MODELLING EPISTEMIC IRRELEVANCE WITH CHOICE FUNCTIONS

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ABSTRACT. We consider coherent choice functions under the recent axiomatisation proposed by De Bock and de Cooman that guarantees a representation in terms of binary preferences, and we discuss how to define conditioning in this framework. In a multivariate context, we propose a notion of marginalisation, and its inverse operation called weak (cylindrical) extension. We combine this with our definition of conditioning to define a notion of irrelevance, and we obtain the irrelevant natural extension in this framework: the least informative choice function that satisfies a given irrelevance assessment.

Keywords: Choice functions, coherence, sets of desirable gambles, natural extension, conditioning, epistemic irrelevance.

1. INTRODUCTION

Consider two random variables X_1 and X_2, a belief model about X_2, and an assessment that X_1 is irrelevant to X_2—meaning that learning about the value of X_1 does not influence our beliefs about X_2. What is the least informative joint belief model about X_1 and X_2 that satisfies this irrelevance assessment and that marginalises to the given belief model about X_2? This belief model is called the "irrelevant natural extension". Having an expression for the irrelevant natural extension is important for inference purposes, as such extensions appear frequently in the context of credal networks [5, 6, 10].

In the framework of sets of desirable gambles, an expression for the natural extension was established by de Cooman and Miranda [15]. In this paper, we extend their result to choice functions.

Choice functions are related to the fundamental problem in decision theory: how to make a choice from within a set of available (uncertain) options. In their seminal book, von Neumann and Morgenstern [39] provided an axiomatisation of choice based on pairwise comparisons between options, which has since received much attention, for instance by Rubin [27] who generalised this idea and proposed a theory of choice functions based on choices between more than two elements. One of the aspects of Rubin's theory [27] is that, between any pair of options, the subject either prefers one of them or is indifferent between them, so two options are never incomparable. However, for instance when the available information does not allow for a complete comparison of the options, the subject may be undecided between two options without being indifferent between them; this will for instance typically be the case when there is little relevant information available. This is one of the motivations for a theory of imprecise probabilities [1, 40], where incomparability and indifference are distinguished. With this idea, Kadane et al. [19] and Seidenfeld et al. [31] generalised Rubin's axioms to allow for incomparability.

The theory of sets of desirable gambles (and as a consequence also relevant particular cases such as coherent lower previsions, belief functions or possibility measures) is embedded into coherent choice functions.The greater generality of the latter is due to the fact that, when choosing between a set of options, not only are assessments of incomparability allowed, as is the case with sets of desirable gambles, but, unlike with these, the choices we make are not uniquely determined by making pairwise comparisons between the options. In other words, the information of our decision model is not uniquely determined by choices on sets of two alternatives. While this greater generality may be of interest in practice, it is also important that such a theory of coherent choice functions is operational, in the sense that it allows for conservative reasoning.

In this paper, we are going to study if it is possible to determine the implications in a choice model of an assessment of irrelevance and of the notion of coherence. The latter shall be modelled by means of the axiomatisation of De Bock and de Cooman [12], which generalises the theory of Seidenfeld et al. [31] in that it does not have an Archimedean axiom. One of the main advantages of the axiomatisation in De Bock and de Cooman [12] above the earlier work by Van Camp in [33] is that it guarantees a representation in terms of pairwise choice (that is, choices between two *sets* of options), in the sense we shall recall below.

The remainder of this paper is organised as follows: in Section 2, we recall the axiomatisation of coherent choice functions in [12] and the connection with pairwise choice. Next, in Section 3, we introduce our conditioning rule for choice functions, and show how it relates with the existing conditioning rule for sets of desirable gambles. We use this definition to define a notion of irrelevance in Section 4, from which we derive a formula for the irrelevant natural extension. Some additional comments are gathered in Section 5. In order to ease the reading, we have relegated the proofs to an Appendix.

2. Sets of desirable gamble sets & sets of desirable gambles

Consider a finite possibility space $\mathcal{X}$ in which a random variable X takes values. We denote by $\mathcal{L}(\mathcal{X})$ the set of all gambles—real-valued functions—on $\mathcal{X}$, often denoted by $\mathcal{L}$ when it is clear from the context what the possibility space is. As one simple example, we have the *indicator* gambles: for any subset E of $\mathcal{X}$, we use $\mathbb{I}_E$ to denote the indicator of E, which is the gamble that assumes the value 1 on E and 0 elsewhere.

We attach the following interpretation to gambles. $f(X)$ is an uncertain reward: if the actual outcome turns out to be x in $\mathcal{X}$, then the subject's capital changes by $f(x)$. For any two gambles f and g, we write $f \leq g$ when $f(x) \leq g(x)$ for all x in $\mathcal{X}$, and we write $f < g$ when $f \leq g$ and $f \neq g$. We identify a real constant α with the (constant) gamble that maps every element of $\mathcal{X}$ to α. We collect all the non-negative gambles—the gambles f for which $f \geq 0$—in the set $\mathcal{L}(\mathcal{X})_{\geq 0}$ (often denoted by $\mathcal{L}_{\geq 0}$) and the positive ones—the gambles f for which $f > 0$—in $\mathcal{L}(\mathcal{X})_{>0}$ (often denoted by $\mathcal{L}_{>0}$). Similarly, we write $f \nleq g$ when $f(x) > g(x)$ for some x in $\mathcal{X}$, and collect all the gambles f for which $f \nleq 0$ in the set $\mathcal{L}(\mathcal{X})_{\nleq 0}$ (often denoted by $\mathcal{L}_{\nleq 0}$).

We denote by $\mathcal{Q}(\mathcal{L}(\mathcal{X}))$ the set of all finite subsets of $\mathcal{L}(\mathcal{X})$—also denoted by $\mathcal{Q}$ when the set of gambles $\mathcal{L}(\mathcal{X})$ is clear from the context. Elements of $\mathcal{Q}$ are the *gamble sets*. We define two special subsets of $\mathcal{Q}$: the collection $\mathcal{Q}_{\overline{\emptyset}} \coloneqq \mathcal{Q} \setminus \{\emptyset\}$ of non-empty gamble sets, and the collection $\mathcal{Q}_0 \coloneqq \{A \in \mathcal{Q} : 0 \in A\} \subseteq \mathcal{Q}_{\overline{\emptyset}}$ of gamble sets that include the status quo 0.

2.1. Sets of desirable gamble sets.

A subject can state his preferences by specifying his rejected gambles from within every gamble set:

Definition 1 (Rejection function). A *rejection function* R on $\mathcal{L}(\mathcal{X})$ is a map

$$R : \mathcal{Q}_{\overline{\emptyset}}(\mathcal{L}(\mathcal{X})) \to \mathcal{Q}(\mathcal{L}(\mathcal{X})) : A \mapsto R(A)$$

with the property that $R(A) \subseteq A$.

The idea of a rejection function R is that it identifies the set of gambles $R(A)$ that a subject rejects from a given set of options A. As we will see underneath, rejecting a gamble f from a set A means that A contains at least one gamble that is preferred over f.

Equivalent to the notion of a rejection function R is that of a choice function C, which identifies the set $C(A) \coloneqq A \setminus R(A)$ of non-rejected or chosen options from every gamble set A.

Example 1. (running example) Consider the situation where you have a coin with two identical sides of unknown type: either both sides are heads (H) or tails (T). The random variable X that represents the outcome of a coin flip assumes a value in the finite possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$. This assessment is important for inference purposes: for instance, in a sequence of outcomes of successive flips from this coin, observing one of the outcomes immediately fixes all the other outcomes.

Let R be a rejection function that describes this situation. If we identify a gamble f with the array $(f(H), f(T))$, then we see that both $\mathbb{I}_{\{\mathrm{H}\}} = (1,0)$ and $\mathbb{I}_{\{\mathrm{T}\}} = (0,1)$ represent a non-negative uncertain reward, with the possibility of yielding a (strictly) positive reward. Therefore it makes sense to require $0 \in R(\{0, \mathbb{I}_{\{\mathrm{H}\}}\})$ or $0 \in R(\{0, \mathbb{I}_{\{\mathrm{T}\}}\})$: both $\mathbb{I}_{\{\mathrm{H}\}}$ and $\mathbb{I}_{\{\mathrm{T}\}}$ are preferred to the status quo 0. Similarly, since $-\mathbb{I}_{\{\mathrm{H}\}} < 0$ and $-\mathbb{I}_{\{\mathrm{T}\}} < 0$, we obtain $0 \in C(\{0, -\mathbb{I}_{\{\mathrm{H}\}}\})$ and $0 \in C(\{0, -\mathbb{I}_{\{\mathrm{T}\}}\})$, and even $0 \in C(\{0, -\mathbb{I}_{\{\mathrm{H}\}}, -\mathbb{I}_{\{\mathrm{T}\}}\})$.

The contemplations above actually hold for any rejection function R that describes a coherent belief. What is specific to the rejection function R in the current situation, is that at least one of the gambles $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ or $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$, is preferred to 0, for any ε and δ in $\mathbb{R}_{>0}$, where we denote by $\mathbb{R}_{>0}$ the set of (strictly) positive real numbers. Indeed, if both the side of the coin are heads, then the gamble $-\mathbb{I}_{\{\mathrm{T}\}}$ can be considered equivalent to 0, since it will always yield the reward 0, and hence $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$ yields the positive reward δ so it is preferred to 0. If both the sides of the coin are tails, a similar argument shows that the gamble $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ is preferred to 0 as well. Therefore, our rejection function will satisfy $0 \in R(\{0, -\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\})$, for any ε and δ in $\mathbb{R}_{>0}$. Note however that we want our rejection function R to satisfy $0 \in C(\{0, -\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon\})$ and $0 \in C(\{0, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\})$ for all ε and δ in the interval $[0,1)$: the presence of only $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ does not allow us to reject 0, since both sides of the coin may well be tails, and similarly for $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$.

It is also interesting to remark that the rejection function we have in this example is not equivalent to the *vacuous* rejection function, which only rejects a gamble f in A when there is some other gamble g in A such that $f < g$: observe for instance that in our case 0 would be rejected in the gamble set $A = \{0, -\mathbb{I}_{\{\mathrm{H}\}} + 1/2, -\mathbb{I}_{\{\mathrm{T}\}} + 1/2\} = \{0, (-1/2, 1/2), (1/2, -1/2)\}$, while it would be chosen in the case of a vacuous model. ⧫

We focus our attention to the special subclass of *coherent* rejection functions, that describe the beliefs of a rational subject:

Definition 2 (Coherent rejection function)**.** We call a rejection function R *coherent* if for all A, A_1 and A_2 in $\mathcal{Q}_{\overline{\varnothing}}$, all $\{\lambda_{f,g}, \mu_{f,g} : f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f and g in $\mathcal{L}$:

R_0. $R(A) \neq A$;
R_1. if $f < g$ then $f \in R(\{f, g\})$;
R_2. if $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$ then $A_1 \subseteq R(A)$;
R_3. if $0 \in R(A_1)$ and $0 \in R(A_2)$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then

$$0 \in R(\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\});$$

R_4. $f \in R(A)$ if and only if $f + g \in R(A + \{g\})$.

In definition, we let $A_1 + A_2 \coloneqq \{f + g : f \in A_1, g \in A_2\}$ be the Minkowski addition of two gamble sets A_1 and A_2, and we define $(\lambda_1, \ldots, \lambda_n) > 0 \Leftrightarrow ((\forall i \in \{1, \ldots, n\})\lambda_i \geq 0 \text{ and } (\exists i \in \{1, \ldots, n\})\lambda_i > 0)$ for any real $\lambda_1, \ldots, \lambda_n$. In other words, this means that $(\lambda_1, \ldots, \lambda_n) > 0 \Leftrightarrow ((\lambda_1, \ldots, \lambda_n) \geq 0 \text{ and } (\lambda_1, \ldots, \lambda_n) \neq 0)$, where we let '$\geq$' and '$=$' work point-wisely on $(\lambda_1, \ldots, \lambda_n)$. This short-hand notation is used in item R_3 of this definition—and will be used in item K_3 of Definition 4 later on—where $(\lambda_{f,g}, \mu_{f,g}) > 0$ means '$\lambda_{f,g} \geq 0$ and $\mu_{f,g} \geq 0$, with at least one of the real numbers $\lambda_{f,g}$ and $\mu_{f,g}$ strictly positive'.

These rationality requirements were introduced by De Bock and de Cooman [12] as a modification of the ones considered in Van Camp's PhD dissertation [33] in order to guarantee a representation of coherent rejection functions in terms of sets of desirable gambles. In turn, Van Camp's representation is based on—after a necessary translation from horse lotteries to options that are represented by elements of a real linear space, such as gambles—the representation of Seidenfeld et al. [31]. Their work is particularly important because they were the first to introduce *imprecise* choice functions—that distinguish between indifference and incomparability—in [19] and proved a representation result in terms of probabilities in [31].

The rationality requirements of Definition 2 are very similar to those of Seidenfeld et al. [31]. There are, however, some differences: (i) [31] considers a strictly weaker version of Axiom R_1; (ii) they use an additional Archimedean axiom that ensures representation in terms of probabilities rather than non-Archimedean structures such as sets of desirable gambles; and (iii) they impose a mixing axiom that rules out maximality as a decision rule. Note that both the coherent choice functions of Seidenfeld et al. [31] and ours obey Aizerman's condition, commonly written as

$$\text{if } A_1 \subseteq R(A_2) \text{ and } A \subseteq A_1 \text{ then } A_1 \setminus A \subseteq R(A_2 \setminus A),$$

for all A, A_1, A_2 in $\mathcal{Q}$. In our setting this is a consequence of Axioms R_2 and R_3.

De Bock and de Cooman [12] established a useful equivalent representation to rejection functions, namely that of a *set of desirable gamble sets*:

Definition 3 (Set of desirable gamble sets)**.** A set of desirable gamble sets K on $\mathcal{L}$ is a subset of $\mathcal{Q}(\mathcal{L})$. We collect all the sets of desirable gamble sets in $\mathbf{K} \coloneqq \mathcal{P}(\mathcal{Q})$.

The idea is that the set of desirable gamble sets K collects all the gamble sets that contain at least one gamble that our subject strictly prefers over the status quo represented by 0, the gamble that will leave your capital unchanged whatever the outcome. A set of desirable gamble sets K is linked with a rejection function R as follows:

$$(\forall A \in \mathcal{Q})(\forall f \in \mathcal{L}) f \in R(A \cup \{f\}) \Leftrightarrow A - \{f\} \in K. \tag{1}$$

Running example. We continue the previous Example 1 where we want to model the subject's belief that a coin has two identical sides of unknown type.

This means that at least one of the gambles $-\mathbb{I}_{\{H\}} + \varepsilon$ and $-\mathbb{I}_{\{T\}} + \delta$, is preferred to 0, for any ε and δ in $\mathbb{R}_{>0}$, or, in other words, that

$$\mathcal{A} \coloneqq \{\{-\mathbb{I}_{\{H\}} + \varepsilon, -\mathbb{I}_{\{T\}} + \delta\} : \varepsilon, \delta \in \mathbb{R}_{>0}\} \subseteq K. \tag{2}$$

Later on, we will look for the (unique) least informative coherent K for which $\mathcal{A} \subseteq K$, and we take this K to be the set of desirable gamble sets that describes the information available in this setting. In order to do so, we will specify in the rest of this section (i) what is the meaning of coherence in this setting, (ii) what we mean by a set of desirable gamble sets to be less informative than another one, and (iii) how to find such least informative coherent sets of desirable gamble sets. ⧫

De Bock and de Cooman [12] gave an axiomatisation of *coherent* sets of desirable gamble sets—sets of desirable gamble sets of rational subjects: [1]

Definition 4 (Coherent set of desirable gamble sets)**.** A set of desirable gamble sets $K \subseteq \mathcal{Q}$ is called *coherent* if for all A, A_1 and A_2 in $\mathcal{Q}$, all $\{\lambda_{f,g}, \mu_{f,g} : f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f in $\mathcal{L}$.

K$_0$. $\varnothing \notin K$;
K$_1$. $A \in K \Rightarrow A \setminus \{0\} \in K$;
K$_2$. $\{f\} \in K$, for all f in $\mathcal{L}_{>0}$;
K$_3$. if $A_1, A_2 \in K$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then

$$\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\} \in K;$$

K$_4$. if $A_1 \in K$ and $A_1 \subseteq A_2$ then $A_2 \in K$, for all A_1 and A_2 in $\mathcal{Q}$.

We collect all the coherent sets of desirable gamble sets in the collection $\overline{\mathbf{K}}(\mathcal{X})$, often simply denoted by $\overline{\mathbf{K}}$ when it is clear from the context what the possibility space $\mathcal{X}$ is.

To give an idea of some of the consequences of coherence, Axioms K$_0$ and K$_1$ imply that $\{0\}$ is never a desirable gamble set, as expected: it does not contain at least one gamble that is strictly preferred over 0. Axiom K$_2$ implies that $\{\mathbb{I}_E\}$ is a desirable gamble set, for every non-empty $E \subseteq \mathcal{X}$. Axiom K$_3$ lets us infer other desirable gamble sets from any two given desirable gamble sets, by considering positive linear combinations of gambles in them: the idea is that, if there is at least one desirable gamble f in A_1 and at least one desirable gamble g in A_2, then the gamble $\lambda f + \mu g$ must be desirable, guaranteeing that the set $\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\}$ is indeed a set of desirable gamble set. Axiom K$_4$ requires that supersets of desirable gamble sets are desirable gamble sets themselves.

Running example. If we return to our running example, we see for instance that both $\{(-1/2, 1/2), (1/2, -1/2)\}$ and $\{(1,0)\}$ belong to K. For notational convenience, we let $f_1 \coloneqq (-1/2, 1/2)$, $f_2 \coloneqq (1/2, -1/2)$ and $g \coloneqq (1,0)$, so that $\{f_1, f_2\}$ and $\{g\}$ belong to K. Applying Axiom K$_3$ with $(\lambda_{f_1,g}, \mu_{f_1,g}) = (1, 1/5) > 0$ and $(\lambda_{f_2,g}, \mu_{f_2,g}) = (1/2, 1/2) > 0$ yields that also $\{\lambda_{f_1,g} f_1 + \mu_{f_1,g} g, \lambda_{f_2,g} f_2 + \mu_{f_2,g} g\} = \{(-3/10, 1/2), (3/4, -1/4)\}$ belongs to K. If we apply now Axiom K$_4$ we deduce that also $\{(-3/10, 1/2), (3/4, -1/4), (-1, -2), (1, -2)\}$ belongs to K. ⧫

Given any rejection function R and any set of desirable gamble sets K that are linked through Equation (1), we have that R is coherent if and only if K is.

Given two sets of desirable gamble sets K_1 and K_2, we follow De Bock & de Cooman [12] in calling K_1 *at most as informative as* K_2 if $K_1 \subseteq K_2$. The resulting partially ordered set $(\mathbf{K}, \subseteq)$ is a complete lattice where intersection serves the role of infimum, and union that of supremum. Furthermore De Bock and de Cooman [12, Theorem 8] show that the partially ordered set $(\overline{\mathbf{K}}, \subseteq)$ of coherent sets of desirable gamble sets is a complete meet-semilattice: given an arbitrary family $\{K_i : i \in I\} \subseteq \overline{\mathbf{K}}$, its infimum $\inf\{K_i : i \in I\} = \bigcap_{i \in I} K_i$ is a coherent set of desirable gamble sets. There therefore is a unique smallest coherent set of desirable gamble sets, which we call the *vacuous* set of desirable gamble sets $K_{\mathrm{v}} \coloneqq \inf \overline{\mathbf{K}}$.

Lemma 1. *The vacuous set of desirable gamble sets* K_{v} *is given by* $\{A : A \in \mathcal{Q} \cap \mathcal{L}_{>0} \neq \varnothing\}$.

Running example. Returning to our running example, we see that our set of desirable gamble sets K is strictly more informative than the vacuous model K_{v}: for instance the gamble set $\{-\mathbb{I}_{\{\mathrm{H}\}} + 1/2, -\mathbb{I}_{\{\mathrm{T}\}} + 1/2\} = \{(-1/2, 1/2), (1/2, -1/2)\}$ belongs to K but has nothing in common with the positive gambles $\mathcal{L}_{>0}$, so it does not belong to K_{v}. ⧫

[1] We refer to their article for a justification of their axioms.

The fact that $(\overline{\mathbf{K}},\subseteq)$ is a complete meet-semilattice allows for conservative reasoning: it makes it possible to extend a partially specified set of desirable gamble sets to the most conservative—least informative—coherent one that includes it. This procedure is called *natural extension*:

Definition 5 ([12, Definition 9])**.** For any assessment $\mathcal{A}\subseteq\mathcal{Q}$, we let $\overline{\mathbf{K}}(\mathcal{A}) \coloneqq \{K\in\overline{\mathbf{K}}\colon \mathcal{A}\subseteq K\}$. We call the assessment $\mathcal{A}$ *consistent* if $\overline{\mathbf{K}}(\mathcal{A})\neq\emptyset$, and we then call $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A})\coloneqq\bigcap\overline{\mathbf{K}}(\mathcal{A})$ the *natural extension* of $\mathcal{A}$.

One of the main results of De Bock and de Cooman [12] is their expression for the natural extension:

Theorem 2 ([12, Theorem 10])**.** *Consider any assessment $\mathcal{A}\subseteq\mathcal{Q}$. Then $\mathcal{A}$ is consistent if and only if $\emptyset\notin\mathcal{A}$ and $\{0\}\notin\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})$. Moreover, if $\mathcal{A}$ is consistent, then* $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A}))$.

Here we used the set $\mathcal{L}^{\mathrm{s}}(\mathcal{X})_{>0}\coloneqq\{\{f\}\colon f\in\mathcal{L}(\mathcal{X})_{>0}\}$—often denoted simply by $\mathcal{L}^{\mathrm{s}}_{>0}$ when it is clear from the context what the possibility space $\mathcal{X}$ is—and the following two operations on $\mathbf{K}$ defined by

$$\mathrm{Rs}(K)\coloneqq\{A\in\mathcal{Q}\colon(\exists B\in K)B\setminus\mathcal{L}_{\leq 0}\subseteq A\} \tag{3}$$

$$\mathrm{Posi}(K)\coloneqq\left\{\left\{\sum_{k=1}^{n}\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\bigtimes_{k=1}^{n}A_k\right\}\colon n\in\mathbb{N},A_1,\ldots,A_n\in K,\left(\forall f_{1:n}\in\bigtimes_{k=1}^{n}A_k\right)\lambda_{1:n}^{f_{1:n}}>0\right\} \tag{4}$$

for all K in $\mathbf{K}$. As usual, we use the short-hand notation $f_{1:n}\coloneqq(f_1,\ldots,f_n)$ for any sequence $(f_1,\ldots,f_n)$.

The rationale behind these two operations is the following: if A contains at least one gamble that is preferred to the zero gamble, then so does the gamble set $A\setminus\mathcal{L}_{\leq 0}$, since any gamble in $A\cap\mathcal{L}_{\leq 0}$ can never be preferred to the zero gamble. Taking also into account Axiom K_4, we deduce that any superset of $A\setminus\mathcal{L}_{\leq 0}$ should also allow to reject the zero gamble. Therefore, a coherent set of desirable gamble sets should be closed under Rs.

That any coherent set of desirable gamble sets should be closed under Posi follows from a finite number of applications of the coherence Axiom K_3.

For arbitrary sets of desirable gamble sets K, we have $K\subseteq\mathrm{Rs}(K)$, since $B\setminus\mathcal{L}_{\leq 0}\subseteq B$ for every B in K. Also, $K\subseteq\mathrm{Posi}(K)$: to see this, it suffices to choose $n\coloneqq 1$, $A_1\coloneqq A\in K$, and $\lambda_{1:1}^{f_{1:1}}\coloneqq 1$ for all $f_{1:1}$ in $\times_{k=1}^{1}A_1=A$ in the definition of the Posi operator. Therefore $K\subseteq\mathrm{Rs}(\mathrm{Posi}(K))$. For coherent sets of desirable gamble sets K however, Theorem 2 implies that $K=\mathrm{Rs}(K)=\mathrm{Posi}(K)=\mathrm{Rs}(\mathrm{Posi}(K))=\mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup K))$. It is also clear from the definitions above that for any sets of desirable gamble sets $K_1\subseteq K_2$, we have $\mathrm{Rs}(K_1)\subseteq\mathrm{Rs}(K_2)$ and $\mathrm{Posi}(K_1)\subseteq\mathrm{Posi}(K_2)$.

In our earlier work [38, Theorem 1], we have found expressions for the characterisation of consistency and the natural extension of rejection functions. Our results in that paper were obtained in a slightly more general setting: instead of requiring Axiom R_3, we required two strictly weaker axioms. For any given assessment $\mathcal{A}$, the resulting natural extension is therefore a less informative—more conservative—rejection function that the one determined by $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A})$. However, this setting was too general to obtain a representation in terms of binary preferences, as our counterexample in [38, Section 5.1] shows. As proved by De Bock and de Cooman [12, Theorem 7], the current axiomatisation does guarantee representation in terms of sets of desirable gambles.

In order to illustrate Theorem 2, consider the following example, which we will also use in Section 2.3 as an example of a non-binary set of desirable gamble sets.

Running example. We revisit our previous running example, and look for the smallest coherent set of desirable gamble sets K that includes $\mathcal{A}$ defined in Equation (2)—the natural extension of $\mathcal{A}$.

Because it will help us show that this assessment $\mathcal{A}$ is consistent, we first find an alternative expression for $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$:

Lemma 3. *For the assessment $\mathcal{A}$ of Equation* (2)*, we have*

$$\begin{aligned}\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}) &= \{A \in \mathcal{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{T}) > 0 \textit{ and } h_2(\mathrm{H}) > 0)) \textit{ or } A \cap \mathcal{L}^{\mathrm{s}}_{>0} \neq \varnothing\} \\ &= \mathrm{Rs}(\{\{h_1, h_2\} \colon h_1, h_2 \in \mathcal{L}_{\nleq 0} \textit{ and } (h_1(\mathrm{T}), h_2(\mathrm{H})) > 0\}).\end{aligned}$$

To prove that $\mathcal{A}$ is a consistent assessment, by Theorem 2 we need to show that $\varnothing \notin \mathcal{A}$ and $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. By its definition, $\varnothing \notin \mathcal{A}$, so we focus on showing that $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. Using Lemma 3 we know that $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$ consists of the supersets of gamble sets $\{h_1, h_2\}$ where none of h_1 and h_2 are equal to 0, so we find immediately that indeed $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. Therefore $\mathcal{A}$ is a consistent assessment, and by Theorem 2 its natural extension is given by the coherent set of desirable gamble sets $\mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$. What is this $\mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$? Thanks to Lemma 3, this can be easily found: since $\mathrm{Rs}(\mathrm{Rs}(K)) = \mathrm{Rs}(K)$ for any gamble set K, we immediately find that the natural extension of $\mathcal{A}$ is $\mathrm{Rs}(\{\{h_1, h_2\} \colon h_1, h_2 \in \mathcal{L}_{\nleq 0} \text{ and } (h_1(\mathrm{T}), h_2(\mathrm{H})) > 0\})$. Thus $K = \{A \in \mathcal{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H}) > 0)) \text{ or } A \cap \mathcal{L}^{\mathrm{s}}_{>0} \neq \varnothing\}$ is the smallest coherent set of desirable gamble sets that corresponds to our belief that the coin has two identical sides of unknown type, and we therefore take this as our model in this running example. ⧫

2.2. **Sets of desirable gambles.** Since we have taken to mean "$A \in K$" that A contains at least one gamble that is desirable, the singleton elements of K play an important role: if $\{f\} \in K$, then the gamble f is desirable. Given a set of desirable gamble sets, we collect in

$$D_K \coloneqq \{f \in \mathcal{L} \colon \{f\} \in K\} \tag{5}$$

the gambles that are considered desirable, and call it the *set of desirable gambles based on K*.

Sets of desirable gambles can therefore be seen as special sets of desirable gamble sets. In the recent years, there has been much interest in sets of desirable gambles on its own, without reference to sets of desirable gamble sets or choice functions (see for instance [1, Chapter 1] or [4, 25, 30]). One reason for this is that they include as particular cases coherent lower previsions, and therefore most models of non-additive measures (belief functions, possibility measures, etc) while at the same time avoiding some of the issues that arise when conditioning on events of (lower) probability zero. A set of desirable gambles D is simply a subset of $\mathcal{L}$; we collect in $\mathbf{D} \coloneqq \mathcal{P}(\mathcal{L})$ all the sets of desirable gambles. We focus on the special subclass of *coherent* sets of desirable gambles:

Definition 6 (Coherent set of desirable gambles)**.** A set of desirable gambles D is called *coherent* if for all f and g in $\mathcal{L}$, and λ and μ in $\mathbb{R}$:

D_1. $0 \notin D$;
D_2. $\mathcal{L}_{>0} \subseteq D$;
D_3. if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.

We collect all the coherent sets of desirable gambles in $\overline{\mathbf{D}}(\mathcal{X})$, often simply denoted by $\overline{\mathbf{D}}$ when it is clear from the context what the possibility space $\mathcal{X}$ is.

Just as we did for sets of desirable gamble sets, we call the set of desirable gambles D_1 *at most as informative as* set of desirable gambles D_2 if $D_1 \subseteq D_2$. The partially ordered set

$(\overline{\mathbf{D}}, \subseteq)$ is a complete meet-semilattice. The natural extension is defined in a similar way as for sets of desirable gamble sets: an assessment $A \subseteq \mathcal{L}$ is called *consistent* if $\overline{\mathbf{D}}(A) \coloneqq \{D \in \overline{\mathbf{D}} : A \subseteq D\}$ is non-empty. If this is the case, $\mathrm{cl}_{\overline{\mathbf{D}}}(A) \coloneqq \bigcap \overline{\mathbf{D}}(A)$ is called the *natural extension* of A. The expression for the natural extension is remarkably similar to the one in Theorem 2:

Theorem 4 ([16, Theorem 1])**.** *Consider any assessment $A \subseteq \mathcal{L}$. Then A is consistent if and only if $0 \notin \mathrm{posi}(\mathcal{L}_{>0} \cup A)$. Moreover, in that case $\mathrm{cl}_{\overline{\mathbf{D}}}(A) = \mathrm{posi}(\mathcal{L}_{>0} \cup A)$.*

In this theorem, we used the operation posi on $\mathbf{D}$:

$$\mathrm{posi}(A) \coloneqq \Big\{ \sum_{k=1}^{n} \lambda_k f_k : n \in \mathbb{N}, f_1, \ldots, f_n \in A, \lambda_{1:n} > 0 \Big\},$$

for all $A \subseteq \mathcal{L}$.

2.3. Connection between sets of desirable gamble sets and sets of desirable gambles.

Given a set of desirable gamble sets K, its corresponding set of desirable gambles D_K is uniquely given by Equation (5), and it is coherent if K is [12, Proposition 6]. On the other hand, a coherent set of desirable gambles D may have multiple sets of desirable gamble sets corresponding to it by Equation (5), in the sense that the collection

$$\overline{\mathbf{K}}_D \coloneqq \{K \in \overline{\mathbf{K}} : D_K = D\}$$

may have more than one element. However, there is always a unique least informative one:

Proposition 5. *Given a coherent set of desirable gambles D, the infimum $\inf \overline{\mathbf{K}}_D$ of its compatible coherent sets of desirable gamble sets is*

$$K_D \coloneqq \{A \in \mathcal{Q} : A \cap D \neq \emptyset\}. \tag{6}$$

The coherent sets of desirable gamble sets of the form K_D with $D \in \overline{\mathbf{D}}$, are particularly important. Since they are completely determined by pairwise comparison (of gambles in D with 0), they are called *binary*. De Bock and de Cooman [12] established a representation result of coherent sets of desirable gamble sets, in terms of binary ones:

Theorem 6 ([12, Theorem 7])**.** *Every coherent set of desirable gamble sets K is dominated by at least one binary set of desirable gamble sets: the set*

$$\overline{\mathbf{D}}(K) \coloneqq \{D \in \overline{\mathbf{D}} : K \subseteq K_D\} \tag{7}$$

is non-empty. Moreover, $K = \bigcap \{K_D : D \in \overline{\mathbf{D}}(K)\}$.

This theorem generalises the important representation result of Seidenfeld et al. [31, Theorem 4] to a non-Archimedean setting, where the atoms that fulfil the representation are now coherent sets of desirable gambles, rather than (Archimedean) probability mass functions. In order to obtain their result, Seidenfeld et al. [31] needed two additional axioms: an Archimedean one, guaranteeing an appropriate level of continuity, and a mixing axiom, which renders Walley–Sen maximality [2] incompatible with coherent choice functions. De Bock and de Cooman [12] let go of these two axioms, and were able to prove the general representation Theorem 6. Additionally, they also considered the effect of adding the mixing axiom of Seidenfeld et al. [19, 31], while still abstaining from Archimedeanity. With this additional axiom, they have established a more specialised representation in terms of *lexicographic* sets of desirable gambles, which are exactly [3, 37] the sets of desirable gambles that correspond to lexicographic probability systems (with no non-trivial Savage-null events)[3]. For a similar study using Van Camp's axiomatisation, we refer to [37].

[2] See [40, Section 3.9] and [32, Section 2.1] for an introduction to this notion.

[3] See [28, Section 2.7] for a definition of Savage-null events.

Running example. We continue with our running example. We derived the coherent set of desirable gamble sets $K \coloneqq \mathrm{Rs}(\{\{h_1,h_2\}: h_1,h_2 \in \mathcal{L}_{\nleq 0} \text{ and } (h_1(\mathrm{T}),h_2(\mathrm{H})) > 0\})$ that describes the subject's belief that the coin has two identical sides of unknown type. In this example we wonder whether this can be retrieved using binary comparisons: is K a binary set of desirable gamble sets? If K was a binary set of desirable gamble sets, then $K = K_D$ for the set of desirable gambles $D \coloneqq D_K$, as is shown in [12, Proposition 5].

So let us first find D_K. Using Equation (5), we find that

$$\begin{aligned}D_K = \{f \in \mathcal{L}: \{f\} \in K\} &= \{f \in \mathcal{L}: (\exists h_1,h_2 \in \{f\}) h_1,h_2 \in \mathcal{L}_{\nleq 0} \text{ and } (h_1(\mathrm{H}),h_2(\mathrm{T})) > 0\}\\ &= \{f \in \mathcal{L}: f \in \mathcal{L}_{\nleq 0}, f > 0\} = \mathcal{L}_{>0},\end{aligned}$$

so $D_K = \mathcal{L}_{>0}$ is the least informative coherent set of desirable gambles, also called the *vacuous* set of desirable gambles: using pairwise comparisons only, we cannot distinguish the current situation with a vacuous belief. This shows why sets of desirable gambles cannot model this in a satisfactory way. Since $K_{D_K} = \{A \in \mathcal{Q}: A \cap \mathcal{L}_{>0} \neq \emptyset\}$ does not contain the gamble set $\{-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}, -\mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{4}\}$ while K does, this also shows that K is a non-binary set of desirable gamble sets.

How can we represent this K? In other words, what is the representing set $\overline{\mathbf{D}}(K)$ of desirable gambles from Theorem 6? To find this set, consider first the two special coherent sets of desirable gambles

$$\begin{aligned}D_{\mathrm{H}} &\coloneqq \{f \in \mathcal{L}: f(\mathrm{H}) > 0\} \cup \mathcal{L}_{>0}\\ D_{\mathrm{T}} &\coloneqq \{f \in \mathcal{L}: f(\mathrm{T}) > 0\} \cup \mathcal{L}_{>0}\end{aligned}$$

which correspond to (practical) certainty about H and T, respectively. Indeed, if the subject is certain about H, then any gamble that yields a positive gain when H occurs, however small, will be desirable. Actually, in very much the same way as we did earlier, D_{H} and D_{T} can be retrieved as the natural extensions of the consistent assessment $A_{\mathrm{H}} \coloneqq \{-\mathbb{I}_{\{\mathrm{T}\}} + \varepsilon: \varepsilon \in \mathbb{R}_{>0}\}$ and $A_{\mathrm{T}} \coloneqq \{-\mathbb{I}_{\{\mathrm{H}\}} + \delta: \delta \in \mathbb{R}_{>0}\}$, respectively.

To find $\overline{\mathbf{D}}(K)$, we need to find all the coherent sets of desirable gambles D such that $K \subseteq K_D$. So consider any A in K. This implies that there is a subset $\{h_1,h_2\} \subseteq A$ such that $h_1,h_2 \in \mathcal{L}_{\nleq 0}$ and $(h_1(\mathrm{T}),h_2(\mathrm{H})) > 0$. Then $A \cap D_{\mathrm{H}} \neq \emptyset$ and $A \cap D_{\mathrm{T}} \neq \emptyset$, so $A \in K_{D_{\mathrm{H}}}$ and $A \in K_{D_{\mathrm{T}}}$. Therefore $D_{\mathrm{H}}, D_{\mathrm{T}} \in \overline{\mathbf{D}}(K)$. But D_{H} and D_{T} are the only elements of $\overline{\mathbf{D}}(K)$: to see this, assume *ex absurdo* that another coherent set of desirable gambles D belongs to $\overline{\mathbf{D}}(K)$, so $K \subseteq K_D$. This would imply that neither $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ nor $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$ belongs to D, for some ε and δ in $\mathbb{R}_{>0}$. But the gamble set $\{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\}$ belongs to K, a contradiction. So we find by Theorem 6 that $K = K_{D_{\mathrm{H}}} \cap K_{D_{\mathrm{T}}}$.

This is an example of a conceptually easy type of belief that cannot be modelled by sets of desirable gambles—and therefore also not by credal sets or lower previsions—in a satisfactory way. The approach proposed in this work, using non-binary sets of desirable gamble sets, provide the proper tools to model the beliefs illustrated in this example. ⧫

We are now able to discuss the interest of coherent sets of desirable gamble sets as a model of the available information. It follows from the results above, and in particular Theorem 6 which was first proved by De Bock & de Cooman [12, Theorem 7] that coherent sets of desirable gamble sets can be equivalently seen as families of binary coherent sets of desirable gamble sets—and therefore also as families of coherent sets of desirable gambles. Therefore, they may be given a sensitivity analysis interpretation, where one of the coherent sets of desirable gambles in this family is the 'true' model; this was for instance the case with our running example.

Translating to rejection functions, a gamble f is rejected in a gamble set when at least one of the compatible models defined in terms of sets of desirable gambles allows us to reject f. In other words, the choice model is represented by a *set of* ensembles of binary comparisons; but it can only be summarised in terms of a *single* ensemble of binary comparisons when one of the representing coherent sets of desirable gambles encompasses all the others.

This viewpoint is similar, if we were to work with sets of probability measures, to the notion of E-admissibility [21], but without the additional assumption of convexity over these sets. Indeed, that convexity may be inadequate in some cases is not a new proposal; this was extensively studied among others by Seidenfeld et al. [29, 30], showing that in some cases the axiom of *mixture dominance* is not reasonable. This was indeed the case for our running example: the degenerate distributions on H and on T were assumed to be valid, but not any non-trivial mixture of them.

The advantage of sets of desirable gamble sets is then to combine (a) the attractive features of sets of desirable gambles to deal more easily with the problem of conditioning, with (b) the flexibility that provides not requiring that the set of representing belief models is convex.

3. CONDITIONING

Consider a variable X that assumes values in a non-empty possibility space $\mathcal{X}$. Suppose that we have a belief model about X, be it a coherent set of desirable gamble sets on $\mathcal{L}$ or a coherent set of desirable gambles on $\mathcal{X}$, or—less general—just a single probability mass function on X, or a set of them. When new information becomes available, in the form of 'X assumes a value in some (non-empty) subset E of X', we can take this into account by conditioning our belief model on E.

For some of these belief models, such as (sets of) probability mass functions, conditioning on events of probability zero can be problematic, because, roughly speaking, Bayes's Rule typically requires to divide by zero in these situations. However, working with sets of desirable gambles is one way of overcoming this problem. In this section, we will see why, and explain that sets of desirable gamble sets do not suffer from this problem either.

We will let any event, except for the (trivially) impossible event $\emptyset$, serve as a conditioning event. We collect the allowed conditioning events in

$$\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}) \coloneqq \{E \subseteq \mathcal{X} : E \neq \emptyset\}.$$

We will first review how conditioning is done using sets of desirable gambles (see [16] for more details). After that, we will introduce conditional sets of desirable gamble sets, and study the connection between both cases. Given the discussion in Section 2.3, this immediately translates to rejection functions and choice functions as well.

There are multiple equivalent definitions for conditional sets of desirable gambles. Most of them, for instance those in [4, 26, 40, 41] result in a conditional set of desirable gambles on $\mathcal{X}$. However, we find it more useful and convenient that a conditional model is defined on gambles on E, rather than on $\mathcal{X}$, because, after getting to know that E occurs, the possibility space becomes effectively E.

Definition 7 ([16, Equation (17)])**.** Consider any set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ and any conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$. We define the *conditional set of desirable gambles* $D\rfloor E \subseteq \mathcal{L}(E)$ as

$$D\rfloor E \coloneqq \{f \in \mathcal{L}(E) : \mathbb{I}_E f \in D\}. \tag{8}$$

In this definition, we let for any E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$ and any gamble f on E its multiplication $\mathbb{I}_E f$ denote the gamble on $\mathcal{X}$ defined by

$$(\mathbb{I}_E f)(x) \coloneqq \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{9}$$

for all x in $\mathcal{X}$. Note that, for any gambles f and g on E, we have $f \neq g \Leftrightarrow \mathbb{I}_E f \neq \mathbb{I}_E g$, and, as a consequence, $f < g \Leftrightarrow \mathbb{I}_E f < \mathbb{I}_E g$.

It was proved by de Cooman and Quaghebeur [16, Proposition 8] that conditioning preserves coherence: if D is a coherent set of desirable gambles, then so is $D\rfloor E$, for any E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$. This explains why sets of desirable gambles do not suffer from conditioning on events of probability zero: $D\rfloor E$ is well-defined and coherent for every conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$, even if E has probability zero according to some, or all, of the probabilities induced by D.

For sets of desirable gamble sets, conditioning can be defined using the same simple underlying ideas:

Definition 8 (Conditioning)**.** Given any set of desirable gamble sets K and any conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$, we define the *conditional set of desirable gamble sets* $K\rfloor E$ on $\mathcal{L}(E)$ as

$$K\rfloor E \coloneqq \{A \in \mathcal{Q}(\mathcal{L}(E)) : \mathbb{I}_E A \in K\}, \tag{10}$$

where for any A in $\mathcal{Q}(\mathcal{L}(E))$ and E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$, we let $\mathbb{I}_E A \coloneqq \{\mathbb{I}_E g : g \in A\}$ be a set of gambles on $\mathcal{X}$.

Running example. Consider again the coin in our running example, and assume that we toss it twice: this means that our possibility space is now $\mathcal{X}_c \coloneqq \{(\mathrm{H},\mathrm{H}),(\mathrm{H},\mathrm{T}),(\mathrm{T},\mathrm{H}),(\mathrm{T},\mathrm{T})\}$. If we believe that the two sides of the coin are either both heads or both tails, this means that, reasoning as in Section 2, our coherent set of desirable gamble sets should be the natural extension of

$$\mathcal{A}' \coloneqq \{\{-\mathbb{I}_{\{(\mathrm{H},\mathrm{H})^c\}} + \varepsilon, -\mathbb{I}_{\{(\mathrm{T},\mathrm{T})^c\}} + \delta\} : \varepsilon, \delta \in \mathbb{R}_{>0}\},$$

that is given by

$$\{A \in \mathcal{Q} : ((\exists h_1, h_2 \in A)(h_1(\mathrm{T},\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H},\mathrm{H}) > 0)) \text{ or } A \cap \mathcal{L}^{\mathrm{s}}_{>0} \neq \emptyset\}.$$

If we now assume that the first toss resulted heads, we would be conditioning on the event $E = \{(\mathrm{H},\mathrm{T}),(\mathrm{H},\mathrm{H})\}$. This produces the coherent set of desirable gamble sets

$$K\rfloor E \coloneqq \{A \in \mathcal{Q}(\mathcal{L}(E)) : \mathbb{I}_E A \in K\} = \{A \in \mathcal{Q}(\mathcal{L}(E)) : A \cap \mathcal{L}^{\mathrm{s}}_{>0} \neq \emptyset\},$$

being the vacuous set of desirable gamble sets on E. ♦

Proposition 7. *Consider any set of desirable gamble sets K on $\mathcal{L}(\mathcal{X})$ and any conditioning event E in $\mathcal{P}_{\overline{\emptyset}}$. If K is coherent, then so is $K\rfloor E$.*

One of the useful properties of our definition of conditioning is that it preserves coherence, as shown in Proposition 7, and therefore sets of desirable gamble sets also do not suffer from conditioning on events of probability zero. But it should also be consistent with the operator given in Definition 7 for sets of desirable gambles; in other words, we should verify whether Definition 8 reduces to Definition 7 when only considering binary choice. Of course, to investigate this, we must keep in mind the connection between sets of desirable gamble sets and sets of desirable gambles, explained in Section 2.3.

For our two conditioning rules—the one in Definition 7 for sets of desirable gambles and the one in Definition 8 for sets of desirable gamble sets—to be a match, we must prove that: (i) the conditioning rule for sets of desirable gamble sets reverts to the known conditioning rule for the corresponding sets of desirable gambles, and (ii) in the special case of purely binary choice, the conditioning for sets of desirable gamble sets coincides with the conditioning rule for desirability. Mathematically, (i) means that $D_K\rfloor E = D_{K\rfloor E}$ for any coherent set of desirable gamble sets K and conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$, and (ii) means that $K_D\rfloor E = K_{D\rfloor E}$, for any coherent set of desirable gambles D, and any conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$. The next proposition guarantees that both these conditions are satisfied:

Proposition 8. *Consider any coherent set of desirable gamble sets K, any coherent set of desirable gambles D, and any conditioning event E in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X})$. Then $D_K\rfloor E = D_{K\rfloor E}$ and $K_D\rfloor E = K_{D\rfloor E}$. Furthermore, $K\rfloor E = \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}$.*

The last statement of Proposition 8 guarantees that the conditional set of desirable gamble sets $K\rfloor E$ can be retrieved by conditioning every element of K's representing set $\overline{\mathbf{D}}(K)$ from Theorem 6. This is illustrated in Figure 1.

$$
\begin{tikzcd}
K \arrow[r, "\text{Eq. (7)}"] \arrow[d, "\text{Eq. (10)}"'] & \overline{\mathbf{D}}(K) \arrow[dd, "\text{Eq. (8)}"] \\
K\rfloor E & \\
\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\} \arrow[u, "\cap"] & \{D\rfloor E : D \in \overline{\mathbf{D}}(K)\} \arrow[l, "\text{Eq. (6)}"]
\end{tikzcd}
$$

FIGURE 1. Commuting diagram for conditioning.

Running example. In the previous part of this running example, we obtained the vacuous set of desirable gamble sets on E as our conditional set of desirable gamble sets $K\rfloor E$. The fact that we end up with a vacuous model (instead of a model that describes certainty about H—after all, we observed that the first toss landed heads) might come as counter-intuitive. Intuition for this result can best be found in Proposition 8.

By a reasoning completely analogous to the part of this running example in Section 2.3, we find that the representing set $\overline{\mathbf{D}}(K)$ of desirable gambles is given by $\{D_{\mathrm{H,H}}, D_{\mathrm{T,T}}\}$, where

$$
\begin{aligned}
D_{\mathrm{H,H}} &:= \{f \in \mathcal{L}(\mathcal{X}_c) : f(\mathrm{H},\mathrm{H}) > 0\} \cup \mathcal{L}(\mathcal{X}_c)_{>0},\\
D_{\mathrm{T,T}} &:= \{f \in \mathcal{L}(\mathcal{X}_c) : f(\mathrm{T},\mathrm{T}) > 0\} \cup \mathcal{L}(\mathcal{X}_c)_{>0}.
\end{aligned}
$$

Proposition 8 then implies that $K\rfloor E = \bigcap\{K_{D_{\mathrm{H,H}}\rfloor E}, K_{D_{\mathrm{T,T}}\rfloor E}\}$. Let us determine the two conditional sets of desirable gambles involved. To find $D_{\mathrm{H,H}}\rfloor E = \{f \in \mathcal{L}(E) : \mathbb{I}_E f \in D_{\mathrm{H,H}}\}$, note that $(\mathrm{H},\mathrm{H}) \in E$, whence $(\mathbb{I}_E f)(\mathrm{H},\mathrm{H}) = f(\mathrm{H},\mathrm{H})$ for any f in $\mathcal{L}(E)$. Therefore $D_{\mathrm{H,H}}\rfloor E = \{f \in \mathcal{L}(E) : f(\mathrm{H},\mathrm{H}) > 0 \text{ or } \mathbb{I}_E f \in \mathcal{L}(\mathcal{X}_c)_{>0}\} = \{f \in \mathcal{L}(E) : f(\mathrm{H},\mathrm{H}) > 0\} \cup \mathcal{L}(E)_{>0}$, which is the set of desirable gambles on E that corresponds to a belief of certainty about (H,H).

To find $D_{\mathrm{T,T}}\rfloor E = \{f \in \mathcal{L}(E) : \mathbb{I}_E f \in D_{\mathrm{T,T}}\}$, note that $(\mathrm{T},\mathrm{T}) \notin E$, whence $(\mathbb{I}_E f)(\mathrm{T},\mathrm{T}) = 0$ for any f in $\mathcal{L}(E)$. Therefore $D_{\mathrm{T,T}}\rfloor E = \{f \in \mathcal{L}(E) : \mathbb{I}_E f \in \mathcal{L}(\mathcal{X}_c)_{>0}\} = \mathcal{L}(E)_{>0}$, the smallest coherent set of desirable gambles on E. This means that $K_{D_{\mathrm{T,T}}\rfloor E} = K_{\mathrm{v}}$ is the vacuous set of desirable gamble sets on E.

Using Proposition 8, we find that $K\rfloor E = \inf\{K_{D_{\mathrm{H,H}}\rfloor E}, K_{D_{\mathrm{T,T}}\rfloor E}\} = \inf\{K_{D_{\mathrm{H,H}}\rfloor E}, K_{\mathrm{v}}\} = K_{\mathrm{v}}$, the vacuous set of desirable gamble sets on E. So via the set of representing sets of binary models, we obtained the same conditional model, as guaranteed by Proposition 8.

The intuition why we do not end up with a model that describes certainty about H is that, for sets of desirable gamble sets that are representable by sets of probabilities, our conditioning rule amounts to *natural extension* as a conditional rule, and not to *regular extension*. For a discussion about natural extension versus regular extension, we refer to [11, 24], [6, Section 2.7 and Chapter 3] and [40, Appendix J]. In short, natural extension considers all the models (probabilities) in the representation, and will result in a vacuous conditional model if there is at least one model (probability) that assigns probability zero to the conditioning event. Loosely speaking, in this example, the culprit in the representation $\overline{\mathbf{D}}(K)$ is $D_{\mathrm{T,T}}$. Regular extension, in contrast, ignores the models (probabilities) in the representation that assign probability zero to the conditioning event, resulting in a vacuous conditional model only if every model (probability) in the representation assigns probability zero to the conditioning event. Loosely speaking, a notion of regular extension as conditioning rule for sets of desirable gamble sets would result in ignoring $D_{\mathrm{T,T}}$, and we would end up with $K_{D_{\mathrm{H,H}}\rfloor E}$, expressing certainty about H. A theory of regular extension as a conditioning rule for sets of desirable gamble sets falls outside the scope of this paper. ♦

4. Multivariate sets of desirable gamble sets

In this section, we will generalise the concepts of marginalisation, weak (cylindrical) extension and irrelevant natural extension introduced by de Cooman and Miranda [15] for sets of desirable gambles to choice models. We will provide the linear space of gambles, on which we define our sets of desirable gamble sets, with a more complex structure: we will consider the vector space of all gambles whose domain is a Cartesian product of a finite number of finite possibility spaces. More specifically, consider $n \in \mathbb{N}$ variables $X_1, \dots, X_n$ that assume values in the finite possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Belief models about these variables $X_1, \dots, X_n$ will be defined on gambles on $\mathcal{X}_1, \dots, \mathcal{X}_n$. We may also consider gambles on the Cartesian product $\times_{k=1}^n \mathcal{X}_k$, giving rise to the $\prod_{k=1}^n |\mathcal{X}_k|$-dimensional linear space $\mathcal{L}(\times_{k=1}^n \mathcal{X}_k)$.

4.1. Basic notation & cylindrical extension.

For every non-empty subset $I \subseteq \{1, \dots, n\}$ of indices, we let X_I be the tuple of variables that takes values in $\mathcal{X}_I \coloneqq \times_{r \in I} \mathcal{X}_r$. We will denote generic elements of $\mathcal{X}_I$ as x_I or z_I, whose components are $x_i \coloneqq x_I(i)$ and $z_i \coloneqq z_I(i)$, for all i in I. As we did before, when $I = \{k, \dots, \ell\}$ for some k, ℓ in $\{1, \dots, n\}$ with $k \leq \ell$, we will use as a short-hand notation $X_{k:\ell} \coloneqq X_{\{k,\dots,\ell\}}$, taking values in $\mathcal{X}_{k:\ell} \coloneqq \mathcal{X}_{\{k,\dots,\ell\}}$ and whose generic elements are denoted by $x_{k:\ell} \coloneqq x_{\{k,\dots,\ell\}} = (x_k, \dots, x_\ell)$.

We assume that the variables $X_1, \dots, X_n$ are *logically independent*, meaning that for each non-empty subset I of $\{1, \dots, n\}$, x_I may assume every value in $\mathcal{X}_I$.

It will be useful for any gamble f on $\mathcal{X}_{1:n}$, any non-empty proper subset I of $\{1, \dots, n\}$ and any x_I in $\mathcal{X}_I$, to interpret the partial map $f(x_I, \bullet)$ as a gamble on $\mathcal{X}_{I^c}$, where $I^c \coloneqq \{1, \dots, n\} \setminus I$. Likewise, for any set A of gambles on $\mathcal{X}_{1:n}$, we let $A(x_I, \bullet) \coloneqq \{f(x_I, \bullet) : f \in A\}$ be a corresponding set of gambles on $\mathcal{X}_I$.

We will need a way to relate gambles on different domains:

Definition 9 (Cylindrical extension)**.** Given two disjoint and non-empty subsets I and I' of $\{1, \dots, n\}$ and any gamble f on $\mathcal{X}_I$, we let its *cylindrical extension* f^* to $\mathcal{X}_{I \cup I'}$ be defined by

$$f^*(x_I, x_{I'}) \coloneqq f(x_I) \text{ for all } x_I \text{ in } \mathcal{X}_I \text{ and } x_{I'} \text{ in } \mathcal{X}_{I'}.$$

Similarly, given any set of gambles $A \subseteq \mathcal{L}(\mathcal{X}_I)$, we let its *cylindrical extension* $A^* \subseteq \mathcal{L}(\mathcal{X}_{I \cup I'})$ be defined as $A^* \coloneqq \{f^* : f \in A\}$.

Remark 1. Formally, f^* belongs to $\mathcal{L}(\mathcal{X}_{I\cup I'})$ while f belongs to $\mathcal{L}(\mathcal{X}_I)$. However, f^* is completely determined by f and vice versa: they clearly only depend on the value of X_I, and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioural point of view. Taking this into account, we shall follow the lead of [13, 15] and we will frequently use the simplifying device of *identifying* a gamble f on $\mathcal{L}(\mathcal{X}_I)$ with its cylindrical extension f^* on $\mathcal{L}(\mathcal{X}_{I\cup I'})$, for any disjoint and non-empty subsets I and I' of the index set $\{1,\dots,n\}$. This convention allows us for instance to identify $\mathcal{L}(\mathcal{X}_I)$ with a subset of $\mathcal{L}(\mathcal{X}_{1:n})$, and, as another example, for any set $A \subseteq \mathcal{L}(\mathcal{X}_{1:n})$, to regard $A \cap \mathcal{L}(\mathcal{X}_I)$ as those gambles in A that depend on the value of $\mathcal{X}_I$ only. Therefore, for any event E in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{E\times\mathcal{X}_{I^c}}$, and hence also the event E with $E\times\mathcal{X}_{I^c}$. This device for instance also allows us to write, for any f on $\mathcal{X}_I$ and g on $\mathcal{X}_{I\cup I'}$, that $f \le g \Leftrightarrow (\forall x_I \in \mathcal{X}_I, x_{I'} \in \mathcal{X}_{I'})\, f(x_I) \le g(x_I, x_{I'}) \Leftrightarrow f^* \le g$. ♦

4.2. **Marginalisation and weak extension.** Suppose we have a set of desirable gamble sets K on $\mathcal{L}(\mathcal{X}_{1:n})$ modelling a subject's beliefs about the variable $X_{1:n}$. What is the information that K contains about X_O, where O is some non-empty subset of the index set $\{1,\dots,n\}$? Finding this information can be done through marginalisation.

Definition 10 (Marginalisation)**.** Given any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable gamble sets K on $\mathcal{L}(\mathcal{X}_{1:n})$, its *marginal set of desirable gamble sets* $\mathrm{marg}_O K$ on $\mathcal{L}(\mathcal{X}_O)$ is defined as

$$\mathrm{marg}_O K \coloneqq \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \in K\} = K \cap \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)). \tag{11}$$

We use the simplifying device of Remark 1 of identifying A with a subset of $\mathcal{L}(\mathcal{X}_{1:n})$. Without resorting to this convention, we can characterise $\mathrm{marg}_O K$ as:

$$(\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)))\ A \in \mathrm{marg}_O K \Leftrightarrow A^* \in K.$$

It follows at once from Definition 10 that marginalisation preserves the order: if $K_1 \subseteq K_2$, then $\mathrm{marg}_O K_1 \subseteq \mathrm{marg}_O K_2$, for all sets of desirable gamble sets K_1 and K_2 on $\mathcal{L}(\mathcal{X}_{1:n})$. Marginalisation also preserves coherence:

Proposition 9. *Consider any set of desirable gamble sets K on $\mathcal{L}(\mathcal{X}_{1:n})$ and any non-empty subset O of $\{1,\dots,n\}$. If K is coherent, then so is* $\mathrm{marg}_O K$.

Let us compare with sets of desirable gambles. De Cooman and Miranda [15] defined, for any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable gambles D, its *marginal set of desirable gambles* $\mathrm{marg}_O D$ on $\mathcal{L}(\mathcal{X}_O)$ as

$$\mathrm{marg}_O D \coloneqq \{f \in \mathcal{L}(\mathcal{X}_O) : f \in D\} = D \cap \mathcal{L}(\mathcal{X}_O). \tag{12}$$

Let us ascertain that the definition of marginalisation reduces, in the case of binary choice, to the one for sets of desirable gambles:

Proposition 10. *Consider any non-empty subset O of $\{1,\dots,n\}$, any set of desirable gamble sets K on $\mathcal{L}(\mathcal{X}_{1:n})$, and any set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{1:n})$. Then*

$$\mathrm{marg}_O D_K = D_{\mathrm{marg}_O K} \text{ and } \mathrm{marg}_O K_D = K_{\mathrm{marg}_O D}.$$

Furthermore, $\mathrm{marg}_O K = \bigcap\{K_{\mathrm{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}$.

The last statement of this proposition guarantees that the marginal set of desirable gamble sets $\mathrm{marg}_O K$ can be retrieved by marginalising every element of K's representing set $\overline{\mathbf{D}}(K)$. This is illustrated in Figure 2.

$$\begin{tikzcd}
K \arrow[rr, "\text{Eq. (7)}"] \arrow[d, "\text{Eq. (11)}"'] & & \overline{\mathbf{D}}(K) \arrow[dd, "\text{Eq. (12)}"] \\
\operatorname{marg}_O D_K & & \\
\{K_{\operatorname{marg}_O D} : D \in \overline{\mathbf{D}}(K)\} \arrow[u, "\cap"] & & \{\operatorname{marg}_O D : D \in \overline{\mathbf{D}}(K)\} \arrow[ll, "\text{Eq. (6)}"]
\end{tikzcd}$$

FIGURE 2. Commuting diagram for marginalisation.

Now that marginalisation is in place, and that we know that it coincides with the eponymous concept for sets of desirable gambles in the case of pairwise choice, we are ready to look for some kind of inverse operation to it. Suppose we have a coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$ modelling a subject's belief about X_O, where O is a non-empty subset of $\{1,\dots,n\}$. We want to extend K_O to a coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$ that represents the same beliefs. This leads to the following definition:

Definition 11 (Weak extension)**.** Given a coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$, the smallest coherent set of desirable gamble sets K on $\mathcal{L}(\mathcal{X}_{1:n})$ such that $\operatorname{marg}_O K = K_O$, if it exists, is called the *weak extension* of K_O.

Let us study this notion in more detail. Given a non-empty subset O of $\{1,\dots,n\}$ and a coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$, an assessment based on it that is important for the weak extension, is

$$\mathcal{A}^{1:n}_{K_O} \coloneqq \{A^*:A \in K_O\} \subseteq \mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n})).$$

To make clear that $\mathcal{A}^{1:n}_{K_O}$ is a collection of sets of gambles on $\mathcal{X}_{1:n}$, we made the cylindrical extension explicit by writing A^*. Using our simplifying device of Remark 1 however, we can equivalently write $\mathcal{A}^{1:n}_{K_O} = K_O$—and we will do this throughout—and interpret it as a collection of sets of gambles on $\mathcal{X}_{1:n}$, and therefore as an assessment for sets of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$.

It turns out that the weak extension always exists:

Proposition 11. *Consider any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$. Then the least informative coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to K_O is given by*

$$\operatorname{ext}_{1:n}(K_O) \coloneqq \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup \mathcal{A}^{1:n}_{K_O})), \tag{13}$$

and it satisfies $\operatorname{marg}_O(\operatorname{ext}_{1:n}(K_O)) = K_O$.

How is this result connected with the weak extension for sets of desirable gambles? De Cooman and Miranda [15, Proposition 7] show that, given any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gambles $D_O \subseteq \mathcal{L}(\mathcal{X}_O)$, its weak extension $\operatorname{ext}^{\mathbf{D}}_{1:n}(D_O) \subseteq \mathcal{L}(\mathcal{X}_{1:n})$—the least informative coherent set of desirable gambles on $\mathcal{X}_{1:n}$ that marginalises to D_O—exists and is given by

$$\operatorname{ext}^{\mathbf{D}}_{1:n}(D_O) \coloneqq \operatorname{posi}(\mathcal{L}(\mathcal{X}_{1:n})_{>0} \cup D_O). \tag{14}$$

We show that the weak extension $\mathrm{ext}_{1:n}(K_O)$ of a coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$ can also be retrieved by taking the weak extension of every element of K_O's representing set $\overline{\mathbf{D}}(K_O)$ from Theorem 6:

Proposition 12. *Consider any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$. Then*

$$\mathrm{ext}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O)} : D_O \in \overline{\mathbf{D}}(K_O)\}.$$

Figure 3 illustrates the result.

$$\begin{tikzcd}
K_O \arrow[rr, "\text{Eq. (7)}"] \arrow[d, "\text{Eq. (13)}"'] & & \overline{\mathbf{D}}(K_O) \arrow[dd, "\text{Eq. (14)}"] \\
\mathrm{ext}_{1:n}(K_O) & & \\
\{K_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O)} : D_O \in \overline{\mathbf{D}}(K_O)\} \arrow[u, "\cap"] & & \{\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O) : D_O \in \overline{\mathbf{D}}(K_O)\} \arrow[ll, "\text{Eq. (6)}"]
\end{tikzcd}$$

FIGURE 3. Commuting diagram for the weak extension.

4.3. **Conditioning on variables.** In Section 3 we have seen how we can condition sets of desirable gamble sets on events. Here, we take a closer look at conditioning in a multivariate context.

Suppose we have a set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$, representing a subject's beliefs about the value of $X_{1:n}$. Assume now that we obtain the information that the I-tuple of variables X_I—where I is a non-empty subset of $\{1,\dots,n\}$—assumes a value in a certain non-empty subset E_I of $\mathcal{X}_I$—so E_I belongs to $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$. There is no new information about the other variables X_{I^c}. How can we condition K_n using this new information?

This is a particular instance of Definition 8, with the following specifications: $\mathcal{X} = \mathcal{X}_{1:n}$ and $E = E_I \times \mathcal{X}_{I^c}$. The indicator $\mathbb{I}_E$ of the conditioning event E satisfies $\mathbb{I}_E(x_{1:n}) = \mathbb{I}_{E_I}(x_I)$ for all $x_{1:n}$ in $\mathcal{X}_{1:n}$, and taking Remark 1 into account, therefore $\mathbb{I}_E = \mathbb{I}_{E_I}$. Equation (9) defines the multiplication of a gamble f on $E_I \times \mathcal{X}_{I^c}$ with $\mathbb{I}_{E_I}$ to be a gamble $\mathbb{I}_{E_I} f$ on $\mathcal{X}_{1:n}$, given by, for all $x_{1:n}$ in $\mathcal{X}_{1:n}$:

$$\mathbb{I}_{E_I} f(x_{1:n}) = \begin{cases} f(x_{1:n}) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \tag{15}$$

and the multiplication of $\mathbb{I}_{E_I}$ with a set A of gambles on $E_I \times \mathcal{X}_{I^c}$ is the set $\mathbb{I}_{E_I} A = \{\mathbb{I}_{E_I} f : f \in A\}$ of gambles on $\mathcal{X}_{1:n}$.

Now that we have instantiated all the relevant aspects of Definition 8, we are ready to find the conditional set of desirable gamble sets $K_n\rfloor E_I$, given a joint set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$:

$$K_n\rfloor E_I = \{A \in \mathcal{Q}(\mathcal{L}(E_I \times \mathcal{X}_{I^c})) : \mathbb{I}_{E_I} A \in K_n\}.$$

The conditional set of desirable gamble sets $K_n\rfloor E_I$ is defined on gambles on $E_I \times \mathcal{X}_{I^c}$. However, usually—see, for instance, [6, 15]—conditioning on information about X_I results in a model on $\mathcal{X}_{I^c}$. We therefore consider

$$\mathrm{marg}_{I^c}(K_n\rfloor E_I)=\{A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I^c})):\mathbb{I}_{E_I}A\in K_n\}$$

as the set of desirable gamble sets that represents the conditional beliefs about X_{I^c}, given that $X_I\in E_I$. In this context, the multiplication $\mathbb{I}_{E_I}f$ of $\mathbb{I}_{E_I}$ and a gamble f in A is defined through Equation (15):

$$\mathbb{I}_{E_I}f(x_{1:n})=\begin{cases}f(x_{I^c}) & \text{if } x_I\in E_I\\ 0 & \text{if } x_I\notin E_I\end{cases}$$

for all $x_{1:n}$ in $\mathcal{X}_{1:n}$.

Note that, in the particular case of conditioning on a singleton—say, $E_I=\{x_I\}$ for some x_I in $\mathcal{X}_I$—the set $K_n\rfloor x_I$ of desirable gamble sets[4] is on $\mathcal{L}(\{x_I\}\times\mathcal{X}_{I^c})$. Every gamble f on $\{x_I\}\times\mathcal{X}_{I^c}$ can be uniquely identified with a gamble $f(x_I,\bullet)$ on $\mathcal{X}_{I^c}$, and therefore $\{x_I\}\times\mathcal{X}_{I^c}$ can be identified with $\mathcal{X}_{I^c}$. Therefore the resulting set of desirable gamble sets $K_n\rfloor x_I$ can be identified with its marginal $\mathrm{marg}_{I^c}(K_n\rfloor x_I)$.

Propositions 7 and 9 guarantee the coherence of $\mathrm{marg}_{I^c}(K_n\rfloor E_I)$ for any coherent K_n.

As is the case for desirability ([15, Proposition 9]), the order of marginalisation and conditioning can be reversed, under some conditions:

Proposition 13. *Consider any coherent set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$, any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$. Then*

$$\mathrm{marg}_O(K_n\rfloor E_I)=\mathrm{marg}_O((\mathrm{marg}_{I\cup O}K_n)\rfloor E_I).$$

4.4. **Irrelevant natural extension.** Now that the basic operations of multivariate sets of desirable gamble sets—marginalisation, weak extension and conditioning—are in place, we are ready to look at a simple type of structural assessment. The assessment that we will consider, is that of *epistemic irrelevance*.

Definition 12 (Epistemic (subset) irrelevance)**.** Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$. We say that X_I is *epistemically irrelevant* to X_O when learning about the value of X_I does not influence or change the subject's beliefs about X_O. A set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$ is said to satisfy *epistemic irrelevance* of X_I to X_O when

$$\mathrm{marg}_O(K_n\rfloor E_I)=\mathrm{marg}_O K_n \text{ for all } E_I \text{ in } \mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I).$$

The idea behind this definition is that observing that X_I belongs to E_I turns K_n into the conditioned set of desirable gamble sets $K_n\rfloor E_I$ on $\mathcal{L}(E_I\times\mathcal{X}_{I^c})\supseteq\mathcal{L}(\mathcal{X}_O)$, so requiring that learning that X_I belongs to E_I does not affect the subject's beliefs about X_O, amounts to requiring that the marginal models of K_n and $K_n\rfloor E_I$ should be equal.

In a similar manner, we may define the more general notion of *conditional epistemic irrelevance*: given any disjoint and non-empty subsets I, O and C of $\{1,\dots,n\}$, a set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$ is said to satisfy *epistemic irrelevance of X_I to X_O conditional on X_C* when

$$\mathrm{marg}_O(K_n\rfloor(E_I\times\{x_C\}))=\mathrm{marg}_O(K_n\rfloor x_C) \text{ for all } E_I \text{ in } \mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I) \text{ and all } x_C \text{ in } \mathcal{X}_C.$$

We refer to [6, 7, 9, 10, 17, 22] for more information about, and some applications of, epistemic irrelevance in graphical models.

Epistemic irrelevance can be reformulated in an interesting and slightly different manner:

[4] Actually, since the conditioning event is $\{x_I\}$, we should write $K_n\rfloor\{x_I\}$ rather than $K_n\rfloor x_I$, but since no confusion can arise, and for notational simplicity, we will use the latter notation. A similar choice has been made by de Cooman and Miranda in [15].

Proposition 14. *Consider any coherent set of desirable gamble sets K_n on $\mathcal{L}(\mathcal{X}_{1:n})$, and any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$. Then the following statements are equivalent:*

(i) $\operatorname{marg}_O(K_n \rfloor E_I) = \operatorname{marg}_O K_n$ *for all E_I in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$;*
(ii) $A \in K_n \Leftrightarrow \mathbb{I}_{E_I} A \in K_n$, *for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and E_I in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$.*

Epistemic irrelevance assessments are useful in constructing sets of desirable gamble sets on larger domains from other ones on smaller domains. Suppose we have a set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$, and an assessment that X_I is epistemically irrelevant to X_O, where I and O are disjoint and non-empty subsets of $\{1,\dots,n\}$. How can we combine K_O and this irrelevance assessment into a coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{I\cup O})$, or more generally, on $\mathcal{L}(\mathcal{X}_{1:n})$? We consider the following definition:

Definition 13 (Irrelevant natural extension)**.** Consider a set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$, and an assessment that X_I is epistemically irrelevant to X_O, where I and O are disjoint and non-empty subsets of $\{1,\dots,n\}$. The least informative coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to K_O and that satisfies epistemic irrelevance of X_I to X_O is called the $X_I - X_O$ *irrelevant natural extension* of K_O.

In order to study the irrelevant natural extension, the following set will play a crucial role:

$$\mathcal{A}^{\mathrm{irr}}_{I\to O} := \{\mathbb{I}_{E_I} A : A \in K_O \text{ and } E_I \in \mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)\} \tag{16}$$

which we will interpret as an assessment on $\mathcal{L}(\mathcal{X}_{I\cup O})$.

Theorem 15. *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$. The $X_I - X_O$ irrelevant natural extension of K_O is given by*

$$\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) := \operatorname{ext}_{1:n}(K^{\mathrm{irr}}_{I\cup O}), \tag{17}$$

where

$$K^{\mathrm{irr}}_{I\cup O} := \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I\cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I\to O})). \tag{18}$$

Furthermore,

$$\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) = \bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}(D)} : D \in \overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})\}.$$

The final statement of this theorem guarantees that the irrelevant natural extension that marginalises to a set of desirable gamble sets K_O can be retrieved by extending every element of $K^{\mathrm{irr}}_{I\cup O}$'s representing set $\overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})$ from Theorem 6, as Figure 4 illustrates.

$$\begin{tikzcd}
K_O \arrow[r, "\text{Eq. (18)}"] \arrow[d, "\text{Eq. (17)}"'] & K^{\mathrm{irr}}_{I\cup O} \arrow[r, "\text{Eq. (7)}"] & \overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O}) \arrow[dd, "\text{Eq. (14)}"] \\
\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) & & \\
\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}(D)} : D \in \overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})\} \arrow[u, "\cap"] & & \{\operatorname{ext}^{\mathbf{D}}_{1:n}(D) : D \in \overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})\} \arrow[ll, "\text{Eq. (6)}"']
\end{tikzcd}$$

FIGURE 4. Commuting diagram for the irrelevant natural extension.

Running example. Let us return to our running example. We now consider a coin of which we may have different beliefs, and require that learning about the outcome of this other coin should not change our beliefs about the coin with two identical sides. Thus, if we index our tosses by $\{1,2\}$, and take $I=\{1\}$, $O=\{2\}$ to be sets of indices of the outcomes of the new coin and of our original one, respectively, then we are interested in the $X_I - X_O$ irrelevant natural extension of K_O. This is given by

$$\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) \coloneqq \mathrm{ext}_{1:n}(K^{\mathrm{irr}}_{I\cup O}),$$

which in this case is equal to

$$\{A \in \mathcal{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{H},\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H},\mathrm{H}) > 0))\} \cup$$
$$\{A \in \mathcal{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{T},\mathrm{T}) > 0 \text{ and } h_2(\mathrm{T},\mathrm{H}) > 0))\} \cup \{A \in \mathcal{Q} \colon A \cap \mathcal{L}^{\mathrm{s}}_{>0} \neq \emptyset\}.$$

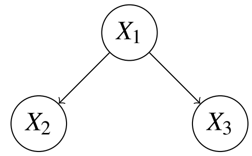
This is to be expected: the O-marginal of this model is given by the model for our original coin, depicted in the part of this running example in Section 2.1, and it furthermore satisfies the epistemic irrelevance of I to O. ⧫

5. Discussion and conclusions

We have studied the irrelevant natural extension in the framework of choice functions. To define this, we introduced conditioning and marginalisation in this framework. We related our definitions and results with the existing definitions and results in the framework of sets of desirable gambles, and showed that they match with each other. The results in this paper are important because they are a first step for establishing a theory of credal networks with choice functions. Besides their generality, such credal networks would have the advantage that the local models are easy to elicit: choice functions can be assessed directly from a subject, simply by collecting the gambles she rejects from within any given set of gambles.

In this respect, the theory of credal networks has been extended to consider the cases where the imprecise models in each node are coherent lower previsions [2, 14] or sets of desirable gambles [10, 17]. We believe that it should be easy to combine our results in order to make a further generalisation to choice functions. Specifically, (i) we should make an assessment of conditional epistemic irrelevance, so that the non-parents non-descendants of a given node are epistemically irrelevant to it, conditional on its parents; (ii) extend the results in this paper to conditional epistemic irrelevance so as to obtain the resulting models in terms of sets of desirable gamble sets; and (iii) use the work on natural extension by De Bock & de Cooman [12] in order to find the natural extension, or least committal extension, of the union of those models.

One important issue that is still open is the lack of an expression for the *independent* natural extension for choice functions. The independent natural extension is a symmetric version of the irrelevant natural extension: if X_I is independent to X_O, then both X_I is irrelevant to X_O, and *vice versa*. This arises typically in credal networks; the easiest credal network in which its graphical interpretation implies symmetrised irrelevance is the following.



In this case, X_2's non-parent non-descendant is X_3, so the common interpretation of the credal network—that for every variable X its non-parent non-descendants are irrelevant to X conditional on the value of X's parent—requires that X_3 is conditionally irrelevant to X_2

given X_1. Similarly, X_3's non-parent non-descendant is X_2, so X_2 is conditionally irrelevant to X_3 given X_1: in other words, X_2 and X_3 are *conditionally epistemic independent given* X_1. We expect the representation result of De Bock and de Cooman [12, Theorem 7] to be crucial for an expression for the (conditionally) epistemic independent natural extension this. A first step in this direction, would be to establish the following representation of the irrelevant natural extension:

$$\mathrm{ext}_{1:n}^{\mathrm{irr}}(K_O) = \bigcap\{K_{\mathrm{ext}_{1:n}^{\overline{\mathbf{D}},\mathrm{irr}}(D_O)} : D_O \in \overline{\mathbf{D}}(K_O)\}$$

where $\mathrm{ext}_{1:n}^{\overline{\mathbf{D}},\mathrm{irr}}(D_O)$ is the irrelevant natural extension for sets of desirable gambles, established in [15]. We suspect furthermore that this independent natural extension will be given by the set of desirable gamble sets $\mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup \mathcal{A}_{I\to O}^{\mathrm{irr}} \cup \mathcal{A}_{O\to I}^{\mathrm{irr}}))$. These are conjectures of us, based on some preliminary insight, but we have no proof as of yet.

In addition, it would also be interesting to consider other, intermediate notions of irrelevance and independence, such as the notion of *subset irrelevance* considered in [8], and more generally, the compatibility of choice functions with other structural assessments, such as weak and strong invariance. In this sense, in [34] and [36] we studied how to embed the notions of exchangeability and indifference within the theory of coherent choice functions; it should not be very difficult to extend those results to the alternative axiomatisation by De Bock and de Cooman [8] we have considered in this paper.

Another future line of research would be the connection of our choice functions with decision trees. These were already extended to the imprecise case in [18, 20]. In this respect, we expect that the study of the connections with the different decision rules made in [36] may be useful.

Finally, it seems that it is only a small step, by combining suitably the operations of epistemic irrelevance, weak extension and natural extension, to establish a marginal extension theorem [23, 40] in the theory of choice functions. This would open the door for a connection between choice functions and stochastic processes.

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Appendix A. Proofs

Proof of Lemma 1. For notational convenience, we let $\tilde{K} \coloneqq \{A \in \mathcal{Q} : A \cap \mathcal{L}_{>0} \neq \emptyset\}$; we will show (i) that any coherent set of desirable gamble sets K must include $\tilde{K}$, and (ii) that $\tilde{K}$ is a coherent set of desirable gamble sets. (i) and (ii) together imply that $\tilde{K}$ is indeed the smallest coherent set of desirable gamble sets, which we indicated by K_{v}.

For (i), consider any coherent set of desirable gamble sets K, and any gamble set A in $\tilde{K}$. This means by the definition of $\tilde{K}$ that $f \in A$ for some f in $\mathcal{L}_{>0}$, whence $\{f\} \in K$ by Axiom K_2. Using Axiom K_4 we infer that indeed $A \in K$. So we see that $\tilde{K} \subseteq K$ for any K in $\overline{\mathbf{K}}$.

For (ii), note that $\emptyset \notin \tilde{K}$ by its definition, so it satisfies Axiom K_0. For Axiom K_1, it suffices to note that $(A \setminus \{0\}) \cap \mathcal{L}_{>0} \neq \emptyset$ whenever $A \cap \mathcal{L}_{>0} \neq \emptyset$. For Axiom K_2, note that $\{f\} \cap \mathcal{L}_{>0} \neq \emptyset \Leftrightarrow f \in \mathcal{L}_{>0}$, whence indeed $\{f\} \in \tilde{K}$ by its definition. For Axiom K_3, consider any A_1 and A_2 in $\tilde{K}$, meaning that $h_1>0$ and $h_2>0$ for some h_1 in A_1 and h_2 in A_2. Then for any choice of $(\lambda_{h_1,h_2},\mu_{h_1,h_2})>0$ we have that $\lambda_{h_1,h_2}h_1+\mu_{h_1,h_2}h_2>0$, which belongs to the gamble set $\{\lambda_{f,g}f+\mu_{f,g}g\colon f\in A_1,g\in A_2\}$. Therefore indeed $\{\lambda_{f,g}f+\mu_{f,g}g\colon f\in A_1,g\in A_2\}\in\tilde{K}$ by its definition. Finally, for Axiom K_4, consider any gamble sets A_1 and A_2 such that $A_1\subseteq A_2$ and $A_1\in\tilde{K}$—meaning that $f\in A_1$ for some f in $\mathcal{L}_{>0}$. Since $A_1\subseteq A_2$, we have also $f\in A_2$ whence indeed $A_2\in\tilde{K}$. □

Proof of Lemma 3. For the sake of brevity, we denote

$$K^1_{\mathrm{H,T}}\coloneqq\{A\in\mathcal{Q}\colon((\exists h_1,h_2\in A)(h_1(\mathrm{T})>0\text{ and }h_2(\mathrm{H})>0))\text{ or }A\cap\mathcal{L}^{\mathrm{s}}_{>0}\neq\varnothing\},$$
$$K^2_{\mathrm{H,T}}\coloneqq\mathrm{Rs}(\{\{h_1,h_2\}\colon h_1,h_2\in\mathcal{L}_{\not\leq 0},(h_1(\mathrm{T}),h_2(\mathrm{H}))>0\}).$$

We will show (i) that $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})\subseteq K^1_{\mathrm{H,T}}$, (ii) that $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$, and (iii) that $K^2_{\mathrm{H,T}}\subseteq\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})$.

For (i)—to show that $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})\subseteq K^1_{\mathrm{H,T}}$—consider any gamble set A in $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})$. This means that there are n in $\mathbb{N}$, A_1, …, A_n in $\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A}$, and, for all $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda^{f_{1:n}}_{1:n}>0$, such that $A=\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^n A_k\right\}$. Without loss of generality, assume that $A_1,\dots,A_\ell\in\mathcal{A}$ and $A_{\ell+1},\dots,A_n\in\mathcal{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0,\dots,n\}$. Therefore, we may denote, also without loss of generality, $A_1=\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_1,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_1\},\dots,A_\ell=\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_\ell,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_\ell\}$, $A_{\ell+1}=\{g_{\ell+1}\},\dots,A_n=\{g_n\}$, where ε_1, δ_1, …, ε_ℓ, δ_ℓ are elements of $\mathbb{R}_{>0}$ and $g_{\ell+1},\dots,g_n$ elements of $\mathcal{L}_{>0}$. If $\ell=0$ or $\lambda^{f_{1:n}}_{1:\ell}=0$ for some $f_{1:n}$ in $\times_{k=1}^n A_k$—and therefore necessarily $\lambda^{f_{1:n}}_{\ell+1:n}>0$—then we have that $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=\sum_{k=\ell+1}^n\lambda_k^{f_{1:n}}g_k$ is a gamble in $\mathcal{L}_{>0}$, so we find that $A\cap\mathcal{L}^{\mathrm{s}}_{>0}\neq\varnothing$. If, on the other hand, $\ell\geq 1$ and $\lambda^{f_{1:n}}_{1:\ell}\neq 0$—and hence $\lambda^{f_{1:n}}_{1:\ell}>0$—for every $f_{1:n}$ in $\times_{k=1}^n A_k$, then for the two sequences of gambles $f^{\mathrm{H}}_{1:n}=(f^{\mathrm{H}}_1,\dots,f^{\mathrm{H}}_n)\coloneqq(-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_1,\dots,-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_\ell,g_{\ell+1},\dots,g_n)$ and $f^{\mathrm{T}}_{1:n}=(f^{\mathrm{T}}_1,\dots,f^{\mathrm{T}}_n)\coloneqq(-\mathbb{I}_{\{\mathrm{T}\}}+\delta_1,\dots,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_\ell,g_{\ell+1},\dots,g_n)$ in $\times_{k=1}^n A_k$ we have that

$$h_1\coloneqq\sum_{k=1}^n\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k=\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k+\sum_{k=\ell+1}^n\lambda_k^{f^{\mathrm{H}}_{1:n}}g_k\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k=-\mathbb{I}_{\{\mathrm{H}\}}\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}+\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}\varepsilon_k$$

and, similarly,

$$h_2\coloneqq\sum_{k=1}^n\lambda_k^{f^{\mathrm{T}}_{1:n}}f^{\mathrm{T}}_k\geq-\mathbb{I}_{\{\mathrm{T}\}}\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}+\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}\delta_k,$$

so $h_1(\mathrm{T})\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}\varepsilon_k>0$ and $h_2(\mathrm{H})\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}\delta_k>0$. Note that both h_1 and h_2 belong to A, so we find that $(\exists h_1,h_2\in A)(h_1(\mathrm{T})>0\text{ and }h_2(\mathrm{H})>0)$. Therefore indeed $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})\subseteq K^1_{\mathrm{H,T}}$.

For (ii)—to show that $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$—consider any gamble set A in $K^1_{\mathrm{H,T}}$. Then (a) $h_1(\mathrm{T})>0$ and $h_2(\mathrm{H})>0$ for some h_1 and h_2 in A, or (b) $A\cap\mathcal{L}^{\mathrm{s}}_{>0}\neq\varnothing$. If (a), then $h_1,h_2\in\mathcal{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}),h_2(\mathrm{H}))>0$, so $A\in K^2_{\mathrm{H,T}}$. If (b), then $h>0$ for some h in A, so for $h_1\coloneqq h_2\coloneqq h$ trivially $h_1,h_2\in\mathcal{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}),h_2(\mathrm{H}))=(h(\mathrm{T}),h(\mathrm{H}))>0$, whence $A\in K^2_{\mathrm{H,T}}$. We conclude that indeed $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$.

For (iii)—to show that $K^2_{\mathrm{H,T}}\subseteq\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0}\cup\mathcal{A})$—consider any gamble set A in $K^2_{\mathrm{H,T}}$. Then $A\supseteq\{h_1,h_2\}\setminus\mathcal{L}_{\leq 0}=\{h_1,h_2\}$ for some h_1 and h_2 in $\mathcal{L}_{\not\leq 0}$ such that $(h_1(\mathrm{T}),h_2(\mathrm{H}))>0$. Without loss of generality, rename the gambles in

$$A=\{f^{\mathrm{I}}_1,\dots,f^{\mathrm{I}}_{n_{\mathrm{I}}},f^{\mathrm{II}}_1,\dots,f^{\mathrm{II}}_{n_{\mathrm{II}}},f^{\mathrm{III}}_1,\dots,f^{\mathrm{III}}_{n_{\mathrm{III}}},f^{\mathrm{IV}}_1,\dots,f^{\mathrm{IV}}_{n_{\mathrm{IV}}}\},$$

with n_{I}, n_{II}, n_{III} and n_{IV} in $\{0\}\cup\mathbb{N}$ such that $n\coloneqq 2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+n_{\mathrm{IV}}\geq 1$, gambles $f_1^{\mathrm{I}}, \dots, f_{n_{\mathrm{I}}}^{\mathrm{I}}$ in the positive quadrant $\mathcal{L}_{>0}$, gambles $f_1^{\mathrm{II}}, \dots, f_{n_{\mathrm{II}}}^{\mathrm{II}}$ in the second quadrant $\mathcal{L}_{\mathrm{II}}\coloneqq\{f\in\mathcal{L}\colon f(\mathrm{H})<0<f(\mathrm{T})\}$, gambles $f_1^{\mathrm{III}}, \dots, f_{n_{\mathrm{III}}}^{\mathrm{III}}$ in the negative quadrant $\mathcal{L}_{\leq 0}$, and gambles $f_1^{\mathrm{IV}}, \dots, f_{n_{\mathrm{IV}}}^{\mathrm{IV}}$ in the fourth quadrant $\mathcal{L}_{\mathrm{IV}}\coloneqq\{f\in\mathcal{L}\colon f(\mathrm{T})<0<f(\mathrm{H})\}$. We must show that A belongs to $\mathrm{Posi}(\mathcal{L}_{>0}^{\mathrm{s}}\cup\mathcal{A})$. To this end, we will construct n gamble sets $A_1, \dots, A_n$ and, for every $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda_{1:n}^{f_{1:n}}>0$ such that $A=\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^n A_k\right\}$.

Let $A_1\coloneqq\{g_1\}\in\mathcal{L}_{>0}^{\mathrm{s}}, \dots, A_{n_{\mathrm{I}}}\coloneqq\{g_{n_{\mathrm{I}}}\}\in\mathcal{L}_{>0}^{\mathrm{s}}$. We consider the additional n_{I} gamble sets $A_{n_{\mathrm{I}}+1}\coloneqq\dots\coloneqq A_{2n_{\mathrm{I}}}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+1\}\in\mathcal{A}$, in order to have enough freedom in selecting the coefficients $\lambda_{1:n}^{f_{1:n}}>0$ later on. For every i in $\{1,\dots,n_{\mathrm{II}}\}$, let $A_{2n_{\mathrm{I}}+i}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_i,-\mathbb{I}_{\{\mathrm{T}\}}+\delta\}\in\mathcal{A}$ with $\varepsilon_i\coloneqq\frac{f_i^{\mathrm{II}}(\mathrm{T})}{f_i^{\mathrm{II}}(\mathrm{T})-f_i^{\mathrm{II}}(\mathrm{H})}>0$ and $\delta\coloneqq\frac{f_1^{\mathrm{IV}}(\mathrm{H})}{f_1^{\mathrm{IV}}(\mathrm{H})-f_1^{\mathrm{IV}}(\mathrm{T})}>0$ if $n_{\mathrm{IV}}\geq 1$, otherwise $\delta\coloneqq 1$. For every i in $\{1,\dots,n_{\mathrm{III}}\}$, if $f_i^{\mathrm{III}}\neq 0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+\frac{1}{4},-\mathbb{I}_{\{\mathrm{T}\}}+1\}\in\mathcal{A}$ and $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{4}\}\in\mathcal{A}$; if $f_i^{\mathrm{III}}=0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i}\coloneqq A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+\frac{1}{2},-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{2}\}\in\mathcal{A}$. Finally, for every i in $\{1,\dots,n_{\mathrm{IV}}\}$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+i}\coloneqq\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_i\}\in\mathcal{A}$ with $\delta_i\coloneqq\frac{f_i^{\mathrm{IV}}(\mathrm{H})}{f_i^{\mathrm{IV}}(\mathrm{H})-f_i^{\mathrm{IV}}(\mathrm{T})}>0$.

The set $\times_{k=1}^n A_k$ contains $2^{n-n_{\mathrm{I}}}=2^{n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+n_{\mathrm{IV}}}$ sequences. Each such sequence $f_{1:n}$ is characterised by a choice of f_i in the binary set A_i—which we will denote by $\{g_i^{\mathrm{H}},g_i^{\mathrm{T}}\}$, where g_i^{H} is the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon$ and g_i^{T} the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{T}\}}+\delta$—, for every i in $\{n_{\mathrm{I}}+1,\dots,n\}$. For the first n_{I} entries $f_{1:n_{\mathrm{I}}}$ of $f_{1:n}$ we have no choice but to chose $f_{1:n_{\mathrm{I}}}=g_{1:n_{\mathrm{I}}}$, since $\times_{k=1}^{n_{\mathrm{I}}}A_k$ is the singleton $\{g_{1:n_{\mathrm{I}}}\}$.

For any sequence $f_{1:n}$ in $\times_{k=1}^n A_k$, define n real coefficients $\lambda_{1:n}^{f_{1:n}}$ as follows:

- Situation (a): If there is an i in $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$ such that

$$\begin{aligned}(&f_{2n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_n)\\ &=(g_{2n_{\mathrm{I}}+1}^{\mathrm{T}},\dots,g_{i-1}^{\mathrm{T}},g_i^{\mathrm{H}},g_{i+1}^{\mathrm{T}},\dots,g_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}^{\mathrm{T}},g_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1}^{\mathrm{H}},\dots,g_n^{\mathrm{H}})\end{aligned}$$

  or, in other words, such that $f_i=g_i^{\mathrm{H}}$, $(\forall k\in\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}\setminus\{i\})f_k=g_k^{\mathrm{T}}$, and $(\forall k\in\{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\})f_k=g_k^{\mathrm{H}}$, then let

$$\begin{aligned}\lambda_i^{f_{1:n}}&\coloneqq f_j^{\mathrm{II}}(\mathrm{T})-f_j^{\mathrm{II}}(\mathrm{H})>0 \text{ for } j\coloneqq i-2n_{\mathrm{I}},\\ \lambda_k^{f_{1:n}}&\coloneqq 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.\end{aligned}$$

- Situation (b): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$ such that

$$\begin{aligned}(&f_{2n_{\mathrm{I}}+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_n)\\ &=(g_{2n_{\mathrm{I}}+1}^{\mathrm{T}},\dots,g_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}^{\mathrm{T}},g_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1}^{\mathrm{H}},\dots,g_{i-1}^{\mathrm{H}},g_i^{\mathrm{T}},g_{i+1}^{\mathrm{H}},\dots,g_n^{\mathrm{H}}),\end{aligned}$$

  or, in other words, such that $f_i=g_i^{\mathrm{T}}$, $(\forall k\in\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\})f_k=g_k^{\mathrm{T}}$, and $(\forall k\in\{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\}\setminus\{i\})f_k=g_k^{\mathrm{H}}$, then let

$$\begin{aligned}\lambda_i^{f_{1:n}}&\coloneqq f_j^{\mathrm{IV}}(\mathrm{H})-f_j^{\mathrm{IV}}(\mathrm{T})>0 \text{ for } j\coloneqq i-2n_{\mathrm{I}}-n_{\mathrm{II}}-2n_{\mathrm{III}},\\ \lambda_k^{f_{1:n}}&\coloneqq 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.\end{aligned}$$

- Situation (c): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$ such that

$$\begin{aligned}(&f_{2n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},\\ &f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{n_{\mathrm{III}}+i-1},f_{n_{\mathrm{III}}+i},f_{n_{\mathrm{III}}+i+1},\dots,f_n)\end{aligned}$$

$$= (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_{i},g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},\\ g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_{n_{\mathrm{III}}+i-1},g^{\mathrm{T}}_{n_{\mathrm{III}}+i},g^{\mathrm{H}}_{n_{\mathrm{III}}+i+1},\dots,g^{\mathrm{H}}_{n}),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$, $f_{n_{\mathrm{III}}+i} = g^{\mathrm{T}}_{n_{\mathrm{III}}+i}$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\} \setminus \{i\}) f_k = g^{\mathrm{T}}_k$ and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\} \setminus \{n_{\mathrm{III}}+i\}) f_k = g^{\mathrm{H}}_k$, then let

$$\begin{aligned}
&\lambda^{f_{1:n}}_k \coloneqq 0 \text{ for all } k \text{ in } \{1,\dots,n\} \setminus \{i, n_{\mathrm{III}}+i\};\\
&\lambda^{f_{1:n}}_i \coloneqq \lambda^{f_{1:n}}_{n_{\mathrm{III}}+i} \coloneqq 1 \text{ if } f^{\mathrm{III}}_{i-2n_{\mathrm{I}}-n_{\mathrm{II}}} = 0,\\
&\lambda^{f_{1:n}}_i \coloneqq -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T})) > 0 \text{ and}\\
&\lambda^{f_{1:n}}_{n_{\mathrm{III}}+i} \coloneqq -\frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) + 3f^{\mathrm{III}}_j(\mathrm{T})) > 0 \text{ if } f^{\mathrm{III}}_{i-2n_{\mathrm{I}}-n_{\mathrm{II}}} \neq 0.
\end{aligned}$$

- Situation (d): If none of the Situations (a), (b) nor (c) apply, and if there is an i in $\{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}$ such that

$$(f_{n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}}) = (g^{\mathrm{T}}_{n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}}),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$ and $(\forall k \in \{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\} \setminus \{i\}) f_k = g^{\mathrm{T}}_k$, then let

$$\begin{aligned}
&\lambda^{f_{1:n}}_{i-n_{\mathrm{I}}} \coloneqq 1,\\
&\lambda^{f_{1:n}}_k \coloneqq 0 \text{ for all } k \text{ in } \{1,\dots,n\} \setminus \{i-n_{\mathrm{I}}\}.
\end{aligned}$$

- Situation (e1): If $A \cap \mathcal{L}_{>0} \neq \emptyset$—so $n_{\mathrm{I}} \geq 1$—and none of the Situations (a), (b), (c) nor (d) apply, then let

$$\lambda^{f_{1:n}}_1 \coloneqq 1 \text{ and } \lambda^{f_{1:n}}_{2:n} \coloneqq 0.$$

- Situation (e2): If $A \cap \mathcal{L}_{>0} = \emptyset$—so $n_{\mathrm{II}} \geq 1$ and $n_{\mathrm{IV}} \geq 1$ because $(h_1(\mathrm{T}), h_2(\mathrm{H})) > 0$—and none of the Situations (a), (b), (c) nor (d) apply, then let, with $i \coloneqq 2n_{\mathrm{I}}+1$,

$$\begin{aligned}
&\lambda^{f_{1:n}}_i \coloneqq f^{\mathrm{II}}_1(\mathrm{T}) - f^{\mathrm{II}}_1(\mathrm{H}) > 0 \text{ if } f_i = g^{\mathrm{H}}_i,\\
&\lambda^{f_{1:n}}_i \coloneqq f^{\mathrm{IV}}_1(\mathrm{H}) - f^{\mathrm{IV}}_1(\mathrm{T}) > 0 \text{ if } f_i = g^{\mathrm{T}}_i,\\
&\lambda^{f_{1:n}}_k \coloneqq 0 \text{ or all } k \text{ in } \{1,\dots,n\} \setminus \{i\}.
\end{aligned}$$

In this way, we have defined coefficients $\lambda^{f_{1:n}}_{1:n} > 0$ for every $f_{1:n}$ in $\times_{k=1}^n A_k$. It only remains to show, with our choices of $\lambda^{f_{1:n}}_{1:n} > 0$, that $A = \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k : f_{1:n} \in \times_{k=1}^n A_k\}$.

We first prove that $A \subseteq \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k : f_{1:n} \in \times_{k=1}^n A_k\}$. We do this by checking that every element of A belongs to $\{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k : f_{1:n} \in \times_{k=1}^n A_k\}$. To show that $f^{\mathrm{II}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k : f_{1:n} \in \times_{k=1}^n A_k\}$ for every j in $\{1,\dots,n_{\mathrm{II}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (a) for $i \coloneqq j + 2n_{\mathrm{I}}$, which is then an element of $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$. Then

$$\begin{aligned}
\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k &= (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H})) g^{\mathrm{H}}_i\\
&= (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H})) \left(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f^{\mathrm{II}}_i(\mathrm{T})}{f^{\mathrm{II}}_i(\mathrm{T}) - f^{\mathrm{II}}_i(\mathrm{H})} \right)\\
&= (f^{\mathrm{II}}_j(\mathrm{H}) - f^{\mathrm{II}}_j(\mathrm{T})) \mathbb{I}_{\{\mathrm{H}\}} + f^{\mathrm{II}}_j(\mathrm{T}) = f^{\mathrm{II}}_j,
\end{aligned}$$

so indeed $f^{\mathrm{II}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k : f_{1:n} \in \times_{k=1}^n A_k\}$.

To show that $f_j^{\mathrm{IV}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$ for every j in $\{1,\dots,n_{\mathrm{IV}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (b) for $i \coloneqq j + 2n_{\mathrm{I}} + n_{\mathrm{II}} + 2n_{\mathrm{III}}$, which is then an element of $\{2n_{\mathrm{I}} + n_{\mathrm{II}} + 2n_{\mathrm{III}} + 1,\dots,n\}$. Then

$$\begin{aligned}\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k &= (f_j^{\mathrm{IV}}(\mathrm{H}) - f_j^{\mathrm{IV}}(\mathrm{T}))g_i^{\mathrm{T}} \\ &= (f_j^{\mathrm{IV}}(\mathrm{H}) - f_j^{\mathrm{IV}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{T}\}} + \frac{f_i^{\mathrm{IV}}(\mathrm{H})}{f_i^{\mathrm{IV}}(\mathrm{H}) - f_i^{\mathrm{IV}}(\mathrm{T})}\right) \\ &= (f_j^{\mathrm{IV}}(\mathrm{T}) - f_j^{\mathrm{IV}}(\mathrm{H}))\mathbb{I}_{\{\mathrm{T}\}} + f_j^{\mathrm{IV}}(\mathrm{H}) = f_j^{\mathrm{IV}},\end{aligned}$$

so indeed $f_j^{\mathrm{IV}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$.

To show that $f_j^{\mathrm{III}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$ for every j in $\{1,\dots,n_{\mathrm{III}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (c) for $i \coloneqq j + 2n_{\mathrm{I}} + n_{\mathrm{II}}$, which is then an element of $\{2n_{\mathrm{I}} + n_{\mathrm{II}} + 1,\dots,2n_{\mathrm{I}} + n_{\mathrm{II}} + n_{\mathrm{III}}\}$. Then

$$\begin{aligned}\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k &= -\frac{1}{2}(3f_j^{\mathrm{III}}(\mathrm{H}) + f_j^{\mathrm{III}}(\mathrm{T}))g_i^{\mathrm{H}} - \frac{1}{2}(f_j^{\mathrm{III}}(\mathrm{H}) + 3f_j^{\mathrm{III}}(\mathrm{T}))g_i^{\mathrm{T}} \\ &= -\frac{1}{2}(3f_j^{\mathrm{III}}(\mathrm{H}) + f_j^{\mathrm{III}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}\right) - \frac{1}{2}(f_j^{\mathrm{III}}(\mathrm{H}) + 3f_j^{\mathrm{III}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{4}\right) \\ &= f_j^{\mathrm{III}}(\mathrm{H})\mathbb{I}_{\{\mathrm{H}\}} + f_j^{\mathrm{III}}(\mathrm{T})\mathbb{I}_{\{\mathrm{T}\}} = f_j^{\mathrm{III}}\end{aligned}$$

if $f_j^{\mathrm{III}} \neq 0$, and

$$\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = g_i^{\mathrm{H}} + g_i^{\mathrm{T}} = -\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{2} - \mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{2} = 0 = f_j^{\mathrm{III}}$$

if $f_j^{\mathrm{III}} = 0$, so indeed $f_j^{\mathrm{III}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$.

To show that $f_j^{\mathrm{I}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$ for every j in $\{1,\dots,n_{\mathrm{I}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (d) for $i \coloneqq j + n_{\mathrm{I}}$, which is then an element of $\{n_{\mathrm{I}} + 1,\dots,2n_{\mathrm{I}}\}$. Then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = g_j = f_j^{\mathrm{I}}$, so indeed $f_j^{\mathrm{I}} \in \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$.

We finally show, conversely, that $A \supseteq \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$. Consider any f in $\left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$. Then $f = \sum_{k=1}^n \lambda_k^{f_{1:n}} f_k$ for some $f_{1:n}$ in $\times_{k=1}^n A_k$. If this $f_{1:n}$ satisfies the conditions of Situation (a) for some i in $\{2n_{\mathrm{I}} + 1,\dots,2n_{\mathrm{I}} + n_{\mathrm{II}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{II}}$ for $j \coloneqq i - 2n_{\mathrm{I}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (b) for some i in $\{2n_{\mathrm{I}} + n_{\mathrm{II}} + 2n_{\mathrm{III}} + 1,\dots,n\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{IV}}$ for $j \coloneqq i - 2n_{\mathrm{I}} - n_{\mathrm{II}} - 2n_{\mathrm{III}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (c) for some i in $\{2n_{\mathrm{I}} + n_{\mathrm{II}} + 1,\dots,2n_{\mathrm{I}} + n_{\mathrm{II}} + n_{\mathrm{III}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{III}}$ for $j \coloneqq i - 2n_{\mathrm{I}} - n_{\mathrm{II}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (d) for some i in $\{n_{\mathrm{I}} + 1,\dots,2n_{\mathrm{I}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{I}}$ for $j \coloneqq i - n_{\mathrm{I}}$, as shown above, so $f \in A$. The only other possibility is that $f_{1:n}$ satisfies the conditions of Situation (e1) or (e2), depending on whether or not $A \cap \mathcal{L}_{>0} \neq \emptyset$. If $A \cap \mathcal{L}_{>0} \neq \emptyset$ (so Situation (e1)), then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_1^{\mathrm{I}}$, which is an element of A since $n_{\mathrm{I}} \geq 1$, so $f \in A$. If $A \cap \mathcal{L}_{>0} = \emptyset$ (so Situation (e2)), then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f_1^{\mathrm{II}}(\mathrm{T}) - f_1^{\mathrm{II}}(\mathrm{H}))\left(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f_1^{\mathrm{II}}(\mathrm{T})}{f_1^{\mathrm{II}}(\mathrm{T}) - f_i^{\mathrm{II}}(\mathrm{H})}\right) = f_1^{\mathrm{II}}$ or $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f_1^{\mathrm{IV}}(\mathrm{H}) - f_1^{\mathrm{IV}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f_1^{\mathrm{IV}}(\mathrm{H})}{f_1^{\mathrm{IV}}(\mathrm{H}) - f_i^{\mathrm{IV}}(\mathrm{T})}\right) = f_1^{\mathrm{IV}}$, which both belong to A

since $n_{\mathrm{II}} \geq 1$ and $n_{\mathrm{IV}} \geq 1$, so $f \in A$. There are no other possibilities, so we conclude that indeed $A \supseteq \left\{\sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k : f_{1:n} \in \times_{k=1}^{n} A_k\right\}$. □

Proof of Proposition 5. By definition, the least informative coherent set of desirable gamble sets that includes $\{\{f\} : f \in D\}$ is the natural extension $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A}_D)$ of the assessment $\mathcal{A}_D \coloneqq \{\{f\} : f \in D\}$.

Let us first show that $\mathcal{A}_D$ is consistent. By Theorem 2, we need to show that $\emptyset \notin \mathcal{A}_D$ and $\{0\} \notin \mathrm{Posi}(\mathcal{L}_{>0}^{\mathrm{s}} \cup \mathcal{A}_D) = \mathrm{Posi}(\mathcal{A}_D)$, where the equality follows from the fact that $\mathcal{L}_{>0}^{\mathrm{s}} \subseteq \mathcal{A}_D$ by Axiom D_2. By definition, $\emptyset \notin \mathcal{A}_D$, so it remains to prove that $\{0\} \notin \mathrm{Posi}(\mathcal{A}_D)$. To this end, consider any singleton $\{g\}$ in $\mathrm{Posi}(\mathcal{A}_D)$. There are n in $\mathbb{N}$, $f_1, \ldots, f_n$ in D, and coefficients $\lambda_{1:n}^{f_{1:n}} > 0$, such that $\sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k = g$. Since $f_1, \ldots f_n$ belong to D, so does $\sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k$, by repeated application of Axiom D_3. So $g \in D$. By Axiom D_1, $0 \notin D$, whence indeed $g \neq 0$.

We now know that $\mathcal{A}_D$ is consistent, so by Theorem 2, its natural extension $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A}_D)$ is equal to $\mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$, since we already know that $\mathcal{L}_{>0}^{\mathrm{s}} \subseteq \mathcal{A}_D$. Let us show that $K_D = \mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$; we prove (i) $K_D \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$ and (ii) $K_D \supseteq \mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$. For (i), consider any A in K_D, so $A \cap D \neq \emptyset$, and therefore $f \in A$ for some f in D. This tells us that $\{f\} \in \mathcal{A}_D$. Since $K \subseteq \mathrm{Posi}(K)$ for any K in $\mathbf{K}$, we find that $\{f\} \in \mathrm{Posi}(\mathcal{A}_D)$. Therefore, any superset of $\{f\}$—and in particular indeed the set A—will belong to $\mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$.

Let us now show that (ii) $K_D \supseteq \mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$. To this end, consider any element A of $\mathrm{Rs}(\mathrm{Posi}(\mathcal{A}_D))$. Then, by the definition of the Rs operator, there is some B in $\mathrm{Posi}(\mathcal{A}_D)$ such that $B \setminus \mathcal{L}_{\leq 0} \subseteq A$. This means that there are n in $\mathbb{N}$, $A_1, \ldots, A_n$ in $\mathcal{L}_{>0}^{\mathrm{s}} \cup \mathcal{A}_D$, and, for all $f_{1:n}$ in $\times_{k=1}^{n} A_k$, coefficients $\lambda_{1:n}^{f_{1:n}} > 0$, such that $\left\{\sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k : f_{1:n} \in \times_{k=1}^{n} A_k\right\} = B$. Since the entries of any sequence $f_{1:n}$ in $\times_{k=1}^{n} A_k$ belong to $\mathcal{L}_{>0} \cup D$, so does $\sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k$, by repeated application of Axiom D_3. So we find $B \subseteq D \cup \mathcal{L}_{>0} = D$, where the equality is a consequence of Axiom D_2, and hence $B \setminus \mathcal{L}_{\leq 0} \subseteq D$. So $A \cap D \neq \emptyset$, and therefore indeed $A \in \mathbf{K}_D$.

We finish the proof by showing that K_D is indeed compatible with D, or, in other words, that $D_{K_D} = D$. Indeed, $D_{K_D} = \{f \in \mathcal{L} : \{f\} \in K_D\} = \{f \in \mathcal{L} : \{f\} \cap D \neq \emptyset\} = D$. □

Proof of Proposition 7. For Axiom K_0, consider any A in $K\rfloor E$. Then $\mathbb{I}_E A \in K$, whence $\mathbb{I}_E A \neq \emptyset$ since K satisfies Axiom K_0. Therefore indeed $A \neq \emptyset$.

For Axiom K_1, consider any A in $K\rfloor E$. Then $\mathbb{I}_E A \in K$, whence $\mathbb{I}_E A \setminus \{0\} \in K$ since K satisfies Axiom K_1. Since $\mathbb{I}_E f \neq 0 \Leftrightarrow f \neq 0$ for any gamble f on E, we find that $\mathbb{I}_E(A \setminus \{0\}) \in K$, whence indeed $A \setminus \{0\} \in K\rfloor E$.

For Axiom K_2, consider any f in $\mathcal{L}(E)_{>0}$. Then $\mathbb{I}_E f \in \mathcal{L}(\mathcal{X})_{>0}$, whence by Axiom K_2 $\{\mathbb{I}_E f\} \in K$. Therefore indeed $\{f\} \in K\rfloor E$.

For Axiom K_3, consider any A_1 and A_2 in $K\rfloor E$, and, for any f in A_1 and g in A_2, any $(\lambda_{f,g}, \mu_{f,g}) > 0$. Then $\mathbb{I}_E A_1 \in K$ and $\mathbb{I}_E A_2 \in K$, whence by Axiom K_3 $\{\lambda_{f,g} f + \mu_{f,g} g : f \in \mathbb{I}_E A_1, g \in \mathbb{I}_E A_2\} = \{\lambda_{f,g} \mathbb{I}_E f + \mu_{f,g} \mathbb{I}_E g : f \in A_1, g \in A_2\} = \mathbb{I}_E \{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\} \in K$, where we identified $(\lambda_{\mathbb{I}_E f, \mathbb{I}_E g}, \mu_{\mathbb{I}_E f, \mathbb{I}_E g})$ with $(\lambda_{f,g}, \mu_{f,g})$, for any f in A_1 and g in A_2. Therefore indeed $\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\} \in K\rfloor E$.

For Axiom K_4, consider any A_1 in $K\rfloor E$ and any A_2 in $\mathcal{Q}(\mathcal{L}(E))$ such that $A_1 \subseteq A_2$. Then $\mathbb{I}_E A_1 \in K$ and $\mathbb{I}_E A_1 \subseteq \mathbb{I}_E A_2$, whence $\mathbb{I}_E A_2 \in K$ by Axiom K_4. Therefore indeed $A_2 \in K\rfloor E$. □

Proof of Proposition 8. For the first statement, consider any f in $\mathcal{L}(E)$, and infer the following chain of equivalences:

$$f \in D_K \rfloor E \Leftrightarrow \mathbb{I}_E f \in D_K \Leftrightarrow \{\mathbb{I}_E f\} \in K \Leftrightarrow \{f\} \in K\rfloor E \Leftrightarrow f \in D_{K\rfloor E},$$

where the first equivalence follows from Definition 7, the second one and the last one are due to Equation (5), and the third one follows from Definition 8.

For the second statement, consider any A in $\mathcal{Q}(\mathcal{L}(E))$ and the following chain of equivalences:

$$A \in K_D\rfloor E \Leftrightarrow \mathbb{I}_E A \in K_D \Leftrightarrow \mathbb{I}_E A \cap D \neq \emptyset \Leftrightarrow (\exists f \in A)\mathbb{I}_E f \in D \Leftrightarrow A \cap D\rfloor E \neq \emptyset \Leftrightarrow A \in K_{D\rfloor E},$$

where the first equivalence follows from Definition 8, the second one and the last one are due to Proposition 5, and the fourth one follows from Definition 7.

We now turn to the last statement. By Theorem 6 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}))$. Therefore in particular, for any A in $\mathcal{Q}(\mathcal{L}(E))$,

$$\begin{aligned} A \in K\rfloor E \Leftrightarrow \mathbb{I}_E A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))\mathbb{I}_E A \in K_D \\ \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D\rfloor E \\ \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{D\rfloor E} \\ \Leftrightarrow A \in \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 8, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $K\rfloor E = \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}$. $\square$

Proof of Proposition 9. The result follows immediately, once we realise that $A_1 \neq \emptyset \Leftrightarrow A_1^* \neq \emptyset$, that $f > 0 \Leftrightarrow f^* > 0$, that $\lambda f + \mu g \in A_1 \Leftrightarrow \lambda f^* + \mu g^* \in A_1^*$, and that $A_1 \subseteq A_2 \Leftrightarrow A_1^* \subseteq A_2^*$, for all f in $\mathcal{L}(\mathcal{X}_O)$ whose cylindrical extension is f^*, all A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ whose cylindrical extensions are A_1^* and A_2^*, and all λ in μ in $\mathbb{R}$ such that $(\lambda, \mu) > 0$. $\square$

Proof of Proposition 10. For the first statement, observe that indeed

$$\begin{aligned} \operatorname{marg}_O D_K = \{f \in \mathcal{L}(\mathcal{X}_O) : f \in D_K\} &= \{f \in \mathcal{L}(\mathcal{X}_O) : \{f\} \in K\} \\ &= \{f \in \mathcal{L}(\mathcal{X}_O) : \{f\} \in \operatorname{marg}_O K\} = D_{\operatorname{marg}_O K}, \end{aligned}$$

where the second and last equalities follow from Equation (5), and the third one follows from Definition 10.

For the second statement, observe that

$$\begin{aligned} \operatorname{marg}_O K_D = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \in K_D\} &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \cap D \neq \emptyset\} \\ &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \cap \operatorname{marg}_O D \neq \emptyset\} \\ &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \in K_{\operatorname{marg}_O D}\} = K_{\operatorname{marg}_O D}, \end{aligned}$$

where the first equality follows from Definition 10 and the second and penultimate equalities follow from Proposition 5.

We now turn to the last statement. By Theorem 6 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n}))$. Therefore in particular, for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$,

$$\begin{aligned} A \in \operatorname{marg}_O K \Leftrightarrow A \in K &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in \operatorname{marg}_O K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{\operatorname{marg}_O D} \Leftrightarrow A \in \bigcap\{K_{\operatorname{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 10, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $\operatorname{marg}_O K = \bigcap\{K_{\operatorname{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}$. $\square$

Proof of Proposition 11. We will first show that any coherent set of desirable gamble sets K' on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to K_O must be at least as informative as the set $\mathrm{ext}_{1:n}(K_O)$ given by Eq. (13). To establish this, since K' marginalises to K_O, note that $A \in K_O \Leftrightarrow A \in K'$, for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$. Therefore, in particular, $A \in K_O \Rightarrow A \in K'$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$, so $K_O \subseteq K'$. This implies that indeed $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup K_O)) \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup K')) = K'$, where the final equality holds because K' is coherent.

So we already know that any coherent set of desirable gamble sets that marginalises to K_O must be at least as informative as $\mathrm{ext}_{1:n}(K_O)$. It therefore suffices to prove that $\mathrm{ext}_{1:n}(K_O)$ is coherent and that it marginalises to K_O. To show that $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup \mathcal{A}_{K_O}^{1:n}))$ is coherent, by Theorem 2 it suffices to show that K_O is a consistent assessment—that is, to show that $\emptyset \notin \mathcal{A}_{K_O}^{1:n}$ and $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0} \cup \mathcal{A}_{K_O}^{1:n})$. That this is indeed the case follows from the coherence of $K_O = \mathcal{A}_{K_O}^{1:n}$.

The proof is therefore complete if we can show that $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O)) = K_O$. Since for any A in K_O it is obvious that both $A \in \mathrm{ext}_{1:n}(K_O)$ and $A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$, we see immediately that $K_O \subseteq \mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, so we concentrate on proving the converse inclusion. Consider any A in $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, meaning that both $A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and $A \in \mathrm{ext}_{1:n}(K_O)$. That $A \in \mathrm{ext}_{1:n}(K_O)$ implies that $B \setminus \mathcal{L}_{\leq 0} \subseteq A$ for some B in $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup K_O)$. Then there are m in $\mathbb{N}$, $A_1, \ldots, A_m$ in $\mathcal{L}^{\mathrm{s}}_{>0} \cup K_O$, and coefficients $\lambda_{1:m}^{f_{1:m}} > 0$ for all $f_{1:m}$ in $\times_{k=1}^m A_k$ such that $B = \left\{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^m A_k\right\}$. Without loss of generality, assume that $A_1, \ldots, A_\ell \in K_O$ and $A_{\ell+1}, \ldots, A_m \in \mathcal{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0, \ldots, m\}$. Consider the special subset $P \coloneqq \left\{f_{1:m} \in \times_{k=1}^m A_k : \lambda_{1:\ell}^{f_{1:m}} = 0\right\}$ of $\times_{k=1}^m A_k$. If $P \neq \emptyset$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^m \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathcal{L}_{>0} \neq \emptyset$. Since $B \setminus \mathcal{L}_{\leq 0} \subseteq A$, also $A \cap \mathcal{L}_{>0}(\mathcal{X}_O) \neq \emptyset$, whence $A \in K_O$ by coherence [more specifically, by Axioms K_2 and K_4]. Therefore, assume that $P = \emptyset$, and define the coefficients

$$\mu_k^{f_{1:m}} \coloneqq \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell + 1 \end{cases}$$

for all $f_{1:m}$ in $\times_{k=1}^m A_k$ and k in $\{1, \ldots, m\}$. Because $P = \emptyset$, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. Also, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ and $k \geq \ell + 1$, the coefficient $\mu_k^{f_{1:m}}$ equals 0, so we identify $\mu_{1:\ell}^{f_{1:\ell}}$ with $\mu_{1:\ell}^{f_{1:m}}$. Then every element of $\left\{\sum_{k=1}^m \mu_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^m A_k\right\} = \left\{\sum_{k=1}^\ell \mu_k^{f_{1:\ell}} f_k : f_{1:\ell} \in \times_{k=\ell+1}^m A_k\right\} \in \mathrm{Posi}(K_O) = K_O$ is dominated by an element of B. Therefore, by Lemma 16 below $B \in K_O$, whence by coherence, indeed also $A \in K_O$. □

Lemma 16. *Consider any coherent set of desirable gamble sets K and any gamble sets A and B in $\mathcal{Q}$. If $A \in K$ and $(\forall f \in A)(\exists g \in B) f \leq g$, then $B \in K$.*

Proof. Let $A \coloneqq \{f_1, \ldots, f_m\}$ for some m in $\mathbb{N}$, and denote the finite possibility space $\mathcal{X} = \{x_1, \ldots, x_\ell\}$ for some ℓ in $\mathbb{N}$. Since $(\forall f \in A)(\exists g \in B) f \leq g$, we have that B is a superset of

$$B' \coloneqq \left\{f_1 + \sum_{k=1}^{\ell} \mu_{k,1} \mathbb{I}_{\{x_k\}}, \ldots, f_m + \sum_{k=1}^{\ell} \mu_{k,m} \mathbb{I}_{\{x_k\}}\right\} = \left\{f_j + \sum_{k=1}^{\ell} \mu_{k,j} \mathbb{I}_{\{x_k\}} : j \in \{1, \ldots, m\}\right\}$$

for some $\mu_{k,j} \geq 0$ for all k in $\{1, \ldots, \ell\}$ and j in $\{1, \ldots, m\}$. Use the definition of the Posi operator, with $A_1 \coloneqq \{\mathbb{I}_{\{x_1\}}\} \in K$, ..., $A_\ell \coloneqq \{\mathbb{I}_{\{x_\ell\}}\} \in K$, $A_{\ell+1} \coloneqq A \in K$, and for all $f_{1:\ell+1}^j \coloneqq$

$(\mathbb{I}_{\{x_1\}},\mathbb{I}_{\{x_2\}},\dots,\mathbb{I}_{\{x_\ell\}},f_j)\in\times_{k=1}^{\ell+1}A_k$, let $\lambda_{1:\ell+1}^{f^j_{1:\ell+1}}\coloneqq(\mu_{1,j},\mu_{2,j},\dots,\mu_{\ell,j},1)>0$, to infer that

$$\left\{\sum_{k=1}^{\ell+1}\lambda_k^{f^j_{1:\ell+1}}f^j_k : f^j_{1:\ell+1}\in\bigtimes_{k=1}^{\ell+1}A_k\right\}=\left\{f_j+\sum_{k=1}^{\ell}\mu_{k,j}\mathbb{I}_{\{x_k\}}:j\in\{1,\dots,m\}\right\}=B'$$

belongs to $\operatorname{Posi}(K)$. Because $B\supseteq B'$, we have that $B\in\operatorname{Rs}(\operatorname{Posi}(K))$ by the definition of Rs. But since $K=\operatorname{Rs}(\operatorname{Posi}(K))$ by coherence, we infer that indeed $B\in K$. □

Proof of Proposition 12. [5] Since we have seen in Proposition 11 that $\operatorname{ext}_{1:n}(K_O)$ is the smallest coherent set of desirable gamble sets that includes K_O, we know by Theorem 6 that

$$\begin{aligned}\operatorname{ext}_{1:n}(K_O)&=\bigcap\{K_D:D\in\overline{\mathbf{D}}(\mathcal{X}_{1:n})\text{ and }K_O\subseteq K_D\}\\&=\bigcap\{K_D:D\in\overline{\mathbf{D}}(\mathcal{X}_{1:n})\text{ and }K_O\subseteq\operatorname{marg}_O K_D=K_{\operatorname{marg}_O D}\}=\bigcap\{K_D:D\in\mathbf{D}^*\}\end{aligned}$$

where we used Definition 10 in the first equality, Proposition 10 in the second equality, and defined $\mathbf{D}^*\coloneqq\{D\in\overline{\mathbf{D}}(\mathcal{X}_{1:n}):K_O\subseteq\operatorname{marg}_O K_D\}$ for brevity. The proof is finished if we show that $\bigcap\{K_D:D\in\mathbf{D}^*\}=\bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}:D_O\in\overline{\mathbf{D}}(K_O)\}$.

We will first show that $\bigcap\{K_D:D\in\mathbf{D}^*\}\subseteq\bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}:D_O\in\overline{\mathbf{D}}(K_O)\}$. To this end, consider any D_O in $\overline{\mathbf{D}}(K_O)$—implying that $K_O\subseteq K_{D_O}$—, and we will show that $K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}\in\{K_D:D\in\mathbf{D}^*\}$. For notational ease, let $D^*\coloneqq\operatorname{ext}^{\mathbf{D}}_{1:n}D_O\in\overline{\mathbf{D}}(\mathcal{X}_{1:n})$. By the marginalisation property of the weak extension for sets of desirable gambles proved by De Cooman and Miranda [15, Proposition 7], which implies that $\operatorname{marg}_O D^*=D_O$, we have that $\operatorname{marg}_O K_{D^*}=K_{D_O}$, and hence, since $K_O\subseteq K_{D_O}$, that $K_O\subseteq\operatorname{marg}_O K_{D^*}$. But this implies that D^* belongs to $\mathbf{D}^*$, whence indeed $K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}=K_{D^*}\in\{K_D:D\in\mathbf{D}^*\}$.

To show, conversely, that $\bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}:D_O\in\overline{\mathbf{D}}(K_O)\}\subseteq\bigcap\{K_D:D\in\mathbf{D}^*\}$, consider any A in $\bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}:D_O\in\overline{\mathbf{D}}(K_O)\}$, meaning that $A\cap\operatorname{ext}^{\mathbf{D}}_{1:n}D_O\neq\emptyset$ for every D_O in $\overline{\mathbf{D}}(K_O)$. We need to show that then $A\in\bigcap\{K_D:D\in\mathbf{D}^*\}$, so consider any D^* in $\mathbf{D}^*$. That D^* belongs to $\mathbf{D}^*$ implies that $K_O\subseteq K_{\operatorname{marg}_O D^*}$ by the definition of $\mathbf{D}^*$, and hence $\operatorname{marg}_O D^*\in\overline{\mathbf{D}}(K_O)$. This implies that $D'\cap A\neq\emptyset$, where we defined $D'\coloneqq\operatorname{ext}^{\mathbf{D}}_{1:n}(\operatorname{marg}_O D^*)$. But De Cooman and Miranda [15, Proposition 7] have shown that D' is the *smallest* coherent set of desirable gambles on $\mathcal{X}_{1:n}$ that marginalises to $\operatorname{marg}_O D^*$, and therefore $D'\subseteq D^*$. This implies that $A\cap D^*\neq\emptyset$, or, in other words, that $A\in K_{D^*}$. Since the choice of D^* in $\mathbf{D}^*$ was arbitrary, this indeed implies that $A\in\bigcap\{K_{D^*}:D^*\in\mathbf{D}^*\}$. □

Proof of Proposition 13. Consider the following chain of equalities:

$$\begin{aligned}\operatorname{marg}_O(K_n\rfloor E_I)=\{A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_O)):A\in K_n\rfloor E_I\}&=\{A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_O)):\mathbb{I}_{E_I}A\in K_n\}\\&=\{A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_O)):\mathbb{I}_{E_I}A\in\operatorname{marg}_{I\cup O}K_n\}\\&=\{A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_O)):A\in(\operatorname{marg}_{I\cup O}K_n)\rfloor E_I\}\\&=\operatorname{marg}_O((\operatorname{marg}_{I\cup O}K_n)\rfloor E_I),\end{aligned}$$

where the third equality holds because $\mathbb{I}_{E_I}A$ is a set of gambles on $\mathcal{X}_{I\cup O}$. □

[5] The following is an expanded and more detailed version of the proof published in the *International Journal of Approximate Reasoning*; specifically, we establish here the equality $\bigcap\{K_D:D\in\mathbf{D}^*\}=\bigcap\{K_{\operatorname{ext}^{\mathbf{D}}_{1:n}D_O}:D_O\in\overline{\mathbf{D}}(K_O)\}$, that was left there implicit.

Proof of Proposition 14. To show that (i) implies (ii), consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$, and recall the following equivalences:

$$\begin{aligned} A \in K_n &\Leftrightarrow A \in \operatorname{marg}_O(K_n \rfloor E_I) && \text{by Definition 10 and (i)} \\ &\Leftrightarrow A \in K_n \rfloor E_I && \text{by Definition 10} \\ &\Leftrightarrow \mathbb{I}_{E_I} A \in K_n && \text{by Definition 8.} \end{aligned}$$

To show that (ii) implies (i), consider any E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$, and recall the following equalities:

$$\begin{aligned} \operatorname{marg}_O(K_n \rfloor E_I) &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \in K_n \rfloor E_I\} \\ &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : \mathbb{I}_{E_I} A \in K_n\} = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) : A \in K_n\} = \operatorname{marg}_O K_n, \end{aligned}$$

where the first and last equalities follow from Definition 10, the second one from Definition 8, and the third one from (ii). □

Proof of Theorem 15. We will first show that any coherent set of desirable gamble sets K' on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to K_O and that satisfies epistemic irrelevance of X_I to X_O must be at least as informative as $\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O)$. To this end, consider any B in $\mathcal{A}^{\mathrm{irr}}_{I\to O}$. Then $B = \mathbb{I}_{E_I} A$ for some E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$ and A in K_O. Since K' marginalises to K_O, infer that $A \in K'$. Furthermore, since K' satisfies epistemic irrelevance of X_I to X_O, by Proposition 14 also $B = \mathbb{I}_{E_I} A \in K'$. We conclude that $B \in \mathcal{A}^{\mathrm{irr}}_{I\to O} \Rightarrow B \in K' \Leftrightarrow B \in \operatorname{marg}_{I\cup O} K'$, for every B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I\cup O}))$, so $\mathcal{A}^{\mathrm{irr}}_{I\to O} \subseteq \operatorname{marg}_{I\cup O} K'$. This implies that $K^{\mathrm{irr}}_{I\cup O} = \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I\cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I\to O})) \subseteq \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I\cup O})_{>0} \cup \operatorname{marg}_{I\cup O} K')) = \operatorname{marg}_{I\cup O} K'$, where the final equality follows from the fact that $\operatorname{marg}_{I\cup O} K'$ is coherent by Proposition 9. Therefore $\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) = \operatorname{ext}_{1:n}(K^{\mathrm{irr}}_{I\cup O}) \subseteq \operatorname{ext}_{1:n}(\operatorname{marg}_{I\cup O} K')$ and since by Proposition 11 $\operatorname{ext}_{1:n}(\operatorname{marg}_{I\cup O} K')$ is the least informative coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to $\operatorname{marg}_{I\cup O} K'$, we have that $\operatorname{ext}_{1:n}(\operatorname{marg}_{I\cup O} K') \subseteq K'$. Therefore indeed $\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) \subseteq K'$.

The proof of the first statement is therefore complete if we could show that $\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O)$ (i) is coherent, (ii) marginalises to K_O, and (iii) satisfies epistemic irrelevance of X_I to X_O.

For (i), it suffices to show that $\emptyset \notin \mathcal{A}^{\mathrm{irr}}_{I\to O}$ and $\{0\} \notin K^{\mathrm{irr}}_{I\cup O} = \operatorname{Rs}(\operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I\cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I\to O}))$ (note that this second condition is equivalent to $\{0\} \notin \operatorname{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I\cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I\to O})$): indeed, if this is the case, then by Theorem 2 $K^{\mathrm{irr}}_{I\cup O}$ is a coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{I\cup O})$, and then by Proposition 11 $\operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O)$ is a coherent set of desirable gamble sets on $\mathcal{L}(\mathcal{X}_{1:n})$. So we will show that $\emptyset \notin \mathcal{A}^{\mathrm{irr}}_{I\to O}$ and $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$. That $\emptyset \notin \mathcal{A}^{\mathrm{irr}}_{I\to O}$ is clear from Equation (16) because K_O is coherent. So we focus on proving that $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$. Assume *ex absurdo* that $\{0\} \in K^{\mathrm{irr}}_{I\cup O}$. By Lemma 17 below we would then infer that $\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in \{0\}\right\} = \{0\} \in K_O$, contradicting the coherence of K_O. Therefore indeed $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$.

For (ii), we need to show that $A \in \operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O) \Leftrightarrow A \in K_O$ for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$. For necessity, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and assume that $A \in \operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O)$. By Lemma 17 then $\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\} \in K_O$. Since A is a set of gambles on $\mathcal{X}_O$, we infer

$$\Big\{ \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A \Big\} = \Big\{ \sum_{x_I \in \mathcal{X}_I} h : h \in A \Big\} = \big\{ |\mathcal{X}_I| h : h \in A \big\} = |\mathcal{X}_I| A,$$

whence by coherence, indeed $A \in K_O$. For sufficiency, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and assume that $A \in K_O$. Then $A = \mathbb{I}_{\mathcal{X}_I} A$ and $\mathcal{X}_I \in \mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$, so $A \in \mathcal{A}^{\mathrm{irr}}_{I\to O}$. Therefore indeed $A \in \operatorname{ext}^{\mathrm{irr}}_{1:n}(K_O)$.

For (iii), by Proposition 14 it suffices to show that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) \Leftrightarrow \mathbb{I}_{E_I} A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$, for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and E_I in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$. For necessity, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and any E_I in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$, and assume that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. Since we just have shown that $\mathrm{marg}_O \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) = K_O$, this implies that $A \in K_O$, whence indeed $\mathbb{I}_{E_I} A \in \mathcal{A}^{\mathrm{irr}}_{I \to O} \subseteq \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. For sufficiency, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and any E_I in $\mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$, and assume that $\mathbb{I}_{E_I} A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. Since by Proposition 11 $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$ marginalises to $K^{\mathrm{irr}}_{I \cup O}$, this implies that $\mathbb{I}_{E_I} A \in K^{\mathrm{irr}}_{I \cup O}$. Use Lemma 17 to infer that then $\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in \mathbb{I}_{E_I} A\right\} = \left\{\sum_{x_I \in \mathcal{X}_I} \mathbb{I}_{E_I} h(x_I, \bullet) : h \in A\right\} = \left\{|E_I| h : h \in A\right\} = |E_I| A \in K_O$, whence by coherence indeed $A \in K_O$.

The second statement is a direct application of Proposition 12. $\square$

Lemma 17. *Consider any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$, and any coherent set of desirable gamble sets K_O on $\mathcal{L}(\mathcal{X}_O)$. Let $\mathcal{A}^{\mathrm{irr}}_{I \to O}$ be given by Eq.* (16)*. Then*

$$A \in \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I \cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I \to O})) \Rightarrow \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\} \in K_O,$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I \cup O}))$.

Proof. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I \cup O}))$ and assume that $A \in \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I \cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I \to O}))$. Then by Eq. (3) $B \setminus \mathcal{L}_{\leq 0} \subseteq A$ for some B in $\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I \cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I \to O})$, implying that

$$B = \left\{\sum_{k=1}^{m} \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \bigtimes_{k=1}^{m} A_k\right\}$$

for some m in $\mathbb{N}$, $A_1, \dots, A_m$ in $\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I \cup O})_{>0} \cup \mathcal{A}^{\mathrm{irr}}_{I \to O}$, and coefficients $\lambda_{1:m}^{f_{1:m}} > 0$ for all $f_{1:m}$ in $\bigtimes_{k=1}^{m} A_k$. Without loss of generality, assume that $A_1, \dots, A_\ell \in \mathcal{A}^{\mathrm{irr}}_{I \to O}$ and $A_{\ell+1}, \dots, A_m \in \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{I \cup O})_{>0}$ for some ℓ in $\{0, \dots, m\}$. Consider the special subset $P := \left\{f_{1:m} \in \bigtimes_{k=1}^{m} A_k : \lambda_{1:\ell}^{f_{1:m}} = 0\right\}$ of $\bigtimes_{k=1}^{m} A_k$. If $P \neq \varnothing$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^{m} \lambda_k^{g_{1:m}} g_k = \sum_{k=l+1}^{m} \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathcal{L}(\mathcal{X}_{I \cup O})_{>0} \neq \varnothing$, and therefore also $A \cap \mathcal{L}(\mathcal{X}_{I \cup O})_{>0} \neq \varnothing$, whence $\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\} \supseteq \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A \cap \mathcal{L}(\mathcal{X}_{I \cup O})_{>0}\right\} \in K_O$, using Axiom K_2. Applying Axiom K_4, we deduce that $\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\} \in K_O$. So we may assume without loss of generality that $P = \varnothing$, and we define the coefficients

$$\mu_k^{f_{1:m}} := \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell + 1 \end{cases}$$

for all $f_{1:m}$ in $\bigtimes_{k=1}^{m} A_k$ and k in $\{1, \dots, m\}$. Because $P = \varnothing$, for every $f_{1:m}$ in $\bigtimes_{k=1}^{m} A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. The sets $A_{\ell+1}, \dots, A_m$ are singletons containing a positive gamble, so every $f_{1:m}$ in $\bigtimes_{k=1}^{m} A_k$ is completely determined by $f_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_k$. This lets us identify $\mu_{1:\ell}^{f_{1:m}}$ with $\mu_{1:\ell}^{f_{1:\ell}}$. Then every element of $B' := \left\{\sum_{k=1}^{m} \mu_k^{f_{1:m}} f_k : f_{1:m} \in \bigtimes_{k=1}^{m} A_k\right\} = \left\{\sum_{k=1}^{\ell} \mu_k^{f_{1:\ell}} f_k : f_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_k\right\}$ is dominated by an element of B. For every k in $\{1, \dots, \ell\}$ the gamble set A_k belongs to $\mathcal{A}^{\mathrm{irr}}_{I \to O}$, so we may write $A_k = \mathbb{I}_{E_k} A_{O,k}$ with $E_k \in \mathcal{P}_{\overline{\varnothing}}(\mathcal{X}_I)$ and $A_{O,k} \in K_O$. Therefore $|A_k| = |A_{O,k}|$, and every f_k in A_k can be uniquely written as $f_k = \mathbb{I}_{E_k} g_k$ with g_k in $A_{O,k}$. So for every $f_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_k$ there is a unique $g_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_{O,k}$ such that $f_k = \mathbb{I}_{E_k} g_k$ for every k in $\{1, \dots, \ell\}$. For every $f_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_k$ and its corresponding unique $g_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_{O,k}$, we define $\mu_{1:\ell}^{g_{1:\ell}} := \mu_{1:\ell}^{f_{1:\ell}}$. Therefore $B' = \left\{\sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k : g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}$, and hence

$$\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in B'\right\} = \left\{\sum_{x_I \in \mathcal{X}_I} \sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k(x_I, \bullet) : g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}$$

$$= \left\{ \sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} |E_k| g_k(x_I, \bullet) : g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k} \right\}$$

belongs to $\mathrm{Posi}(K_O) = K_O$, using Eq. (4). Since every element of B' is dominated by an element of B, we have that every element of $\left\{ \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in B' \right\}$ is dominated by an element of $\left\{ \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in B \right\}$, so by Lemma 16 $\left\{ \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in B \right\} \in K_O$. By K_4 we have that also indeed $\left\{ \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A \right\} \in K_O$. □

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