}^{\alpha} \not\leq \pi_{\text{X}}^{\alpha}$, a contradiction. Finally, it only remains to see if $l_{\text{X}}(t) \leq l_{\text{Y}}(t)$ for any $t \in [\delta_1, a]$. Take $\alpha = l_{\text{Y}}(t)$. Then, we know that:

$$t = l_{\text{Y}}^{-1]}(\alpha) \leq l_{\text{X}}^{-1]}(\alpha) = l_{\text{X}}^{-1]}(l_{\text{Y}}(t)) \Rightarrow l_{\text{X}}(t) \leq l_{\text{Y}}(t).$$

We conclude that $\overline{F}_{\text{X}} \succeq_{\text{SD}} \overline{F}_{\text{Y}}$ and therefore $\Pi_{\text{X}} \succeq_{\text{SD}_5} \Pi_{\text{Y}}$. □

5. Comparison of fuzzy random variables

Next, we shall apply the results in the previous section to the comparison of fuzzy random variables. Through this section we shall assume that the images of the fuzzy random variables are fuzzy numbers[2], and also that they are *uniformly bounded*, meaning that for each fuzzy random variable $\widetilde{X}$ there is a compact interval $[a, b]$ such that the support $\widetilde{X}(\omega)$ is included in $[a, b]$ for every ω.

[2] These are also called fuzzy random variables of the L-R type in [Aiche and Dubois, 2010, Aiche et at., 2013].

We shall consider a number of possibilities, and in particular we are going to discuss how the stochastic orders given by stochastic dominance, statistical preference and expected utility can be generalized to the comparison of fuzzy random variables.

5.1. **Lower/upper probabilities.** The fuzzy set $\mu_{\widetilde{X}}(U)$ defined in Eq. (2) measures how compatible is the random variable U with the unknown random variable that $\widetilde{X}$ is modeling. In a similar manner we can define a fuzzy set on the set of probabilities:

$$\mu'_{\widetilde{X}}(P) = \sup\{\mu_{\widetilde{X}}(U) : P_U = P\}.$$

Following the same interpretation, $\mu'_{\widetilde{X}}(P)$ measures how compatible is P with the probability induced by the unknown random variable that $\widetilde{X}$ is modeling; as such, $\mu'_{\widetilde{X}}$ is a second order possibility over a set of probabilities. Using the notion of *natural extension*, Walley [1997] introduced a method to reduce $\mu'_{\widetilde{X}}$ to a first order model. Next, we shall show how the resulting model can be used for the comparison of fuzzy random variables, by establishing links with well-known criteria from Imprecise Probability Theory.

Walley defines the lower and upper previsions $\underline{P}_\alpha, \overline{P}^\alpha$ on the set $\mathcal{L}(\mathbb{R})$ by

$$\underline{P}_\alpha(f) = \inf\{P(f) : \mu'_{\widetilde{X}}(P) \geq \alpha\} \text{ and } \overline{P}_\alpha(f) = \sup\{P(f) : \mu'_{\widetilde{X}}(P) \geq \alpha\}, \quad (18)$$

and then he derives the (first order) lower and upper previsions by:

$$\underline{P}^W(f) = \int_0^1 \underline{P}_\alpha(f)\ \mathrm{d}\alpha \text{ and } \overline{P}^W(f) = \int_0^1 \overline{P}_\alpha(f)\ \mathrm{d}\alpha. \quad (19)$$

When we consider fuzzy random variables whose images are fuzzy numbers, it holds that $\mu'_{\widetilde{X}}(P) \geq \alpha$ if and only if there exists a random variable U such that $P_U = P$ and $\mu_{\widetilde{X}}(U) \geq \alpha$. On the other hand:

$$\begin{aligned} \mu'_{\widetilde{X}}(P_U) \geq \alpha \quad &\Leftrightarrow \quad \inf\{\widetilde{X}(\omega)(U(\omega)) \geq \alpha\ \forall \omega \in \Omega\} \Leftrightarrow \forall \omega \in \Omega, \widetilde{X}(\omega)(U(\omega)) \geq \alpha \\ &\Leftrightarrow \quad \forall \omega \in \Omega, U(\omega) \in \widetilde{X}_\alpha(\omega) \Leftrightarrow U \in S(\widetilde{X}_\alpha). \end{aligned}$$

Then, $\{P : \mu'_{\widetilde{X}}(P) \geq \alpha\} = \{P_U : U \in S(\widetilde{X}_\alpha)\}$. In other words, the coherent lower previsions $\underline{P}_\alpha$ and $\overline{P}_\alpha$ given by Eq. (18) correspond to the lower and upper expectation functionals of the random sets determined by the α-cuts.

In particular, if we use the notation $\widetilde{X}_\alpha(\omega) = [l_\alpha(\omega), r_\alpha(\omega)]$, we can use a procedure analogous to Eq. (19) and define the lower and upper expectation associated with $\mu'_{\widetilde{X}}$ by:

$$\underline{E}^W = \int_0^1 E(l_\alpha)\ \mathrm{d}\alpha \text{ and } \overline{E}^W = \int_0^1 E(r_\alpha)\ \mathrm{d}\alpha. \quad (20)$$

Using the notions of interval dominance [Zaffalon et al., 2003], minimax [Satia and Lave, 1973], maximax [Gilboa and Schmeidler, 1989] and E-admissibility [Levi, 1980] of Imprecise Probability Theory, we can compare fuzzy random variables in the following manner.

Definition 8. Let $\widetilde{X}$ and $\widetilde{Y}$ be two uniformly bounded fuzzy random variables whose images are fuzzy numbers. Denote by $\underline{E}^W_{\widetilde{X}}, \overline{E}^W_{\widetilde{X}}$ and $\underline{E}^W_{\widetilde{Y}}, \overline{E}^W_{\widetilde{Y}}$ the lower and upper expectations obtained by Eq. (20). We say that $\widetilde{X}$ is preferred to $\widetilde{Y}$ by:

- **Interval dominance** when $\underline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.

- **Maximin** when $\underline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.
- **Maximax** when $\overline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.
- **E-admissibility** when $\overline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.

In this manner, we can compare fuzzy random variables using some of the generalizations of expected utility to the imprecise case. We refer to Troffaes [2007] for a detailed comparison of these notions. As we shall see next, these conditions are particular cases of fuzzy expected utility.

5.2. **Fuzzy expected utility.** The stochastic order given by expected utility considers preferable the random variable with the higher expectation, i.e., $X \succeq_{\mathrm{E}} Y \Leftrightarrow E(X) \geq E(Y)$. When we want to compare fuzzy random variables, their expectations are given by fuzzy sets (see Eq. (5)), and as a consequence we must consider a fuzzy ranking on them.

It follows from Eq. (6) that the α-cuts of the expectation are given by the Aumann integral of the random set $\widetilde{X}_\alpha$. When the images of the fuzzy random variable are fuzzy numbers, it follows that $\widetilde{X}_\alpha$ is a random closed interval for every $\alpha \in (0,1]$. Since by [Castaldo et al., 2004, Theorem 4.1] we have that

$$\min(A)\int \widetilde{X}_\alpha \, dP = \int id \, dP_{*\widetilde{X}_\alpha} \text{ and } \max(A)\int \widetilde{X}_\alpha \, dP = \int id \, dP^*_{\widetilde{X}_\alpha},$$

the linearity of the integral implies that

$$(A)\int \widetilde{X}_\alpha \, dP = \left[\int id \, dP_{*\widetilde{X}_\alpha}, \int id \, dP^*_{\widetilde{X}_\alpha}\right],$$

and as a consequence the expectation $E(\widetilde{X})$ is a fuzzy number.

From the point of view of imprecise probabilities, we can also consider the possibility measures associated with these fuzzy sets, and one of the imprecise stochastic orders we have discussed in Section 3.3. The following example illustrates both these possibilities.

Example 1. Assume we are trying to model two unknown random variables defined in $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ by means of the fuzzy random variables $\widetilde{X}, \widetilde{Y} : [0,1] \to \mathcal{F}([0,1])$ given by:

$$\widetilde{X}(\omega)(r) = \begin{cases} 2r & \text{if } r \leq \frac{1}{2}. \\ 2-2r & \text{if } r > \frac{1}{2}. \end{cases}$$

$$\widetilde{Y}(\omega)(r) = \begin{cases} 0 & \text{if } \omega \in [0,\frac{1}{2}] \text{ and } r \neq 0. \\ 1 & \text{if } \omega \in [0,\frac{1}{2}] \text{ and } r = 0. \\ 1 & \text{if } \omega \in (\frac{1}{2},1] \text{ and } r \geq \frac{1}{2}. \\ 0 & \text{if } \omega \in (\frac{1}{2},1] \text{ and } r < \frac{1}{2}. \end{cases}$$

The graphical representation of $\widetilde{X}(\omega), \widetilde{Y}(\omega)$ are depicted in Figure 6.

Note that the images of these fuzzy random variables are fuzzy numbers. The expectations of these fuzzy random variables, $E(\widetilde{X})$ and $E(\widetilde{Y})$, depicted in Figure 7,

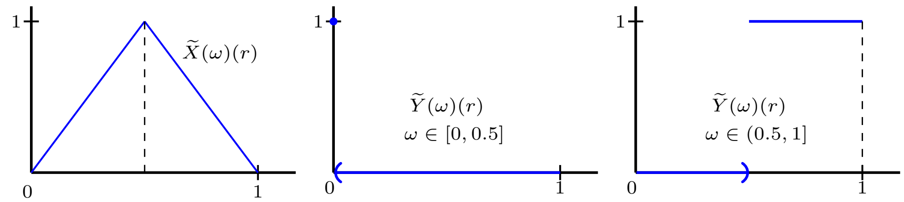

FIGURE 6. Images of $\widetilde{X}(\omega)$ and $\widetilde{Y}(\omega)$.

are the fuzzy sets given by:

$$E(\widetilde{X})(r) = \begin{cases} 2r & \text{if } r \in [0, \frac{1}{2}). \\ 2 - 2r & \text{if } r \in [\frac{1}{2}, 1]. \end{cases}$$
$$E(\widetilde{Y})(r) = \begin{cases} 1 & \text{if } r \in [\frac{1}{4}, \frac{1}{2}]. \\ 0 & \text{otherwise.} \end{cases}$$

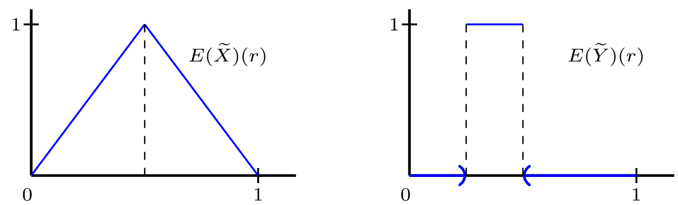

FIGURE 7. Graphical representation of the expectations of $\widetilde{X}$ and $\widetilde{Y}$.

Let us for instance compare these two sets by means of the fuzzy ranking defined by de Campos and González Muñoz [1989], given by

$$A \succeq_{CM} B \iff CM(A) := \int_0^1 \frac{a_\alpha^- + a_\alpha^+}{2} d\alpha \geq \int_0^1 \frac{b_\alpha^- + b_\alpha^+}{2} d\alpha := CM(B), \quad (21)$$

where a_α^-, a_α^+ (resp., b_α^-, b_α^+) denote the infimum and the supremum of the α-cut of A (resp., B). Here we are considering the particular version of the fuzzy ranking where the optimism-pessimism index 0.5. Other fuzzy rankings with a similar idea can be found in Ezzati et al. [2012], Yu and Dat [2014]. We obtain that

$$CM(E(\widetilde{X})) = \frac{1}{2} > \frac{3}{8} = CM(E(\widetilde{Y})),$$

and as a consequence we would conclude that the fuzzy random variable $\widetilde{X}$ is preferred to $\widetilde{Y}$.

On the other hand, we can also interpret these expectations as possibility distributions, and we can thus compare them by means of imprecise expected utility. Denote by Π_{X} and N_{X} the possibility and necessity distributions associated with $E(\widetilde{X})$ and by Π_{Y} and N_{Y} the possibility and necessity distributions associated with $E(\widetilde{Y})$. We have that

$$\int id \, \mathrm{d}F_{\Pi_{\mathrm{X}}} = \frac{1}{4}, \quad \int id \, \mathrm{d}F_{N_{\mathrm{X}}} = \frac{3}{4}, \quad \int id \, \mathrm{d}F_{\Pi_{\mathrm{Y}}} = \frac{1}{4}, \quad \int id \, \mathrm{d}F_{N_{\mathrm{Y}}} = \frac{1}{2}.$$

Applying Corollary 1, we conclude that $\widetilde{X} \succ_{\mathrm{E}_2} \widetilde{Y}$, $\widetilde{X} \equiv_{\mathrm{E}_4,\mathrm{E}_5} \widetilde{Y}$, and that they are incomparable with respect to the first definition. ⧫

This example also helps illustrating how the different versions of imprecise expected utility put the focus on different features of the fuzzy set, and thus end up producing different orders.

Next result establishes a connection between fuzzy expected utility, when the comparison is made by means of the imprecise expected utility, and the comparison of the lower/upper expectation given by Walley's procedure for fuzzy random variables whose images are fuzzy numbers. A somewhat similar result has been established in [Couso and Sánchez, 2011, Section 7] [3].

Theorem 2. *Let $\widetilde{X}$ be a fuzzy random variable with bounded support whose images are fuzzy numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be a bounded random variable. Consider $\underline{P}^W, \overline{P}^W$ given by Eq.* (19)*, let Π the possibility measure associated with $E(\widetilde{X})$, and define*

$$\underline{E}_\Pi(f) := \inf\left\{\int f dP : P \leq \Pi\right\} \quad \textit{and} \quad \overline{E}_\Pi(f) := \sup\left\{\int f dP : P \leq \Pi\right\}$$

Then, it holds that:

$$\underline{P}^W(f) = \underline{E}_\Pi(f) \ \textit{and}\ \overline{E}^W(f) = \overline{E}_\Pi(f).$$

Proof. Let us establish the result for the lower previsions; the proof for the upper previsions follows by duality. Let $[a, b]$ denote the support of $\widetilde{X}$, and define $\mathcal{K} = \{f \in \mathcal{L}([a, b]) \text{ measurable}\}$.

By construction, the coherent lower prevision $\underline{E}_\Pi$ is minimum-preserving on $\mathcal{L}([a, b])$. As a consequence, we deduce from [de Cooman et al., 2008, Theorem 11] that $\underline{E}_\Pi$ is completely monotone. This means in particular [de Cooman et al., 2008, Theorem 15] that it is the Choquet integral with respect to its restriction to events.

On the other hand, since $\widetilde{X}_\alpha$ is a random closed interval for every α, it follows from [Miranda et al., 2010, Theorem 14] that for any $f \in \mathcal{K}$ it holds that $\underline{P}_\alpha(f) = \int f d(P_{\widetilde{X}_\alpha})_*$, where $(P_{\widetilde{X}_\alpha})_*$ denotes the lower probability associated with the random set $\widetilde{X}_\alpha$. As a consequence, the restriction of $\underline{P}_\alpha$ to $\mathcal{K}$ is also completely monotone. From this we can deduce, using the linearity of the integral, that the

[3] Our result is more general because we establish the equality between these two lower previsions on gambles, and not only events, and for this we need to establish the complete monotonicity of these previsions; for clarity, note that in Couso and Sánchez [2011] the functional $\underline{P}_\Pi$ is expressed by means of Dubois and Prade's mean value of a fuzzy number [Dubois and Prade, 1987].

lower prevision $\underline{P}^W$ is also completely monotone: given $f_1, ..., f_p \in \mathcal{K}$, it holds that

$$\begin{aligned}
&\underline{P}^W\left(\bigvee_{i=1}^{p} f_i\right) = \int_0^1 \underline{P}_\alpha\left(\bigvee_{i=1}^{p} f_i\right) d\alpha \\
&\geq \int_0^1 \left(\sum_{i=1}^{p} \underline{P}_\alpha(f_i) - \sum_{i,j} \underline{P}_\alpha(f_i \wedge f_j) + \cdots + (-1)^{p+1} \underline{P}_\alpha\left(\bigwedge_{i=1}^{p} f_i\right)\right) d\alpha \\
&= \sum_{i=1}^{p} \int_0^1 \underline{P}_\alpha(f_i) d\alpha - \sum_{i,j} \int_0^1 \underline{P}_\alpha(f_i \wedge f_j) d\alpha + \cdots + (-1)^{p+1} \int_0^1 \underline{P}_\alpha\left(\bigwedge_{i=1}^{p} f_i\right) d\alpha \\
&= \sum_{i=1}^{p} \underline{P}^W(f_i) - \sum_{i,j} \underline{P}^W(f_i \wedge f_j) + \cdots + (-1)^{p+1} \underline{P}^W\left(\bigwedge_{i=1}^{p} f_i\right),
\end{aligned}$$

where the inequality follows using that $\underline{P}_\alpha$ is completely monotone in $\mathcal{K}$ for every α.

Applying again [de Cooman et al., 2008, Theorem 15], we deduce that for every $f \in \mathcal{K}$ it holds that

$$\underline{P}^W(f) = (C)\int f d\underline{P}^W = \inf f + \int_{-\infty}^{+\infty} \underline{P}^W(f \geq t) dt$$

Now, it has been established in [Couso and Sánchez, 2011, Section 7] that $\underline{P}^W$ and $\underline{P}_\Pi$ coincide on measurable events. As a consequence,

$$\inf f + \int_{-\infty}^{+\infty} \underline{P}^W(f \geq t) dt = \inf f + \int_{-\infty}^{+\infty} \underline{P}_\Pi(f \geq t) dt = (C)\int f d\underline{P}_\Pi = \underline{P}_\Pi(f).$$

Thus, $\underline{P}^W = \underline{P}_\Pi$ on $\mathcal{K}$. □

In particular, we deduce that the result holds when f is the identity function. This allows us to establish the following:

Theorem 3. *Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables with bounded support whose images are fuzzy numbers. Denote by $\underline{E}^W_{\widetilde{X}}, \overline{E}_{\widetilde{X}}, \underline{E}^W_{\widetilde{Y}}, \overline{E}^W_{\widetilde{Y}}$ the lower and upper expectations given by Eq.* (20)*. Then:*

(1) $E(\widetilde{X}) \succeq_{\mathrm{E}_1} E(\widetilde{Y}) \Leftrightarrow \underline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.
(2) $E(\widetilde{X}) \succeq_{\mathrm{E}_2} E(\widetilde{Y}) \Leftrightarrow \underline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.
(3) $E(\widetilde{X}) \succeq_{\mathrm{E}_4} E(\widetilde{Y}) \Leftrightarrow \overline{E}^W_{\widetilde{X}} \geq \underline{E}^W_{\widetilde{Y}}$.
(4) $E(\widetilde{X}) \succeq_{\mathrm{E}_5} E(\widetilde{Y}) \Leftrightarrow \overline{E}^W_{\widetilde{X}} \geq \overline{E}^W_{\widetilde{Y}}$.

Proof. This follows from the previous theorem and Corollary 1, taking into account that $E(\widetilde{X}), E(\widetilde{Y})$ are fuzzy numbers. □

According to Definition 8, the comparison of expectations with respect to imprecise expected utility is also related to the notions of minimax, maximax, interval dominance and E-admissibility.

Example 2. Consider again the fuzzy random variables from Example 1. There, we have seen that $E(\widetilde{X}) \succ_{\mathrm{E}_2} E(\widetilde{Y})$ and $E(\widetilde{X}) \sim_{\mathrm{E_i}} E(\widetilde{Y})$ for $i = 4, 5$, while the are incomparable for $\succeq_{\mathrm{E}_1}$. Using Theorem 3, we conclude that $\widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to the maximin criterion, while they are equivalent with respect to maximax and E-admissibility, and incomparable with respect to interval dominance. ♦

5.3. **Fuzzy stochastic dominance.** Next we consider the extension of stochastic dominance to the imprecise case. Recall that given two random variables X, Y, X is said to stochastically dominate Y when $F_X(t) \leq F_Y(t)$ for every t. When we consider two fuzzy random variables $\widetilde{X}, \widetilde{Y}$, it follows from Eq. (3) that for every real number t, $F_{\widetilde{X}}(t), F_{\widetilde{Y}}(t)$ are fuzzy sets on $[0,1]$. Hence, we should compare them by means of a fuzzy ranking, and this gives rise to the following definition:

Definition 9. Let $\succsim$ be a fuzzy ranking, and consider two fuzzy random variables $\widetilde{X}, \widetilde{Y}$. We say that $\widetilde{X}$ $\succsim$*-stochastically dominates* $\widetilde{Y}$ when $F_{\widetilde{Y}}(t) \succsim F_{\widetilde{X}}(t)$ for every real number t.

As we discussed in Section 2.2, stochastic dominance is quite a strong requirement, and gives rise to many instances of incomparable random variables. Because of this, some weaker versions of stochastic dominance, such as the second, third,...-order stochastic dominance have been proposed [Levy, 1998].

When we consider the extension of stochastic dominance to the fuzzy case in the definition above, we end up with quite a stringent condition when the fuzzy ranking $\succsim$ we consider does not produce a complete order, as is for instance the case with some versions of imprecise stochastic dominance or imprecise expected utility. Because of this, we think it makes more sense to use fuzzy stochastic dominance with respect to a complete fuzzy ranking. The following example illustrates the procedure:

Example 3. Consider again the fuzzy random variables $\widetilde{X}, \widetilde{Y}$ from Example 1. $F_{\widetilde{X}}$ is given in any point ω by the membership function:

$$F_{\widetilde{X}}(\omega)(r) = \begin{cases} 2\omega & \text{if } r \in (0,1], \\ 1 & \text{if } r = 0, \end{cases}$$

when $\omega \in [0, 0.5)$, and by

$$F_{\widetilde{X}}(\omega)(r) = \begin{cases} 2 - 2\omega & \text{if } r \in [0,1), \\ 1 & \text{if } r = 1, \end{cases}$$

when $\omega \in [0.5, 1]$. Similarly, $F_{\widetilde{Y}}$ is given by:

$$F_{\widetilde{Y}}(\omega)(r) = \begin{cases} 1 & \text{if } r = 0.5, \\ 0 & \text{if } r \neq 0.5, \end{cases}$$

for $\omega \in [0, 0.5)$, by:

$$F_{\widetilde{Y}}(\omega)(r) = \begin{cases} 1 & \text{if } r \in [0.5, 1], \\ 0 & \text{if } r \in [0, 0.5), \end{cases}$$

when $\omega \in [0.5, 1)$ and, finally, by:

$$F_{\widetilde{Y}}(\omega)(r) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \in [0, 1), \end{cases}$$

for $\omega = 1$. Figure 8 is a graphical representation of $F_{\widetilde{X}}(\omega)(r)$ and $F_{\widetilde{Y}}(\omega)(r)$.

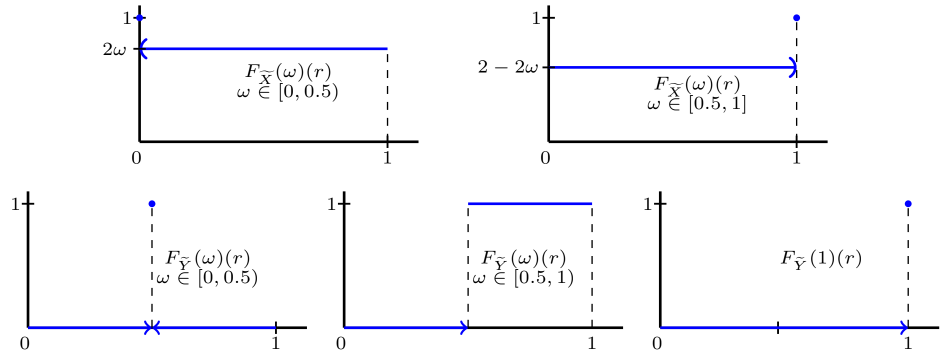

FIGURE 8. Graphical representation of $F_{\widetilde{X}}(\omega)(r)$ and $F_{\widetilde{Y}}(\omega)(r)$.

If we compare them by means of the fuzzy ranking in Eq. (21), we obtain that $CM(F_{\widetilde{X}}(\omega)) = \omega$ for every $\omega \in [0,1]$ while

$$CM(F_{\widetilde{Y}}(\omega)) = \begin{cases} 0.5 & \text{if } \omega \in [0, 0.5) \\ 0.75 & \text{if } \omega \in [0.5, 1) \\ 1 & \text{if } \omega = 1. \end{cases}$$

As a consequence, $\widetilde{X}$ and $\widetilde{Y}$ are incomparable with respect to $\succsim_{CM}$-stochastic dominance. ⧫

Finally, we would like to remark that another extension of stochastic dominance to the comparison of fuzzy random variables has been proposed by Aiche, Abbas and Dubois in [Aiche and Dubois, 2010, 2012, Aiche et at., 2013]. As in our definition above, their idea is to require that, for every real number t, $P(\widetilde{X} \geq t) \geq P(\widetilde{Y} \geq t)$. However, the comparisons $(\widetilde{X} \geq t), (\widetilde{Y} \geq t)$ are made by means of two orders for random intervals μ_1, μ_2, that provide a degree of dominance: then for a fixed threshold $\beta \in [0,1)$ they compute the sets $\{\omega : \mu_1(\widetilde{X}(\omega), t) \geq \beta\}$ and $\{\omega : \mu_2(\widetilde{Y}(\omega), t) \geq \beta\}$, and then compare these by means of the initial probability measure P. This is somewhat related to the ideas in Chanas et al. [1993] and to the indices proposed by Dubois and Prade [1983]. Note that, unlike our approach, that is based on an epistemic interpretation of fuzzy random variables, the aforementioned work makes more sense from the ontic point of view. This can be seen for instance in their allowance for a partial ordering between the intervals.

5.4. **Fuzzy statistical preference.** From Section 2.2, given two random variables X, Y defined on the same probability space $(\Omega, \mathcal{A}, P)$, X is said to be statistically preferred to Y when $P(X \geq Y) \geq P(Y \geq X)$.

When the images of X, Y are fuzzy sets, the comparison between them must be made by means of a fuzzy ranking. In our view, only complete fuzzy rankings make sense in this context. This gives rise to the following definition[4]:

Definition 10. Let $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{F}(\mathbb{R})$ be two fuzzy random variables on a probability space $(\Omega, \mathcal{A}, P)$, and let $\succsim$ be a complete fuzzy ranking. We say that $\widetilde{X}$ is $\succsim$-P

[4]We refer to [Aiche and Dubois, 2012, Section 5] for an alternative definition based on an imprecise ranking between random intervals, that makes more sense under an ontic interpretation.

statistically preferable to $\widetilde{Y}$, and denote it $\widetilde{X} \succsim^P \widetilde{Y}$, if

$$P(\{\omega \in \Omega : \widetilde{X}(\omega) \succsim \widetilde{Y}(\omega)\}) \geq P(\{\omega \in \Omega : \widetilde{Y}(\omega) \succsim \widetilde{X}(\omega)\}).$$

This extension of statistical preference is intermediate between two other notions that in our view are either too strong or too weak, meaning that they lead too often to an assessment of incomparability or indifference:

- $\widetilde{X}$ is called $\succsim$-*strongly statistically preferable* to $\widetilde{Y}$, and we denote it $\widetilde{X} \succsim^s \widetilde{Y}$ if $\widetilde{X}(\omega) \succsim \widetilde{Y}(\omega)$ for any $\omega \in \Omega$.
- $\widetilde{X}$ is called $\succsim$-*weakly statistically preferable* to $\widetilde{Y}$, and we denote it $\widetilde{X} \succsim^w \widetilde{Y}$ if $\widetilde{X}(\omega) \succsim \widetilde{Y}(\omega)$ for some $\omega \in \Omega$.

It is immediate to show that

$$\widetilde{X} \succsim^s \widetilde{Y} \Rightarrow \widetilde{X} \succsim^P \widetilde{Y} \Rightarrow \widetilde{X} \succsim^w \widetilde{Y}$$

for any pair of fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$.

It is also straightforward that, when the fuzzy ranking $\succsim$ agrees with the natural order when restricted to the real numbers (as is virtually the case of almost all fuzzy rankings in the literature), then the notion defined above is indeed an extension of statistical preference: for any pair of random variables X, Y, it holds that

$$X \succsim^P Y \Leftrightarrow X \succeq_{\text{SP}} Y.$$

The main advantage of fuzzy statistical preference is that it allows us to give a degree of preference between the fuzzy random variables $\widetilde{X}, \widetilde{Y}$. In this sense, it is similar to other fuzzy rankings considered in the literature, such as Chen [1985], Kerre [1982]. Another strength of statistical preference is that it has been deemed interesting within decision making with qualitative utilities (and in particular with those in a linguistic scale), and as such it links straightforwardly with fuzzy set theory.

Example 4. Consider again the fuzzy random variables $\widetilde{X}, \widetilde{Y}$ from Example 1, and the fuzzy ranking given by Eq. (21). Then we obtain that:

$$CM(\widetilde{X}(\omega)) = 0.5 > 0 = CM(\widetilde{Y}(\omega)) \ \forall \omega \in [0, 0.5] \text{ and}$$
$$CM(\widetilde{X}(\omega)) = 0.5 < 0.75 = CM(\widetilde{Y}(\omega)) \ \forall \omega \in (0.5, 1],$$

whence $P(\widetilde{X} \succeq_{CM} \widetilde{Y}) = P(\widetilde{Y} \succeq_{CM} \widetilde{X})$ and as a consequence $\widetilde{X}, \widetilde{Y}$ would be indifferent according to statistical preference. ♦

5.5. **Comparison by means of α-cuts.** Finally, let us consider an alternative ranking method for fuzzy random variables, based on the comparison of their α-cuts. Recall that for any fuzzy random variable $\widetilde{X}$ and any $\alpha \in [0, 1]$, its α-cut

$$\widetilde{X}_\alpha : \quad \Omega \quad \longrightarrow \quad \mathcal{P}(\Omega')$$

is a random set. Thus, one possible way of comparing two fuzzy random variables is by considering a ranking on their α-cuts, in the manner we have discussed in Section 3.1. This gives rise to the following definitions:

Definition 11. Consider two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$, and let $\succeq$ be a stochastic order on random sets. We say that:

- $\widetilde{X}$ is $\succeq$-*strongly preferred* to $\widetilde{Y}$, denoted by $\widetilde{X} \succeq^s \widetilde{Y}$, if $\widetilde{X}_\alpha \succeq \widetilde{Y}_\alpha$ for every $\alpha \in [0, 1]$.

- $\widetilde{X}$ is $\succeq$-*weakly preferred* to $\widetilde{Y}$, denoted by $\widetilde{X} \succeq^w \widetilde{Y}$, if $\widetilde{X}_\alpha \succeq \widetilde{Y}_\alpha$ for some $\alpha \in [0,1]$.
- $\widetilde{X}$ is α $\succeq$-*preferred* to $\widetilde{Y}$, denoted by $\widetilde{X} \succeq^\alpha \widetilde{Y}$, if $\widetilde{X}_{\alpha^*} \succeq \widetilde{Y}_{\alpha^*}$ for any $\alpha^* \in [\alpha, 1]$.

The last of the possibilities in the above definition corresponds to an scenario where we fix a threshold α and consider that only the information encompassed by the fuzzy random variable with a certainty level greater than α is of relevance.

It follows immediately from the definition that for any pair of fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ and any stochastic order $\succeq$ on random sets, it holds that:

$$\widetilde{X} \succeq^s \widetilde{Y} \Rightarrow \widetilde{X} \succeq^\alpha \widetilde{Y} \Rightarrow \widetilde{X} \succeq^w \widetilde{Y}.$$

Next we illustrate these notions in the example we have discussed throughout this section, by considering the two fuzzy rankings we have introduced in this paper by means of imprecise stochastic dominance, imprecise expected utility and imprecise statistical preference:

Example 5. Consider again the fuzzy random variables $\widetilde{X}, \widetilde{Y}$ from Example 1. Their α-cuts are given, for any $\alpha \in [0,1]$, by:

$$\widetilde{X}_\alpha(\omega) = \left[\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right].$$
$$\widetilde{Y}_\alpha(\omega) = \begin{cases} \{0\} & \text{if } \omega \in [0, 0.5). \\ [0.5, 1] & \text{if } \omega \in [0.5, 1]. \end{cases}$$

Fix $\alpha \in [0,1]$, and let us compare these two random sets by means of a stochastic order. First of all, if we compare them by means of imprecise stochastic dominance, we must determine their associated p-boxes. These are given by:

$$\underline{F}_{\widetilde{X}_\alpha}(t) = \begin{cases} 0 & \text{if } t \in [0, 1 - \frac{\alpha}{2}). \\ 1 & \text{if } t \in [1 - \frac{\alpha}{2}, 1]. \end{cases}$$
$$\overline{F}_{\widetilde{X}_\alpha}(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{\alpha}{2}). \\ 1 & \text{if } t \in [\frac{\alpha}{2}, 1]. \end{cases}$$
$$\underline{F}_{\widetilde{Y}_\alpha}(t) = \begin{cases} 0.5 & \text{if } t \in [0, 1). \\ 1 & \text{if } t = 1. \end{cases}$$
$$\overline{F}_{\widetilde{Y}_\alpha}(t) = \begin{cases} 0.5 & \text{if } t \in [0, 0.5). \\ 1 & \text{if } t \in [0.5, 1]. \end{cases}$$

Now, if we take into account Eqs. (SD1)– (SD5-6), we conclude that $\widetilde{X}_\alpha \succ_{\text{SD}_4} \widetilde{Y}_\alpha$ and that they are incomparable with respect to the other definitions.

Next, if we use imprecise expected utility, we obtain that:

$$\int id \, d\overline{F}_{\widetilde{X}_\alpha} = \frac{\alpha}{2}, \qquad \int id \, d\underline{F}_{\widetilde{X}_\alpha} = 1 - \frac{\alpha}{2}.$$
$$\int id \, d\overline{F}_{\widetilde{Y}_\alpha} = \frac{1}{4}, \qquad \int id \, d\underline{F}_{\widetilde{Y}_\alpha} = \frac{1}{2}.$$

Applying Proposition 2, we conclude that $\widetilde{X} \equiv^s_{\text{E}_4} \widetilde{Y}$, $\widetilde{X} \equiv^w_{\text{E}_5} \widetilde{Y}$ and $\widetilde{X} \succ^s_{\text{E}_2} \widetilde{Y}$.

Finally, if we use imprecise statistical preference, we need to consider the sets of measurable selections of these α-cuts. They are given by:

$$S(\widetilde{X}_\alpha) = \{U \text{ r.v.} : U(\omega) \in [\tfrac{\alpha}{2}, 1 - \tfrac{\alpha}{2}]\}.$$
$$S(\widetilde{Y}_\alpha) = \{V \text{ r.v.} : V(\omega) = 0 \ \forall \omega \in [0, 0.5), V(\omega) \in [0.5, 1] \ \forall \omega \in [0.5, 1]\}.$$

The maximum and minimum measurable selections are given by:

$$U_1(\omega) = \max_{U \in S(\widetilde{X}_\alpha)} U(\omega) = 1 - \tfrac{\alpha}{2}, \qquad U_2(\omega) = \min_{U \in S(\widetilde{X}_\alpha)} U(\omega) = \tfrac{\alpha}{2}.$$
$$V_1(\omega) = \max_{V \in S(\widetilde{Y}_\alpha)} V(\omega) = I_{(0.5,1]}, \qquad V_2(\omega) = \min_{V \in S(\widetilde{Y}_\alpha)} V(\omega) = \tfrac{1}{2} I_{(0.5,1]}.$$

From this we obtain the following values for $\alpha \in (0,1)$:

$$Q(U_1,V_1) = \tfrac{1}{2}, \qquad Q(U_1,V_2) = 1,$$
$$Q(U_2,V_1) = \tfrac{1}{2}, \qquad Q(U_2,V_2) = \tfrac{1}{2},$$

whence $\widetilde{X}_\alpha \succ_{\mathrm{SP}_1} \widetilde{Y}_\alpha$ for any $\alpha \in (0,1)$. On the other hand, taking into account that $\widetilde{X}_1$ is the constant random variable in $\frac{1}{2}$, we also deduce that $\widetilde{X}_1 \succ_{\mathrm{SP}_5} \widetilde{Y}_1$ and $\widetilde{X}_1 \sim_{\mathrm{SP}_2,\mathrm{SP}_4} \widetilde{Y}_1$. ♦

5.6. **Particular case: trapezoidal fuzzy random variables.** In this section we study the particular case where the images of $\widetilde{X}$ and $\widetilde{Y}$ are trapezoidal fuzzy numbers. We refer to these as *trapezoidal fuzzy random variables.*

The first possibility we have considered in this section is the comparison of fuzzy random variables by means of their expectations. Let $\widetilde{X}$ be a fuzzy random variable such that $\widetilde{X}(\omega)$ is the trapezoidal fuzzy number $(a_1^\omega, a_2^\omega, a_3^\omega, a_4^\omega)$ for any ω. Note that it follows by the definition of fuzzy random variables that the maps $a_1, \dots, a_4 : \Omega \to \mathbb{R}$ are measurable. Moreover, its α-cuts were given in Eq. (11).

Proposition 10. *Consider a fuzzy random variable $\widetilde{X}$ such that $\widetilde{X}(\omega)$ is a trapezoidal fuzzy number $(a_1^\omega, a_2^\omega, a_3^\omega, a_4^\omega)$ for any ω. Consider the functions $f_i : \Omega \to \mathbb{R}$ given by $f_i(\omega) = a_i^\omega$ for $i = 1, \dots, 4$. Then, $E(\widetilde{X}) = (e_1, e_2, e_3, e_4)$ is also a trapezoidal fuzzy number, where $e_i = E(f_i)$ for $i = 1, \dots, 4$.*

Proof. Applying Eq. (6), we know that:

$$\begin{aligned}
E(\widetilde{X})_\alpha &= (A)\int \widetilde{X}_\alpha \mathrm{d}P \\
&= \left\{ \int U \mathrm{d}P : U(\omega) \in [a_1^\omega + \alpha(a_2^\omega - a_1^\omega), a_4^\omega - \alpha(a_4^\omega - a_3^\omega)] \ \forall \omega \right\} \\
&= \left[\int f_1 + \alpha(f_2 - f_1)\mathrm{d}P, \int f_4 - \alpha(f_4 - f_3)\mathrm{d}P \right] \\
&= [E(f_1) + \alpha(E(f_2) - E(f_1)), E(f_4) - \alpha(E(f_4) - E(f_3))] \\
&= [e_1 + \alpha(e_2 - e_1), e_4 - \alpha(e_4 - e_3)].
\end{aligned}$$

From this we deduce that $E(\widetilde{X})$ is a trapezoidal fuzzy number. □

This result extends earlier work for triangular fuzzy random variables by Dengjie [2009] and Loquin and Dubois [2010].

Taking into account Proposition 5 and Theorem 3, we obtain the following simple characterization of the comparison of trapezoidal fuzzy random variables by means of imprecise expected utility. Given $\widetilde{X} = (a_{X_1}, a_{X_2}, a_{X_3}, a_{X_4})$ and $\widetilde{Y} = (a_{Y_1}, a_{Y_2}, a_{Y_3}, a_{Y_4})$ with respective expectations $E(\widetilde{X}) = (e_{X_1}, e_{X_2}, e_{X_3}, e_{X_4})$ and $E(\widetilde{Y}) = (e_{Y_1}, e_{Y_2}, e_{Y_3}, e_{Y_4})$, it holds that:

- $\widetilde{X} \succeq_{\mathrm{E}_1} \widetilde{Y} \Leftrightarrow \underline{E}_{\widetilde{X}}^W = \frac{e_{X_1}+e_{X_2}}{2} \geq \frac{e_{Y_3}+e_{Y_4}}{2} = \overline{E}_{\widetilde{Y}}^W \Leftrightarrow \widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to interval dominance.
- $\widetilde{X} \succeq_{\mathrm{E}_2} \widetilde{Y} \Leftrightarrow \overline{E}_{\widetilde{X}}^W = \frac{e_{X_3}+e_{X_4}}{2} \geq \frac{e_{Y_3}+e_{Y_4}}{2} = \overline{E}_{\widetilde{Y}}^W \Leftrightarrow \widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to maximax.

- $\widetilde{X} \succeq_{\mathrm{E}_4} \widetilde{Y} \Leftrightarrow \overline{E}_{\widetilde{X}} = \frac{e_{X_3}+e_{X_4}}{2} \geq \frac{e_{Y_1}+e_{Y_2}}{2} = \underline{E}^W_{\widetilde{Y}} \Leftrightarrow \widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to E-admissibility.
- $\widetilde{X} \succeq_{\mathrm{E}_5} \widetilde{Y} \Leftrightarrow \underline{E}^W_{\widetilde{X}} = \frac{e_{X_1}+e_{X_2}}{2} \geq \frac{e_{Y_1}+e_{Y_2}}{2} = \underline{E}^W_{\widetilde{Y}} \Leftrightarrow \widetilde{X}$ is preferred to $\widetilde{Y}$ with respect to maximin.

Now, we compare trapezoidal fuzzy random variables by means of fuzzy stochastic dominance. For this, we need the expression of the cumulative distribution function. It turns out that if the initial probability space is non-atomic, the images of this fuzzy distribution function are fuzzy numbers.

Proposition 11. *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space, and $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ a fuzzy random variable whose images are fuzzy numbers.*

(1) For any $t \in \mathbb{R}, F_{\widetilde{X}}(t)$ is a fuzzy number.
(2) If $\widetilde{X}$ is the trapezoidal fuzzy random variable determined by the maps $f_i : \Omega \to \mathbb{R}, i = 1, \ldots, 4$, then for every $t \in \mathbb{R}$ the α-cut of $F_{\widetilde{X}}(t)$ is

$$[F_{\alpha f_3+(1-\alpha)f_4}(t), F_{\alpha f_2+(1-\alpha)f_1}(t)].$$

Proof. (1) Note that this first statement is mentioned without proof in [Couso, 1999, Proposition 5.2.1], so here we add the proof for the aim of completeness. For the first statement it suffices to show that the α-cuts of $F_{\widetilde{X}}(t)$ are closed intervals for all $\alpha \in [0, 1]$.

From the definition, it follows that $\mu_{\widetilde{X}}(U) \geq \alpha$ if and only if $U \in S(\widetilde{X}_\alpha)$, and by Eq. (3) we deduce that the α-cut of $F_{\widetilde{X}}(t)$ is given by

$$\{F_U(t) : U \in S(\widetilde{X}_\alpha)\}.$$

Since the images of $\widetilde{X}$ are fuzzy numbers, it follows that $\widetilde{X}_\alpha$ is a random closed interval, whence the set above is closed [Couso et al., 2002]. Since moreover the non-atomicity of $(\Omega, \mathcal{A}, P)$ guarantees that it is convex [Couso, 1999, Proposition 2.1.6], we deduce that it is a closed interval.

(2) This follows from the first statement, taking also into account that

$$\widetilde{X}_\alpha = [\alpha f_2 + (1-\alpha)f_1, \alpha f_3 + (1-\alpha)f_4]. \qquad \square$$

Note that this does not imply that $F_{\widetilde{X}}(t)$ is a trapezoidal fuzzy number, as we can see from Example 3.

The proposition above, together with Proposition 9, allows us to establish the following result:

Proposition 12. *Let $\widetilde{X}$ and $\widetilde{Y}$ be two trapezoidal fuzzy random variables on a non-atomic probability space such that $\widetilde{X}(\omega) = (a_1^\omega, a_2^\omega, a_3^\omega, a_4^\omega)$ and $\widetilde{Y}(\omega) = (b_1^\omega, b_2^\omega, b_3^\omega, b_4^\omega)$, and consider the functions $f_i(\omega) = a_i^\omega$ and $g_i(\omega) = b_i^\omega$ for $i = 1, \ldots, 4$. Then:*

(1) $\alpha f_3 + (1-\alpha)f_4 \succeq_{\mathrm{SD}} \alpha g_3 + (1-\alpha)g_4 \ \forall\alpha \in (0, 1] \Rightarrow \widetilde{X} \succeq_{\mathrm{SD}_1} \widetilde{Y} \Rightarrow \alpha f_3 + (1-\alpha)f_4 \succeq_{\mathrm{SD}} \alpha g_3 + (1-\alpha)g_4 \ \forall\alpha \in (0.5, 1]$.
(2) $\widetilde{X} \succeq_{\mathrm{SD}_2} \widetilde{Y} \Leftrightarrow \alpha f_2 + (1-\alpha)f_1 \succeq_{\mathrm{SD}} \alpha g_2 + (1-\alpha)g_1 \forall\alpha \in (0, 1]$.
(3) $f_4 \succeq_{\mathrm{SD}} g_1 \Rightarrow \widetilde{X} \succeq_{\mathrm{SD}_4} \widetilde{Y}$.
(4) $\widetilde{X} \succeq_{\mathrm{SD}_5} \widetilde{Y} \Leftrightarrow \alpha f_3 + (1-\alpha)f_4 \succeq_{\mathrm{SD}} \alpha g_3 + (1-\alpha)g_4 \ \forall\alpha \in (0, 1]$.

Another possibility is to compare the images of the trapezoidal fuzzy random variables by means of a fuzzy statistical preference, in the manner we have discussed in Section 5.4. As an example, we consider the fuzzy rankings associated to the comparison indices of Dubois and Prade [1983].

Definition 12. Let A and B be two fuzzy numbers, and define:

- Possibility of Dominance: $PD(A,B) = \sup_{x\geq y}(\min(A(x), B(y)))$.
- Possibility of Strict Dominance: $PSD(A,B) = \sup_x \inf_{y\geq x}(\min(A(x), 1 - B(y)))$.
- Necessity of Dominance: $ND(A,B) = \inf_x \sup_{y\leq x}(\max(1 - A(x), B(y)))$.
- Necessity of Strict Dominance: $NSD(A,B) = 1 - \sup_{x\leq y}(\min(A(x), B(y)))$.

Each of these indices allows to determine a fuzzy ranking: for instance, we shall say $A \succeq_{\text{PD}} B$ when $PD(A,B) \geq PD(B,A)$. When the rankings determined by these indices differ, Dubois and Prade advocate leaving the final ranking in the hands of the decision maker.

In the case of trapezoidal fuzzy numbers, some of these definitions can be simplified:

Lemma 2. *Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. Then:*

$$A \succ_{\text{PD}} B \Leftrightarrow A \succ_{\text{NSD}} B \Leftrightarrow a_2 > b_3$$
$$A \sim_{\text{PD}} B \Leftrightarrow A \sim_{\text{NSD}} B \Leftrightarrow [a_2, a_3] \cap [b_2, b_3] \neq \emptyset.$$

Proof. From the definition, it follows that

$$PD(A,B) = \sup_{x\geq y}(\min(A(x), B(y))) = \begin{cases} 1 & \text{if } b_2 \leq a_3 \\ \alpha < 1 & \text{if } b_2 > a_3, \end{cases}$$

and similarly

$$PD(B,A) = \sup_{x\geq y}(\min(B(x), A(y))) = \begin{cases} 1 & \text{if } a_2 \leq b_3 \\ \beta < 1 & \text{if } a_2 > b_3. \end{cases}$$

Since moreover either $b_2 \leq a_3$ or $a_2 \leq b_3$ (or both), we deduce that $A \succeq_{\text{PD}} B \Leftrightarrow PD(A,B) = 1 \Leftrightarrow b_2 \leq a_3$. As a consequence, $A \sim_{\text{PD}} B \Leftrightarrow [a_2, a_3] \cap [b_2, b_3] \neq \emptyset$ and $A \succ_{\text{PD}} B \Leftrightarrow a_2 > b_3$.

Analogously, we deduce from its definition that

$$NSD(A,B) = 1 - \sup_{x\leq y}(\min(A(x), B(y))) = \begin{cases} 0 & \text{if } a_2 \leq b_3 \\ \alpha > 0 & \text{if } a_2 > b_3 \end{cases}$$

and similarly

$$NSD(B,A) = 1 - \sup_{x\leq y}(\min(B(x), A(y))) = \begin{cases} 0 & \text{if } b_2 \leq a_3 \\ \beta > 0 & \text{if } b_2 > a_3; \end{cases}$$

and again since either $b_2 \leq a_3$ or $a_2 \leq b_3$ (or both), we deduce that $A \succ_{\text{NSD}} B \Leftrightarrow NSD(A,B) > 0 \Leftrightarrow a_2 > b_3$ and $A \sim_{NSD} B \Leftrightarrow [a_2, a_3] \cap [b_2, b_3] \neq \emptyset$. □

Using this lemma, we can simplify Definition 10 for these fuzzy rankings.

Proposition 13. *Given two trapezoidal fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ such that $\widetilde{X}(\omega) = (a_1^\omega, a_2^\omega, a_3^\omega, a_4^\omega)$ and $\widetilde{Y}(\omega) = (b_1^\omega, b_2^\omega, b_3^\omega, b_4^\omega)$ for any $\omega \in \Omega$,*

$$\widetilde{X} \succsim_{\text{PD}}^P \widetilde{Y} \Leftrightarrow \widetilde{X} \succsim_{\text{NSD}}^P \widetilde{Y} \Leftrightarrow P(\{\omega \in \Omega : a_3^\omega \geq b_2^\omega\}) \geq P(\{\omega \in \Omega : b_3^\omega \geq a_2^\omega\}).$$

Proof. From Lemma 2,

$$P(\{\omega \in \Omega : \widetilde{X}(\omega) \succsim_{PD} \widetilde{Y}(\omega)\}) = P(\{\omega \in \Omega : a_3^\omega \geq b_2^\omega\})$$

and

$$P(\{\omega \in \Omega : \widetilde{Y}(\omega) \succsim_{PD} \widetilde{X}(\omega)\}) = P(\{\omega \in \Omega : b_3^\omega \geq a_2^\omega\}),$$

and similarly for NSD. Then, the result trivially follows. □

Finally, and similarly to Section 5.5, we can compare two triangular fuzzy random variables by means of their α-cuts, following the ideas in Section 3.1. In the case of a trapezoidal fuzzy random variable $\widetilde{X}$, its α-cuts are given by:

$$\widetilde{X}_\alpha(\omega) = [a_1^\omega + \alpha(a_2^\omega - a_1^\omega), a_4^\omega - \alpha(a_4^\omega - a_3^\omega)].$$

These are random closed intervals that depend only on the mappings f_1, f_2, f_3, f_4 and on the value of α. If we fix α and compare $\widetilde{X}_\alpha \equiv [f_1 + \alpha(f_2 - f_1), f_4 - \alpha(f_4 - f_3)]$ with $\widetilde{Y}_\alpha \equiv [g_1 + \alpha(g_2 - g_1), g_4 - \alpha(g_4 - g_3)]$ by means of imprecise expected utility, we deduce from Proposition 1 that

(1) $\widetilde{X}_\alpha \succeq_1 \widetilde{Y}_\alpha \Leftrightarrow \alpha E(f_2) + (1-\alpha)E(f_1) \geq \alpha E(g_3) + (1-\alpha)E(g_4)$.
(2) $\widetilde{X}_\alpha \succeq_2 \widetilde{Y}_\alpha \Leftrightarrow \widetilde{X}_\alpha \succeq_3 \widetilde{Y}_\alpha \Leftrightarrow \alpha E(f_3) + (1-\alpha)E(f_4) \geq \alpha E(g_3) + (1-\alpha)E(g_4)$.
(3) $\widetilde{X}_\alpha \succeq_4 \widetilde{Y}_\alpha \Leftrightarrow \alpha E(f_3) + (1-\alpha)E(f_4) \geq \alpha E(g_1) + (1-\alpha)E(g_2)$.
(4) $\widetilde{X}_\alpha \succeq_5 \widetilde{Y}_\alpha \Leftrightarrow \widetilde{X}_\alpha \succeq_6 \widetilde{Y}_\alpha \Leftrightarrow \alpha E(f_1) + (1-\alpha)E(f_2) \geq \alpha E(g_1) + (1-\alpha)E(g_2)$.

6. Decision making application

This section presents an application of the previous definitions to a decision making problem. We use the setting considered in [Merigó et al., 2014, Section 6.2]: a company operating in UK is considering the possibility of expanding to new markets. They consider four alternatives:

A$_1$: Expand to the French market. **A$_3$**: Expand to the Italian market.
A$_2$: Expand to the German market. **A$_4$**: Expand to the Spanish market.

The evaluation of the strategies depends on the economic situation for the next year, which may take four different values:

S$_1$: Bad economic situation. **S$_3$**: Good economic situation.
S$_2$: Regular economic situation. **S$_4$**: Very good economic situation.

The probabilities for each state are estimated as 0.1, 0.3, 0.3 and 0.3, respectively. Then, we can define the probability space $(\Omega, \mathcal{P}(\Omega), P)$, where $\Omega = \{S_1, S_2, S_3, S_4\}$, and model each alternative as a fuzzy random variable taking the following values, which represent the expected benefits:

	S_1	S_2	S_3	S_4
A_1	$(0.2, 0.3, 0.4)$	$(0.6, 0.7, 0.8)$	$(0.2, 0.3, 0.4)$	$(0.5, 0.6, 0.7)$
A_2	$(0.5, 0.5, 0.5)$	$(0.3, 0.4, 0.5)$	$(0.4, 0.5, 0.7)$	$(0.4, 0.5, 0.6)$
A_3	$(0.1, 0.2, 0.4)$	$(0.6, 0.8, 0.9)$	$(0.8, 0.9, 1)$	$(0.7, 0.8, 0.9)$
A_4	$(0.3, 0.4, 0.5)$	$(0.3, 0.4, 0.6)$	$(0.5, 0.5, 0.5)$	$(0.3, 0.4, 0.5)$

Since these alternatives are triangular fuzzy random variables, we can apply the results from Section 5.6.

Let us compare these alternatives by means of the different possibilities we have discussed in the previous section.

6.1. **Comparison by fuzzy expected utility.** A first possibility is to compare these alternatives by means of a fuzzy ranking on their expectations. In the case of the example, we deduce from Proposition 10 that the expectations of $A_1, \dots, A_4$ are the triangular fuzzy numbers given by:

$$E_{A_1} = (0.41, 0.51, 0.61) \qquad E_{A_2} = (0.38, 0.47, 0.59).$$
$$E_{A_3} = (0.64, 0.77, 0.88) \qquad E_{A_4} = (0.36, 0.43, 0.53).$$

If for instance we compare these by means of imprecise expected utility, Proposition 5 allows us to establish the following results:

	A_1	A_2	A_3	A_4
A_1	$\cdot$	$\succ_{\mathrm{E}_{2,5}}$	$-$	$\succ_{\mathrm{E}_{2,5}}$
A_2	$\equiv_{\mathrm{E}_4}$	$\cdot$	$-$	$\succ_{\mathrm{E}_{2,5}}$
A_3	$\succ_{\mathrm{E}_1}$	$\succ_{\mathrm{E}_1}$	$\cdot$	$\succ_{\mathrm{E}_1}$
A_4	$\equiv_{\mathrm{E}_4}$	$\equiv_{\mathrm{E}_4}$	$-$	$\cdot$

A_3 seems to be the most adequate option, because it is preferable to the other alternatives with respect to the first extension of the expected utility (and as a consequence also with respect to any of the other extensions). Taking into account Theorem 3, the same result is obtained when we apply Walley's approach since $E(\widetilde{X})$ and $E(\widetilde{Y})$ are continuous.

We can also consider other fuzzy rankings. Let us for instance take the fuzzy ranking defined by Adamo [1980], which is one of the most interesting according to the study in Wang and Kerre [2001a]. It fixes some value $\alpha \in (0,1)$ and defines the value $AD^\alpha(A) = \sup\{x : \mu_A(x) \geq \alpha\}$. Then the ranking between two fuzzy sets A, B is based on the comparison of the values $AD^\alpha(A), AD^\alpha(B)$. Given $\alpha \in (0,1)$, it follows that

$$AD^\alpha(E_{A_1}) = 0.61 - 0.1\alpha \quad AD^\alpha(E_{A_2}) = 0.59 - 0.12\alpha,$$
$$AD^\alpha(E_{A_3}) = 0.88 - 0.11\alpha \quad AD^\alpha(E_{A_4}) = 0.53 - 0.1\alpha,$$

whence $E_{A_3} \succ_{\mathrm{AD}} E_{A_1} \succ_{\mathrm{AD}} E_{A_2} \succ_{\mathrm{AD}} E_{A_4}$. Note that this order does not depend on the value of α we are considering.

6.2. **Comparison by fuzzy stochastic dominance.** Let us now use fuzzy stochastic dominance to compare the alternatives. For this aim we have to compare $F_{A_i}(t)$ and $F_{A_j}(t)$ for any $t \in [0,1]$, and we shall use imprecise stochastic dominance to do that.

Let us start comparing A_1 and A_2, whose associated fuzzy distribution functions on 0.65 are given by:

$$F_{A_1}(0.65)(p) = \begin{cases} 0.5 & \text{if } p = 0.1, 0.4. \\ 1 & \text{if } p = 0.7. \\ 0 & \text{otherwise.} \end{cases}$$
$$F_{A_2}(0.65)(p) = \begin{cases} 0.125 & \text{if } p = 0.7. \\ 1 & \text{if } p = 1. \\ 0 & \text{otherwise.} \end{cases}$$

If we denote by $(\underline{F}, \overline{F})$ and $(\underline{G}, \overline{G})$ the p-boxes associated with $F_{A_1}(0.65)$ and $F_{A_2}(0.65)$, respectively, it holds that:

$$\underline{F}(t) = \begin{cases} 0 & \text{if } t < 0.7. \\ 1 & \text{if } t \geq 0.7. \end{cases} \qquad \overline{G}(t) = \begin{cases} 0 & \text{if } t < 0.7. \\ 0.125 & \text{if } t \in [0.7, 1). \\ 1 & \text{if } t = 1. \end{cases}$$

Then, $\underline{F}(t) \geq \overline{G}(t)$ for any $t \in [0, 1]$, with strict preference for $t \in [0.7, 1)$. Then, according to Proposition 4, $F_{A_2}(0.65) \succ_{\text{SD}_1} F_{A_1}(0.65)$.

With similar computations, it holds that $F_{A_2}(0.35) \succ_{\text{SD}_1} F_{A_1}(0.35)$, and this allows us to conclude that A_1 and A_2 are incomparable with respect to $\succeq_{\text{SD}_\text{i}}$ for any i.

Following a similar procedure, we can establish the following:

A_1 Vs A_4: They are incomparable under all versions of imprecise stochastic dominance, because

$$F_{A_4}(0.65) \succ_{\text{SD}_1} F_{A_1}(0.65) \text{ and } F_{A_1}(0.35) \succ_{\text{SD}_1} F_{A_4}(0.65).$$

A_2 Vs A_3: They are incomparable under all versions of imprecise stochastic dominance, because

$$F_{A_3}(0.25) \succ_{\text{SD}_1} F_{A_2}(0.25) \text{ and } F_{A_2}(0.8) \succ_{\text{SD}_1} F_{A_3}(0.8).$$

A_3 Vs A_4: They are incomparable under all versions of imprecise stochastic dominance, because

$$F_{A_3}(0.25) \succ_{\text{SD}_1} F_{A_4}(0.25) \text{ and } F_{A_4}(0.8) \succ_{\text{SD}_1} F_{A_3}(0.8).$$

A_1 Vs A_3: It can be proved that $F_{A_1}(t) \succeq_{\text{SD}_4} F_{A_3}(t)$, with strict inequality for some $t \in [0, 1]$, and then $A_3 \succ_{\text{SD}_4} A_1$.

A_2 Vs A_4: It can be proved that $A_2 \equiv_{\text{SD}} A_4$.

Our next table summarizes the obtained relationships:

	A_1	A_2	A_3	A_4
A_1	$\cdot$	$-$	$\succ_{\text{SD}_4}$	$-$
A_2	$-$	$\cdot$	$-$	$\equiv_{\text{SD}_4}$
A_3	$-$	$-$	$\cdot$	$-$
A_4	$-$	$\equiv_{\text{SD}_4}$	$-$	$\cdot$

This illustrates the fact that fuzzy stochastic dominance is usually too stringent to produce meaningful comparisons. This is due to the fact that stochastic dominance is already strong in the case of (precise) random variables, and it gives rise quite often to instances of incomparability. This feature is exacerbated in the imprecise case.

6.3. **Comparison by fuzzy statistical preference.** Our generalization of statistical preference to the comparison of fuzzy random variables consists in considering a fuzzy ranking $\succeq$, and to use it for the comparison of two fuzzy random variables $\widetilde{X}, \widetilde{Y}$, by considering $P(\widetilde{X} \succeq \widetilde{Y})$ and $P(\widetilde{Y} \succeq \widetilde{X})$. In this example, we are going to consider the fuzzy ranking associated with the four indices defined by Dubois and Prade (see Definition 12).

First of all, if we compare them pairwisely by means of PD and NSD, Lemma 2 assures that the two fuzzy rankings reduce to the comparison of the modal points of the triangular fuzzy numbers. The resulting preference degrees are summarized in the following table:

	A_1	A_2	A_3	A_4
A_1	·	0.6	0.1	0.6
A_2	0.4	·	0.1	0.8
A_3	0.9	0.9	·	0.9
A_4	0.4	0.2	0.1	·

With respect to PSD, we obtain the following:

	A_1	A_2	A_3	A_4
A_1	·	0.6	0.1	0.6
A_2	0.4	·	0.1	0.9
A_3	0.9	0.9	·	0.9
A_4	0.4	0.1	0.1	·

However, when using ND the preference degrees change a little bit:

	A_1	A_2	A_3	A_4
A_1	·	0.6	0.1	0.6
A_2	0.4	·	0.1	0.4
A_3	0.9	0.9	·	0.9
A_4	0.4	0.6	0.1	·

To sum up, the four indexes agree in that $A_3 \succ_{\text{SP}} A_1 \succ_{\text{SP}} A_2, A_4$. With respect to these last two alternatives, $A_2 \succ_{\text{SP}} A_4$ with respect to PD, PSD and NSD, while $A_4 \succ_{\text{SP}} A_2$ with respect to ND. In any case, we conclude again that the best alternative is A_3, that is, to invest into the Italian market.

6.4. **Comparison by means of the α-cuts.** Finally, we shall compare the alternatives by considering statistical preference on the α-cuts. For this aim, note that for any $\alpha \in (0, 1]$,

	S_1	S_2
$A_{1,\alpha}$	$[0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$	$[0.6 + 0.1\alpha, 0.8 - 0.1\alpha]$
$A_{2,\alpha}$	$[0.5, 0.5]$	$[0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$
$A_{3,\alpha}$	$[0.1 + 0.1\alpha, 0.4 - 0.2\alpha]$	$[0.6 + 0.2\alpha, 0.9 - 0.1\alpha]$
$A_{4,\alpha}$	$[0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$	$[0.3 + 0.1\alpha, 0.6 - 0.2\alpha]$

and

	S_3	S_4
$A_{1,\alpha}$	$[0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$	$[0.5 + 0.1\alpha, 0.7 - 0.1\alpha]$
$A_{2,\alpha}$	$[0.4 + 0.1\alpha, 0.7 - 0.2\alpha]$	$[0.4 + 0.1\alpha, 0.6 - 0.1\alpha]$
$A_{3,\alpha}$	$[0.8 + 0.1\alpha, 1 - 0.1\alpha]$	$[0.7 + 0.1\alpha, 0.9 - 0.1\alpha]$
$A_{4,\alpha}$	$[0.5, 0.5]$	$[0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$

Denote by $S(A_{i,\alpha})$ the set of measurable selections associated with $A_{i,\alpha}$ for $i = 1, 2, 3, 4$. It holds that:

$$
\begin{aligned}
&P(\min S(A_{3,\alpha}) > \max S(A_{1,\alpha})) \geq P(S_3 \cup S_4) = 0.6 > 0.5.\\
&P(\min S(A_{3,\alpha}) > \max S(A_{2,\alpha})) \geq P(S_2 \cup S_3 \cup S_4) = 0.9 > 0.5.\\
&P(\min S(A_{3,\alpha}) > \max S(A_{4,\alpha})) \geq P(S_2 \cup S_3 \cup S_4) = 0.9 > 0.5.
\end{aligned}
$$

This means that in the worst situation, $S(A_{3,\alpha})$ is preferred to the best alternative in $S(A_{i,\alpha})$, for $i = 1, 2, 4$, whence $A_{3,\alpha} \succeq_{\text{SP}_1} A_{i,\alpha}$ for $i = 1, 2, 4$. Since this holds for any $\alpha \in [0, 1]$, we conclude that $A_3 \succeq^s_{\text{SP}_1} A_i$, for $i = 1, 2, 4$.

Similarly:

$$P(\min S(A_{1,\alpha}) > \max S(A_{4,\alpha})) \geq P(S_2 \cup S_4) = 0.6 > 0.5.$$

Then, we conclude that $A_1 \succeq^s_{\mathrm{SP}_1} A_4$. If we compare A_2 and A_4, we obtain the following:

$$P(\min S(A_{2,\alpha}) > \max S(A_{4,\alpha})) = \begin{cases} P(S_1) = 0.1 & \text{if } \alpha \leq 0.5. \\ P(S_1 \cup S_4) = 0.4 & \text{if } \alpha > 0.5. \end{cases}$$
$$P(\min S(A_{4,\alpha}) > \max S(A_{2,\alpha})) = 0.$$

Then, also $A_2 \succeq^s_{\mathrm{SP}_1} A_4$. Finally, let us compare A_1 and A_2:

$$P(\min S(A_{1,\alpha}) > \max S(A_{2,\alpha})) = \begin{cases} P(S_2) = 0.3 & \text{if } \alpha \leq 0.5 \\ P(S_2 \cup S_4) = 0.6 & \text{if } \alpha > 0.5. \end{cases}$$
$$P(\min S(A_{2,\alpha}) > \max S(A_{1,\alpha})) = P(S_1 \cup S_3) = 0.4.$$

Then, we conclude that $A_1 \succeq^{0.5}_{\mathrm{SP}_1} A_2$, but there is not strong preference. Finally, it can also be proven that $A_1 \succeq^s_{\mathrm{SP}_i} A_2$ for $i = 2, 5$. Next table summarizes the relationships:

	A_1	A_2	A_3	A_4
A_1	$\cdot$	$\succ^{0.5}_{\mathrm{SP}_1}, \succ^{2}_{\mathrm{SP}_2}, \succ^{s}_{\mathrm{SP}_5}$	–	$\succ^s_{\mathrm{SP}_1}$
A_2	–	$\cdot$	–	$\succ^s_{\mathrm{SP}_1}$
A_3	$\succ^s_{\mathrm{SP}_1}$	$\succ^s_{\mathrm{SP}_1}$	$\cdot$	$\succ^s_{\mathrm{SP}_1}$
A_4	–	–	–	$\cdot$

We see then that there is a strong statistical preference between A_3 and A_1 and between A_2 and A_4, and a slightly weaker one between A_1 and A_2.

7. Conclusions

The results in this paper show that a theory of fuzzy decision making under uncertainty can be established through the generalizations of stochastic orders to a fuzzy framework. We have provided a number of extensions of the most important stochastic orders in the literature: expected utility and stochastic dominance. We have also investigated statistical preference, that we find particularly interesting in a context of fuzzy information.

We have proposed several fuzzy stochastic orders in this paper; the choice of one particular order over the others can be made according to a number of criteria: on the one hand, for those based on the imprecise stochastic orders in Definition 6, it should be remarked that the different extensions take into account different underlying criteria, such as robustness or risk aversion; a more thorough discussion of this topic has been made in Montes et al. [2014a,b].

On the other hand, many of the orders, such as fuzzy expected utility or fuzzy statistical preference, require the use of an underlying fuzzy ranking. This second choice is a problem that has been widely discussed, and we refer to Wang and Kerre [2001a] for a critical review. One possibility is to make the choice by means of desirable axiomatic properties of the fuzzy ranking, such as the ones we have investigated in Section 4 for imprecise expected utility and imprecise stochastic dominance. We should also take into account the interpretation of the fuzzy information we have in the particular problem under consideration. Finally, let us remark that some of the new fuzzy rankings we have introduced in this paper do not produce a complete order. We think that this is a feature that may be interesting in contexts where the information is vague or scarce.

Our interpretation of fuzzy random variables in this paper has been an *epistemic* one, as a model for the imprecise knowledge of a random variable; using this interpretation, Walley defined the lower and upper probability models associated with a fuzzy random variable, that can be used to establish a fuzzy ranking. We have proven that this procedure is equivalent to our notion of fuzzy expected utility, where the fuzzy ranking we apply on the expectations is imprecise expected utility.

The more stringent notion of fuzzy stochastic dominance turns out to produce incomparability in quite a few cases, and we have proposed two solutions for this: one is to consider a complete fuzzy ranking when comparing the images of the fuzzy distribution functions; the other to use instead weaker versions of stochastic dominance, such as the stochastic dominance of the n-th order.

As future lines of research, we would like to point out the following: on the one hand, we would like to deepen in the comparison between the different fuzzy stochastic orderings, by studying their behavior in a number of real-life examples; from a more theoretical point of view, it would be interesting to generalize some of the results in this paper to arbitrary fuzzy random variables, and to study the comparisons between n-tuples of fuzzy random variables, instead of just pairs of them. Finally, it would be useful to obtain an axiomatic characterization of some of these orders, in the vein of the existing ones for the precise case.

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