

# A random set characterization of possibility measures

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## Abstract

Several authors have pointed out the relationship between consonant random sets and possibility measures. However, this relationship has only been proven for the finite case, where the inverse Möbius of the upper probability induced by the random set simplifies the computations to a great extent. In this paper, we study the connection between both concepts for arbitrary referential spaces. We complete existing results about the lack of an implication in general with necessary and sufficient conditions for the most interesting cases.

**Keywords:** Possibility measures, maxitive measures, nested random sets, upper probabilities, condensability.

## 1 Introduction

There has been a long interest in the literature concerning supremum preserving upper probabilities. They had already appeared under different names ([27, 30, 31]) before Zadeh called them possibility measures ([32]), and claimed that they were an important tool for modeling linguistic uncertainty. Since then, they have become one of the main elements of the fuzzy theory ([10]), and have also been studied from the measure and theoretical point of view ([2, 3]). Possibility measures constitute a special case of maxitive measures, which have been used in integration ([24, 25]), in extremal theory ([26]) or in connection with the fuzzy theory ([12]).

In the finite context of Evidence Theory, Shafer ([28]) defined a consonant plausibility function as the one whose focal elements are nested. This concept

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is equivalent, in a finitary setting, to those of possibility and maxitive measure. The author argued that these set functions were interesting to model consonant evidence, which is the evidence given by non-contradictory sources of information. On the other hand, Shafer's theory is a reinterpretation of Dempster's work on random sets ([6]). Formally, a random set is a measurable multi-valued mapping. Given a probability space  $(\Omega, \mathcal{A}, P)$ , a measurable space  $(X, \mathcal{A}')$  and a mapping  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$ ,  $\Gamma$  is said to be *strongly measurable* [15] when for every  $A \in \mathcal{A}'$ ,  $\Gamma^*(A) := A^* = \{\omega \in \Omega : \Gamma(\omega) \cap A \neq \emptyset\}$  belongs to  $\mathcal{A}$ .  $A^*$  is called the *upper inverse* of  $A$ . Similarly, the *lower inverse* of  $A$  is defined by  $A_* := \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset, \Gamma(\omega) \subseteq A\}$ . It is  $A_* = (A^{c*})^c \cap X^*$ ; hence, if  $\Gamma$  is strongly measurable,  $A_* \in \mathcal{A} \forall A \in \mathcal{A}'$ .

Using the upper and lower inverses, Dempster ([6]) defined the *upper probability* induced by  $\Gamma$  by  $P^*(A) = \frac{P(A^*)}{P(X^*)}$ , and the *lower probability* by  $P_*(A) = \frac{P(A_*)}{P(X^*)}$ . The restriction of  $P^*$  to the singletons of  $X$  is called the *one-point coverage function* of the random set.

In Evidence Theory, focal elements play the same role as the images of random sets in Dempster's theory, while plausibility measures are analogous to Dempster's upper probabilities. This leads us to seek a connection between the nesting of the images of a random set and the supremum-preserving property of its upper probability. This is interesting not only from a purely theoretical point of view (to obtain an equivalence between both structures) but also from a more practical perspective (to give an interpretation to consonant random sets and see if the one-point coverage function suffices to characterize the random set). In this paper, we investigate the relationship between both models, completing the results obtained in a previous article ([20]).

Special attention has been put on random sets defined on *Polish spaces*. A Polish space  $(X, \tau)$  ([17, 21]) is a topological metrizable space which is complete and separable. We will denote  $\beta(\tau)$  the Borel  $\sigma$ -field generated by the topology. Polish spaces generalize the Euclidean space  $(\mathbb{R}^n, d)$ , while keeping most of the good properties we have there. Moreover, they also generalize the finite spaces  $(X, \mathcal{P}(X))$ . Also, some authors consider only closed or compact-valued random sets ([18]), because they have in most aspects better properties than other classes of random sets. We will study the behaviour of these types of multi-valued

mappings for our problem.

The paper is organized as follows: In Sections 2 and 3, we study thoroughly the concepts of possibility and maxitive measure, and consonant random set. Then, the mentioned problem is discussed; first, we study if a random set inducing a possibility measure is consonant (Section 4) and next we investigate if a consonant random set induces a possibility measure (Section 5). Finally, in Section 6 we give some additional comments on the subject.

## 2 Possibility measures and maxitive capacities

Before introducing the concepts of possibility and maxitive measure, let us define some general properties of set functions.

**Definition 2.1.** [7] Consider  $\mathcal{S}$  a class of sets. A set function is a mapping  $\mu : \mathcal{S} \rightarrow [0, \infty]$  s.t.  $\mu(\emptyset) = 0$ . It is called:

- monotone when  $A \subset B \Rightarrow \mu(A) \leq \mu(B)$ .
- continuous from above when for every decreasing sequence  $(A_n)_n \subset \mathcal{S}$  with  $\bigcap_n A_n \in \mathcal{S}$ , it is  $\mu(\bigcap_n A_n) = \lim_n \mu(A_n)$ .
- continuous from below if  $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$  for every increasing sequence  $(A_n)_n \subset \mathcal{S}$  s.t.  $\bigcup_n A_n \in \mathcal{S}$ .

We will consider throughout a measurable space  $(X, \mathcal{A})$  and a monotone set function  $\mu : \mathcal{A} \rightarrow [0, 1]$  satisfying  $\mu(X) = 1$  (i.e., *normalized*). Let us give another preliminary definition:

**Definition 2.2.** [29] Given a  $\sigma$ -field  $\mathcal{A}$ , an upward net is a subclass  $\mathcal{C} \subset \mathcal{A}$  s.t.  $\forall A_1, A_2 \in \mathcal{C}, \exists A \in \mathcal{C}$  with  $A_1 \cup A_2 \subset A$ .

**Definition 2.3.** ([3, 7, 16, 29]) Let  $(X, \mathcal{A})$  be a measurable space. A monotone and normalized set function  $\mu : \mathcal{A} \rightarrow [0, 1]$  is called:

- $\infty$ -alternating if  $\mu(A_1 \cap \dots \cap A_n) \leq \sum_{i=1}^n \mu(A_i) - \sum_i \sum_j \mu(A_i \cup A_j) + \sum_i \sum_j \sum_k \mu(A_i \cup A_j \cup A_k) - \dots + (-1)^{n+1} \mu(\bigcup_{i=1}^n A_i)$  for any  $A_1, \dots, A_n \in \mathcal{A}, n \in \mathbb{N}$ .
- maxitive when  $\mu(A \cup B) = \max\{\mu(A), \mu(B)\} \forall A, B \in \mathcal{A}$ .

- possibility measure when for every  $(A_i)_{i \in I} \subset \mathcal{A}$  s.t.  $\cup_{i \in I} A_i \in \mathcal{A}$ , it is  $\mu(\cup_{i \in I} A_i) = \sup_{i \in I} \mu(A_i)$ .
- Choquet capacity<sup>1</sup> if it is continuous for decreasing sequences of closed sets and continuous from below (note that in this case  $\mu$  has to be defined on a Borel  $\sigma$ -field  $\beta(\tau)$ ).
- condensable if for every upward net  $\mathcal{C} \subset \mathcal{A}$  such that  $\cup_{A \in \mathcal{C}} A \in \mathcal{A}$  it is  $\mu(\cup_{A \in \mathcal{C}} A) = \sup_{A \in \mathcal{C}} \mu(A)$ .

The term *condensable capacity* was coined by Shafer ([29]) in the context of  $\infty$ -alternating capacities, and it has also been used by Nguyen ([22, 23]), among others. It is a stronger condition than continuity from below. When  $\mathcal{A}$  contains the singletons<sup>2</sup>, a condensable capacity satisfies the following property:

**Lemma 2.1.** [29] Take  $\mu : \mathcal{A} \rightarrow [0, 1]$ . Then,  $\mu$  is condensable if and only if  $\mu(A) = \sup\{\mu(J) : J \subset A \text{ finite}\}$ .

Let us also introduce the notion of *capacitability*, which, as we will show, is related to the condensability of a set function.

**Definition 2.4.** [17] Let  $(X, \tau)$  be a Hausdorff topological space,  $\mu$  a set function on  $\beta(\tau)$ . A set  $A \in \beta(\tau)$  is called  $\mu$ -capacitable when  $\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\}$ .

Some authors [16] also require  $\mu(A) = \inf_{A \subset G \text{ open}} \mu(G)$ . This is not going to be necessary for most of this paper, and we will explicitly state it otherwise. We deduce from lemma 2.1 that a condensable capacity is determined by its values on the finite sets. As these sets are compact, we conclude that given a condensable capacity  $\mu : \beta(\tau) \rightarrow [0, 1]$ , where  $\tau$  is a Hausdorff topology on  $X$ , every Borel set is  $\mu$ -capacitable.

Let us give now a property of maxitive capacities, which will help us to analyze their relationship with possibility measures. We prove previously a lemma for metric spaces:

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<sup>1</sup>We are following the definition from [16]. Other authors [18] only require continuity for decreasing sequences of *compact* sets.

<sup>2</sup>This will be no essential requirement, for in most of the propositions and examples we give in this section we work with the Borel  $\sigma$ -field generated by a metric or a Hausdorff topology.

**Lemma 2.2.** *Let  $(X, \tau)$  be a Hausdorff topological space, and consider a decreasing sequence  $(K_n)_n$  of non-empty compact sets. Then,  $\bigcap_n K_n \neq \emptyset$ . If in addition  $X$  is metric and  $\delta(K_n) \downarrow 0$ , there exists some  $x$  s.t.  $\bigcap_n K_n = \{x\}$ .*

**Proof:** The sequence  $(K_n)_n$  is a sequence of closed sets contained in the compact set  $K_1$ . Moreover, every finite subsequence has a non-empty intersection. Hence, the global intersection  $\bigcap_n K_n$  is non-empty. Now, if there were  $x \neq y \in \bigcap_n K_n$ , it would be  $\delta(K_n) \geq d(x, y) \forall n$ , and this would contradict  $\delta(K_n) \downarrow 0$ . ■

**Proposition 2.3.** *Let  $(X, d)$  be a metric space, and consider a monotone set-function  $\mu$  on the Borel  $\sigma$ -field  $\beta(d)$  continuous for decreasing sequences of compact sets. If  $\mu$  is maxitive, then  $\mu(K) = \max_{x \in K} \mu(\{x\})$  for every  $K$  compact.*

**Proof:** Consider  $K$  compact, and take  $\epsilon_1 = \frac{1}{2}$ . Then,  $K \subset \bigcup_{x \in K} B(x; \epsilon_1)$ , whence  $\exists J \subseteq K$  finite s.t.  $K = \bigcup_{x \in J} (B(x; \epsilon_1) \cap K)$ . As  $\mu$  is maxitive, there is some  $x_1 \in J$  s.t.  $\mu(K) = \mu(B(x_1; \epsilon_1) \cap K)$ . Take  $K_1 = \overline{B(x_1; \epsilon_1)} \cap K$  compact. By the monotonicity of  $\mu$ ,  $\mu(K) = \mu(K_1)$ . Given  $\epsilon_2 = \frac{1}{4}$ ,  $K_1 \subset \bigcup_{x \in K_1} B(x; \epsilon_2)$ . Following the previous reasoning, there exists some  $x_2 \in K_1$  s.t.  $\mu(K_1) = \mu(B(x_2; \epsilon_2) \cap K_1)$ . Take  $K_2 = \overline{B(x_2; \epsilon_2)} \cap K_1$  compact. Then,  $\mu(K_2) = \mu(K)$ , and  $\delta(K_2) \leq \frac{1}{2}$ .

If we repeat this process, we obtain a decreasing sequence  $(K_n)_n$  of compact sets s.t.  $\mu(K_n) = \mu(K) \forall n$ , and  $\delta(K_n) \leq \frac{1}{2^{n-1}}$ . By the previous lemma, there is some  $x \in X$  s.t.  $\bigcap_n K_n = \{x\}$ . As  $\mu$  is continuous for decreasing sequences of compact sets, it is  $\mu(\{x\}) = \lim_n \mu(K_n) = \mu(K)$ . ■

In particular, the result holds when  $\mu$  is a Choquet capacity. Note that the converse is not true in general:

**Example 2.1.** *Consider  $(\mathbb{N}, d)$  with the discrete metric, and  $\mu : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$  given by  $\mu(B) = 0$  if  $B$  is finite,  $\mu(B) = 0.5$  if  $|B| = \infty$ ,  $B \not\subseteq \mathbb{N}$ , and  $\mu(\mathbb{N}) = 1$ . With the discrete topology, the compact sets are finite, and hence  $\mu(K) = 0 = \mu(\{x\})$  for every  $x \in K$  compact. Consequently,  $\mu$  is continuous for decreasing sequences of compact sets. However, given  $A = \{1, 3, 5, \dots\}$ ,  $B = \{2, 4, 6, \dots\}$ , it is  $\mu(A \cup B) = 1 > \max\{\mu(A), \mu(B)\}$ , whence  $\mu$  is not maxitive.*

When a possibility measure is defined on a  $\sigma$ -field that contains the singletons, it is determined by its *possibility distribution*  $\pi : X \rightarrow [0, 1]$ , which is

defined by  $\pi(x) = \mu(\{x\})$ . Then, it is  $\mu(A) = \sup_{x \in A} \pi(x) \forall A \in \mathcal{A}'$ , and this can be given as an alternative definition of possibility measure. It is clear that a possibility measure is always maxitive. The converse, however, does not hold in general: think for instance ([8]) of a set function  $\mu$  on  $\mathcal{P}(\mathbb{R})$  s.t.  $\mu(A) = 1$  if  $A$  is infinite,  $\mu(A) = 0$  if  $A$  is finite. This example shows that for a maxitive measure to be a possibility, we need some additional requirement of continuity, which prevents the jump we had there between  $\sup\{\mu(\{x\}) : x \in A\}$  and  $\mu(A)$ . However, continuity from below is not sufficient: we only need to modify the previous  $\mu$  and make  $\mu(A) = 1$  if  $A$  is uncountable,  $\mu(A) = 0$  otherwise to find a counterexample. An interesting study about maxitive measures (called there *generalized possibility measures*) and possibility measures from the point of view of fuzzy set theory can be found in [12].

In the following theorem we give implications among possibility, maxitive and condensable set functions. We also establish a link with the concept of capacitability. We show in particular that we need to require the continuity for upward nets (i.e., the condensability) in order to obtain the equivalence between possibility and maxitive measures:

**Theorem 2.4.** *Let  $(X, \mathcal{A})$  be a measurable space, and consider  $\mu : \mathcal{A} \rightarrow [0, 1]$ . Let us consider the following conditions:*

1.  $\mu$  is a possibility measure.
2.  $\mu$  is maxitive and condensable.
3.  $\mu(K) = \max_{x \in K} \mu(\{x\}) \forall K$  compact and it is condensable.
4.  $\mu$  is maxitive and every  $A \in \beta(d)$  is  $\mu$ -capacitable.
5.  $\mu(K) = \max_{x \in K} \mu(\{x\}) \forall K$  compact and every  $A \in \beta(d)$  is  $\mu$ -capacitable.

*Then, we have the following relationships:*

- (a) In general,  $1 \Leftrightarrow 2$ .
- (b) If  $(X, \tau)$  is a Hausdorff topological space and  $\mu : \beta(\tau) \rightarrow [0, 1]$ , then  $3 \Leftrightarrow 5 \Rightarrow 1 \Leftrightarrow 2 \Rightarrow 4$ .

(c) If  $(X, d)$  is a metric space and  $\mu : \beta(d) \rightarrow [0, 1]$  is continuous for decreasing sequences of compact sets, then  $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$ .

**Proof:**

(a) Let us prove the first equivalence. It is clear that any possibility measure is maxitive. On the other hand, given an upward net  $\mathcal{C} \subset \mathcal{A}$ , it is  $\mu(\cup_{A \in \mathcal{C}} A) = \sup_{A \in \mathcal{C}} \mu(A)$ , and hence  $\mu$  is condensable.

Conversely, take  $(A_i)_{i \in I} \subset \mathcal{A}$  s.t.  $\cup_{i \in I} A_i \in \mathcal{A}$ , and define  $\mathcal{C} = \{A_{i_1} \cup \dots \cup A_{i_m} : \{i_1, \dots, i_m\} \subset I, m \in \mathbb{N}\}$ . Then,  $\mathcal{C}$  is an upward net and  $\cup_{A \in \mathcal{C}} A = \cup_{i \in I} A_i$ . The condensability of  $\mu$  implies  $\mu(\cup_{i \in I} A_i) = \sup_{A \in \mathcal{C}} \mu(A)$ . As  $\mu$  is maxitive, for every  $A = A_{i_1} \cup \dots \cup A_{i_m} \in \mathcal{C}$ , there exists  $j \in \{i_1, \dots, i_m\}$  s.t.  $\mu(A) = \mu(A_j)$ . Hence,  $\sup_{A \in \mathcal{C}} \mu(A) = \sup_{i \in I} \mu(A_i)$  and consequently  $\mu(\cup_{i \in I} A_i) = \sup_{i \in I} \mu(A_i)$ . Therefore,  $\mu$  is a possibility measure.

(b) We will prove  $3 \Rightarrow 5 \Rightarrow 1 \Leftrightarrow 2 \Rightarrow 4$ , and  $5 \Rightarrow 3$ :

–  $3 \Rightarrow 5$ . It follows from lemma 2.1 and the fact that finite sets are compact.

–  $5 \Rightarrow 1$ . Consider  $(A_i)_{i \in I} \subset \beta(\tau)$  s.t.  $\cup_{i \in I} A_i \in \beta(\tau)$ . Then, given  $\epsilon > 0$ ,  $\exists K \subseteq \cup_{i \in I} A_i$  compact s.t.  $\mu(\cup_{i \in I} A_i) \leq \mu(K) + \epsilon$ . For this  $K$ , there is some  $x \in K$  with  $\mu(K) = \mu(\{x\})$ , whence  $\mu(\cup_{i \in I} A_i) \leq \mu(\{x\}) + \epsilon \leq \sup_{i \in I} \mu(A_i) + \epsilon \forall \epsilon > 0$ . Hence,  $\mu(\cup_{i \in I} A_i) = \sup_{i \in I} \mu(A_i)$ , and  $\mu$  is a possibility measure.

–  $1 \Leftrightarrow 2$ . It is a consequence of part [a].

–  $2 \Rightarrow 4$ . It is analogous to  $3 \Rightarrow 5$ .

–  $5 \Rightarrow 3$ . Take  $A \in \beta(\tau)$ . Then, because of the capacitability, given  $\epsilon > 0$ , there exists  $K \subseteq A$  compact s.t.  $\mu(A) - \mu(K) < \epsilon$ . Now,  $\exists x \in K$  s.t.  $\mu(K) = \mu(\{x\})$ , whence  $\mu(A) - \mu(\{x\}) < \epsilon$ . From lemma 2.1,  $\mu$  is condensable.

(c) It suffices to show that under this additional hypotheses we have  $4 \Rightarrow 5$ , and this follows from proposition 2.3. ■

We can easily find counterexamples showing that  $1 \Rightarrow 5$  and  $4 \Rightarrow 2$  do not hold under the hypotheses stated in part [b] of this theorem.

We see that the concept of capacitability is very important in the context of possibility and maxitive measures. As we have already noted, it is a weaker notion than condensability. To see that they are not equivalent in metric spaces, consider Lebesgue measure  $\lambda$  on  $\beta_{[0,1]}$ . Then, all the Borel sets are  $\lambda$ -capacitable, but  $\lambda$  is not determined by its value on the finite sets and hence it is not condensable.

On the other hand, in [16] it was checked that when  $\mu$  is defined on a complete and separable metric space and it is a Choquet capacity, the Borel sets are capacitable. Formally,

**Proposition 2.5.** *[16] Let  $(X, \tau)$  be a Polish space and let  $\mu$  be a Choquet capacity on  $\beta(\tau)$ . Then every Borel set is  $\mu$ -capacitable. Moreover, we also have  $\mu(A) = \inf\{\mu(G) : A \subset G \text{ open}\} \forall A \in \beta(\tau)$ .*

However, Choquet capacities are not equivalent to condensable set-functions on Polish spaces: Lebesgue measure  $\lambda$  on  $[0, 1]$  is a Choquet capacity that is not condensable. Conversely, consider the Borel  $\sigma$ -field  $\beta$  on  $\mathbb{R}$ , and  $\mu : \beta \rightarrow [0, 1]$  given by  $\mu(A) = 1$  if  $A \cap \mathbb{Q} \neq \emptyset$ ,  $\mu(A) = 0$  otherwise. Then, we can check, using the equivalent condition given in lemma 2.1, that  $\mu$  is condensable; however,  $\mu([\pi - 1/n, \pi + 1/n]) = 1 \forall n$  whereas  $\mu(\bigcap_n [\pi - 1/n, \pi + 1/n]) = \mu(\{\pi\}) = 0$ . Thus,  $\mu$  is not a Choquet capacity.

We can extend the capacitability property given by proposition 2.5 for more general spaces, not necessarily Polish. A Polish space is a complete and separable metric space. In particular,  $(\mathbb{R}^n, \beta(d))$  and a compact metric space are Polish, but a  $\sigma$ -compact metric space is not necessarily Polish (it does not need to be complete). A locally compact Polish space is equivalent to a locally compact  $\sigma$ -compact metric space, and also to a locally compact, Hausdorff and separable topological space. These are the LCS spaces from Matheron ([18]). See [17] for a complete review.

The following theorem will be very useful in connection with the upper probabilities of random sets, as we will later show:



**Proposition 2.6.** *Let  $(X, d)$  be a  $\sigma$ -compact metric space, and consider  $\mu : \beta(d) \rightarrow [0, 1]$  a set function continuous for decreasing sequences of compact sets and continuous from below. Then, every Borel set is  $\mu$ -capacitable. If in addition  $(X, d)$  is locally compact and  $\mu$  is  $\infty$ -alternating, we also have  $\mu(A) = \inf_{A \subseteq G \text{ open}} \mu(G) \forall A \in \beta(d)$ .*

**Proof:**

- Let us start by the first statement. Consider  $A \in \beta(d)$ ,  $\epsilon > 0$ . As  $(X, d)$  is a  $\sigma$ -compact metric space, there exists an increasing sequence  $(K_n)_n$  of compact sets s.t.  $X = \cup_n K_n$ . Because of the continuity from below, it is  $\mu(A) = \sup_n \mu(A \cap K_n)$ , whence  $\exists n_0 \in \mathbb{N}$  s.t.  $\mu(A) - \mu(A \cap K_{n_0}) < \frac{\epsilon}{2}$ .

Consider the compact metric space  $(K_{n_0}, d \upharpoonright_{K_{n_0}})$  (which is in particular Polish), and let  $\mu_{n_0} : \beta(d \upharpoonright_{K_{n_0}}) \rightarrow [0, 1]$  be the restriction of  $\mu$ . Then,  $\mu_{n_0}$  is continuous for decreasing sequences of closed sets, because closed sets in  $(K_{n_0}, d \upharpoonright_{K_{n_0}})$  are compact on  $(X, d)$ . Moreover, it is also continuous from below. Hence, it is a Choquet capacity. From proposition 2.5, we deduce that given  $A \cap K_{n_0} \in \beta(d \upharpoonright_{K_{n_0}})$ , there exists a compact subset  $K \subset A \cap K_{n_0}$  with  $\mu_{n_0}(A \cap K_{n_0}) - \mu_{n_0}(K) \leq \frac{\epsilon}{2}$ . Hence,

$$\begin{aligned} \mu(A) - \mu(K) &= \mu(A) - \mu(A \cap K_{n_0}) + \mu(A \cap K_{n_0}) - \mu(K) = \\ &\mu(A) - \mu(A \cap K_{n_0}) + \mu_{n_0}(A \cap K_{n_0}) - \mu_{n_0}(K) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and  $K \subseteq A \cap K_{n_0} \subseteq A$  is also compact on  $(X, d)$ . Therefore,  $\mu(A) = \sup_{K \subseteq A \text{ compact}} \mu(K)$ .

- We turn now to the second part. Note that the concepts of locally compact  $\sigma$ -compact metric space and locally compact Polish space are equivalent. Take  $A \in \beta(d)$ ,  $\epsilon > 0$ .

If  $(X, \tau)$  is a locally compact Polish space, there exists an increasing sequence of open sets  $(U_n)_n \uparrow X$  with  $U_n \supseteq \overline{U_{n-1}}$  compact for every  $n$ .

Let us denote  $K_n = \overline{U_n}$ , and define  $\mu_n$  the restriction of  $\mu$  to the Polish space  $(K_n, d \upharpoonright_{K_n})$ . As we showed in the previous point,  $\mu_n$  is continuous for decreasing sequences of closed sets on  $(K_n, d \upharpoonright_{K_n})$ . Hence, it is a Choquet capacity, and we can apply proposition 2.5. Given the set  $A_n :=$

$A \cap U_n \in \beta(d \upharpoonright_{K_n})$ , there exists some open set  $G_n$ , (which we may in particular assume open in  $(X, d)$  and also  $G_n \subset U_n$ ), with  $A_n \subset G_n$ ,  $\mu_n(G_n) - \mu_n(A_n) \leq \frac{\epsilon}{2^n}$ , or, equivalently,  $\mu(G_n) - \mu(A_n) \leq \frac{\epsilon}{2^n}$ .

Define  $G'_n = \cup_{i=1}^n G_i$ . We can deduce from the  $\infty$ -alternating property of  $\mu$  that  $\mu(G'_n) - \mu(A_n) \leq \sum_{i=1}^n [\mu(G_i) - \mu(A \cap U_i)] \leq \sum_{i=1}^n \frac{\epsilon}{2^i}$  (see [19, th.21]).

Now,  $\mu(\cup_n G'_n) - \mu(A) = \lim_n [\mu(G'_n) - \mu(A \cap U_n)] \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$ , and  $A = \cup_n (A \cap U_n) \subseteq \cup_n G_n = \cup_n G'_n$  open. Hence,  $\mu(A) = \inf_{A \subseteq G \text{ open}} \mu(G)$  for every  $A \in \beta(\tau)$ . This completes the proof. ■

We are now going to apply these results in the context of random sets. As we will show in this paper, the upper probability of a random set can be maxitive and not a possibility measure. Hence, it is important to clarify when the upper probability of a random set satisfies the conditions of condensability or capacitability. As we have showed in the previous proposition, this is implied under some hypotheses by the continuity of  $P^*$  for certain classes of sets. Let us also remark that the upper probability of a random set is always continuous from below and  $\infty$ -alternating ([22]):

**Theorem 2.7.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \tau)$  a Hausdorff space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. Then:*

1. *If  $\Gamma$  is compact-valued, then  $P^*$  is a Choquet capacity.*
2. *If  $\Gamma$  is closed-valued, then  $P^*$  is continuous for decreasing sequences of compact sets.*
3. *If  $\Gamma$  is closed-valued and  $X$  is a  $\sigma$ -compact metric space, then  $P^*(A) = \sup_{K \subseteq A \text{ compact}} P^*(K) \forall A \in \beta(d)$ .*
4. *If in addition  $X$  is locally compact, then for every  $A \in \beta(d)$ ,  $P^*(A) = \sup_{K \subseteq A \text{ compact}} P^*(K) = \inf_{A \subseteq G \text{ open}} P^*(G)$ .*

**Proof:** Consider a decreasing sequence  $(A_n)_n$ . To prove the first two statements, we are going to show that if  $\Gamma(\omega) \cap A_n$  is compact for every  $n$ , it is  $\cap_n (A_n)^* = (\cap_n A_n)^*$ :

$\bigcap_n (A_n)^* = \bigcap_n \{\omega \in \Omega : \Gamma(\omega) \cap A_n \neq \emptyset\}$ . From lemma 2.2, a decreasing sequence of non-empty compact sets has a non-empty intersection. Hence,  $\bigcap_n \{\omega \in \Omega : \Gamma(\omega) \cap A_n \neq \emptyset\} = \{\omega \in \Omega : \bigcap_n (\Gamma(\omega) \cap A_n) \neq \emptyset\} = \{\omega \in \Omega : \Gamma(\omega) \cap (\bigcap_n A_n) \neq \emptyset\} = (\bigcap_n A_n)^*$ .

1. Now, note that  $P^*$  will be a Choquet capacity iff it is continuous for decreasing sequences of closed sets. Given such a sequence  $(A_n)_n$ , if  $\Gamma$  is compact-valued we have  $\Gamma(\omega) \cap A_n$  compact for every  $\omega, n$ , whence  $(\bigcap_n A_n)^* = \bigcap_n (A_n)^*$  and consequently  $P^*(\bigcap_n A_n) = \lim_n P^*(A_n)$ .
2. Similarly, if  $\Gamma$  is closed-valued, given  $(A_n)_n$  a decreasing sequence of compact sets, it is  $\Gamma(\omega) \cap A_n$  compact  $\forall \omega, n$ , whence  $P^*(\bigcap_n A_n) = \lim_n P^*(A_n)$ .
3. 4. They follow from the previous point and proposition 2.6. ■

The third part of this theorem generalizes a result from Matheron ([18]).

**Remark 2.1.** *Note that for closed random sets on a locally compact Polish space, the upper probability is not necessarily a Choquet capacity: take for instance  $\Gamma : [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$  given by  $\Gamma(\omega) = \mathbb{R} \forall \omega$ . Then,  $P^*([n, \infty)) = 1 \forall n$ , whereas  $P^*(\bigcap_n [n, \infty)) = P^*(\emptyset) = 0$ . Therefore,  $P^*$  is not continuous for decreasing sequences of closed sets. This example shows that upper probabilities induced by closed random sets on locally compact Polish spaces (and hence on  $\sigma$ -compact metric spaces) do not possess the same properties as those induced by compact random sets on Polish spaces. We will see, however, that they have the same behaviour in relation to the problem treated in this paper.*

Using theorems 2.4 and 2.7, we can deduce the following result:

**Theorem 2.8.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, d)$  a metric space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. If any of the following conditions hold:*

- $\Gamma$  is compact-valued and  $X$  is Polish.
- $\Gamma$  is closed-valued and  $X$  is  $\sigma$ -compact,

*then  $P^*$  is a possibility measure  $\Leftrightarrow P^*$  is maxitive  $\Leftrightarrow P^*(K) = \max_{x \in K} P^*({x})$  for every  $K$  compact.*

**Proof:** Note first that under any of the hypotheses of this theorem,  $P^*$  is continuous for decreasing sequences of compact sets, whence the third part of theorem 2.4 is applicable. Also, we have showed in proposition 2.5 and theorem 2.7 that under these conditions, the Borel sets are  $P^*$ -capacitable. Using the equivalence  $1 \Leftrightarrow 4 \Leftrightarrow 5$  from theorem 2.4, we complete the proof. ■

This result will acquire importance in the following sections, when we study the relationship between possibility measures and consonant random sets. It is interesting to remark that although in this paper we are trying to be as general as possible, some authors only consider closed or compact random sets on Polish spaces ([1, 14, 18]), for they have good mathematical properties while working on a (relatively) general framework. It is therefore of special importance to study the behaviour of this type of random sets for our problem.

### 3 Nested random sets

Let us take a closer look at consonant (or nested) random sets. They will be used to represent what Shafer calls *consonant evidence*, that is, evidence where different sources of information do not contradict each other.

As we said in the introduction, focal sets and images of random sets play the same role on different contexts. Hence, the nesting of the focal elements on Shafer's theory should be equivalent to some kind of nesting among the images of the random set. In [20], we considered the following:

**Definition 3.1.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \mathcal{A}')$  a measurable space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. We say that  $\Gamma$  is consonant of type:*

- C1 if any  $\omega_1, \omega_2 \in \Omega$  satisfy either  $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$  or  $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$ .
- C2 if there exists  $N \subset \Omega$  null s.t. any  $\omega_1, \omega_2 \in \Omega \setminus N$  satisfy the previous relation.
- C3 if for any  $\omega_1, \omega_2 \in \Omega$ , either  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = 0$  or  $P^*(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) = 0$ .
- C4 when there exists  $N \subset \Omega$  null s.t. any  $\omega_1, \omega_2 \in \Omega \setminus N$  satisfy the previous relation.

- *C5* if there exists a null set  $N \subset \Omega$  s.t. for any  $\omega_1 \in \Omega \setminus N$  exists  $\omega_2 \neq \omega_1$  s.t.  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = 0$  or  $P^*(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) = 0$ .
- *C6* if any pair  $x_1, x_2 \in X$  satisfies  $P(\{x_1\}^* \setminus \{x_2\}^*) = 0$  or  $P(\{x_2\}^* \setminus \{x_1\}^*) = 0$ .<sup>3</sup>

Conditions *C1* – *C2* call the images of the random set nested when one of them is included in the other. They coincide with the notion of nesting mostly used in the literature. On the other hand, conditions *C3* – *C4* only require that one of the differences  $\Gamma(\omega_1) \setminus \Gamma(\omega_2)$  and  $\Gamma(\omega_2) \setminus \Gamma(\omega_1)$  is a set of null upper probability. The distinction between *C1* and *C2* (or equivalently, *C3* and *C4*) comes from the fact that the behaviour of the random set on a null set does not affect its upper probability, and hence a random set inducing a possibility measure could be *C2* and not *C1* (or *C4* and not *C3*). When condition *C5* does not hold, we can argue that the random set is not consonant ‘at all’: there is a set of positive probability such that the image of any of its elements is not nested with the image of any other element in  $\Omega$ . Finally, condition *C6* is related to the upper inverses. We refer to [20] for a more comprehensive motivation and interpretation of these definitions.

All these conditions are different from one another in general, even though some of them will have a similar behaviour; for instance, *C1* and *C2* will have the same implications on the upper probability of the random set, and the same applies to *C3* and *C4*. We can easily prove the implication relationships drawn in figure 1, and we refer to [20] for a detailed proof.

Figure 1: Relationships among the conditions.

In [28], Shafer introduces the concept of consonant plausibility function as one whose focal elements are nested. This concept coincides with that of a possibility measure in the case of a finitary setting. In that case it is also equivalent to a maxitive set function. If we study the conditions given on definition 3.1 for the case of a finite final space, we can prove the following result:

<sup>3</sup>Of course, these conditions are only applicable when  $\mathcal{A}$  and  $\mathcal{A}'$  satisfy certain hypotheses: in the case of conditions *C3* and *C4*,  $\mathcal{A}'$  must contain the set differences  $\Gamma(\omega_1) \setminus \Gamma(\omega_2)$ , etc.

**Proposition 3.1.** [20] *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \mathcal{P}(X))$  a measurable space, with  $X$  finite, and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. The following conditions are equivalent:*

1.  $P^*$  is a possibility measure.
2.  $\Gamma$  is C2.
3.  $\Gamma$  is C4.
4.  $\Gamma$  is C6.

This proposition is an immediate consequence of the equivalence, valid for the finite case, between the focal elements of the upper probability and the images of the random set. This connection is also studied in [11].

We can easily find examples showing that, even in the finite case, conditions C1 and C3 are not necessary for  $P^*$  to be a possibility measure, and condition C5 is not sufficient.

This proposition is one of the arguments which makes us wonder about the existence of an equivalence between nested random sets and possibility measures in the case of an infinite referential. This is also sustained by some results established for particular infinite spaces ([5, 9, 13]) where both concepts are related. However, as we will show in the following sections, neither of the implications holds for arbitrary referential spaces, and additional requirements (though not very restrictive ones) are necessary in order to obtain the equivalence.

## 4 $P^*$ possibility implies $\Gamma$ nested?

In [20], we considered the direct and inverse problems for this equivalence. That is, on one hand we studied if any random set inducing a possibility measure is consonant; and we also investigated whether any consonant random set induces a possibility measure.

Let us begin with the results for the direct problem. Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \mathcal{A}')$  a measurable space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set, and assume that  $P^*$  is a possibility measure. In general, the maxitivity of  $P^*$  suffices to guarantee that  $\Gamma$  is consonant C6 (see [20]), and consequently if  $P^*$

is a possibility measure  $\Gamma$  will be  $C6$ . The following example shows that the other nesting conditions do not necessarily hold:

**Example 4.1.** Consider  $\Gamma : [0, 1] \rightarrow \mathcal{P}([0, 1])$  given by  $\Gamma(\omega) = [0, 1] \setminus \{\omega\} \forall \omega \in [0, 1]$ . Then,  $P^*(\{x\}) = 1 \forall x \in [0, 1]$ , and consequently  $P^*$  is a possibility measure. However,  $\Gamma$  is not consonant  $C5$ : given  $\omega_1 \neq \omega_2 \in [0, 1]$ ,  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = P^*(\{\omega_2\}) = 1$ , and  $P^*(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) = P^*(\{\omega_1\}) = 1$ . Hence, it is not  $C1, \dots, C4$  either, for these conditions are stronger than  $C5$ .

Even if this example shows the lack of an implication in general, it would be interesting to see whether there is a nested random set inducing this possibility measure, or if additional conditions on  $\Gamma$  guarantee the equivalence. As we saw in the finite case,  $P^*$  possibility does not imply that  $\Gamma$  is  $C1$  or  $C3$ , because we can modify  $\Gamma$  on a null set so that it is not  $C1$  (or  $C3$ ) and this does not alter the upper probability. Then, we only need to clarify under which conditions  $P^*$  possibility implies that  $\Gamma$  is  $C2, C4$  or  $C5$ . Let us show that when  $P^*$  is a possibility on  $\mathcal{P}(X)$  conditions  $C2$  and  $C4$  become equivalent:

**Proposition 4.1.** Consider a random set  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  s.t.  $P^*$  is a possibility measure on  $\mathcal{P}(X)$ . Then,  $\Gamma$  is  $C2$  if and only if it is  $C4$ .

**Proof:** It is clear that if  $\Gamma$  is  $C2$ , then it is  $C4$ . Conversely, assume that  $\Gamma$  is a  $C4$  random set whose upper probability is a possibility measure. Then, there exists  $N \subset \Omega$  null s.t.  $\forall \omega_1, \omega_2 \in \Omega \setminus N$ , it is  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = 0$  or  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_1)) = 0$ . Consider now  $\omega_1, \omega_2 \in \Omega \setminus N$  s.t.  $\Gamma(\omega_1) \not\subseteq \Gamma(\omega_2)$  and  $\Gamma(\omega_2) \not\subseteq \Gamma(\omega_1)$ . If it is  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = 0$ , we take  $x_1 \in \Gamma(\omega_1) \setminus \Gamma(\omega_2)$ . Otherwise, we take  $x_2 \in \Gamma(\omega_2) \setminus \Gamma(\omega_1)$ .

Denote by  $A$  the set of elements  $x$  selected through this process. As  $P^*$  is a possibility measure, it must be  $P^*(A) = 0$ , for every  $x \in A$  satisfies  $P^*(\{x\}) = 0$ . Take now  $\omega_1, \omega_2 \in \Omega \setminus (A^* \cup N)$ . Then, if we had  $\Gamma(\omega_1) \not\subseteq \Gamma(\omega_2)$  and  $\Gamma(\omega_2) \not\subseteq \Gamma(\omega_1)$ , one of them should be in  $A^*$ , a contradiction. Therefore,  $\Gamma$  is  $C2$ . ■

**Remark 4.1.** There is no additional implication among the nesting conditions when  $P^*$  is a possibility measure on  $\mathcal{P}(X)$ . To see this, consider the following examples:

1. Let  $([0, 1], \beta_{[0,1]}, \lambda)$  be a probability space,  $([0, 1], \mathcal{P}([0, 1]))$  a measurable space and consider  $\Gamma_1 : [0, 1] \rightarrow \mathcal{P}([0, 1])$  given by  $\Gamma_1(0) = [0, 1], \Gamma_1(\omega) = [0, 1] \setminus \{\omega\}$  if  $\omega \in (0, 1]$ . Then,  $P^*$  is a possibility measure on  $\mathcal{P}([0, 1])$  and  $\Gamma_1$  is C5 but not C4.
2. Consider the measurable space  $(\{-1, 0, 1\}, \mathcal{P}(\{-1, 0, 1\}))$ , and  $\Gamma_2 : [0, 1] \rightarrow \mathcal{P}(\{-1, 0, 1\})$  given by  $\Gamma_2(\omega) = \{0\} \forall \omega \in (0, 1), \Gamma_2(0) = \{-1\}, \Gamma_2(1) = \{1\}$ . Then,  $\Gamma_2$  is C3 but not C1, and  $P^*$  is a possibility measure.
3. Take  $(\{0, 1\}, \mathcal{P}(\{0, 1\}))$ , and  $\Gamma_3 : [0, 1] \rightarrow \mathcal{P}(\{0, 1\})$  given by  $\Gamma_3(\omega) = \{0, 1\} \forall \omega \in (0, 1), \Gamma_3(0) = \{0\}, \Gamma_3(1) = \{1\}$ . Then,  $\Gamma_3$  is C4 but not C3, and  $P^*$  is a possibility measure.

We summarize the relationships among the nesting conditions when  $P^*$  is a possibility measure in figure 2.

Figure 2: Relationships among the conditions when  $P^*$  is a possibility on  $\mathcal{P}(X)$ .

Last proposition is also interesting for it helps us understand the situation for the finite case. We can check that condition C6 is equivalent to  $P^*(J) = \max_{x \in J} P^*(\{x\}) \forall J$  finite, and this is equivalent in the case of  $X$  finite to  $P^*$  maxitive. This does not hold for an infinite referential, as we deduce from example 4.1.

Let us give now a couple of results concerning the images of the random set and their relation with the upper probability.

**Proposition 4.2.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, d)$  a compact metric space,  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a closed-valued random set. If  $P^*$  is maxitive, then  $\Gamma$  is C2.*

**Proof:** As  $X$  is compact, for every  $n$  there is a finite measurable partition of  $X$   $\{F_1^n, \dots, F_{m_n}^n\}$  s.t.  $\delta(F_i^n) \leq \frac{1}{2^n} \forall i = 1, \dots, m_n$ . Let us define recursively  $\{A_1^n, \dots, A_{k_n}^n\} := \{F_i^n \cap A_j^{n-1} : i = 1, \dots, m_n, j = 1, \dots, k_{n-1}\}$ ,  $n \geq 2$ , and  $\{A_1^1, \dots, A_{k_1}^1\} := \{F_1^1, \dots, F_{m_1}^1\}$ ; we obtain a sequence of measurable partitions s.t. each of them is finer than the precedent.



Define  $\Gamma_n : \Omega \rightarrow \mathcal{P}(X)$  by  $\Gamma_n(\omega) = \cup\{A_i^n : \omega \in \Gamma^*(A_i^n)\}$ . It is  $\Gamma_n(\omega) = \emptyset \Leftrightarrow \Gamma(\omega) = \emptyset$ , whence  $\Gamma_n^*(X) = \Gamma^*(X)$ . Moreover,  $x \in \Gamma_n(\omega)$  implies  $d(x, \Gamma(\omega)) \leq \frac{1}{2^n}$ , because  $\delta(A_i^n) \leq \frac{1}{2^n} \forall i$ .

- Given  $A \in \beta(d)$ , it is  $\Gamma_n^*(A) = \cup\{\Gamma^*(A_i^n) : A \cap A_i^n \neq \emptyset\}$ . Hence,  $\Gamma_n$  is strongly measurable.
- $\Gamma_n$  is simple, because  $k_n$  is finite.
- For every  $\omega \in \Omega$ ,  $\Gamma(\omega) = \cap_n \Gamma_n(\omega)$ : the construction of the partitions  $\{A_i^n : i = 1, \dots, k_n\}$  implies  $\Gamma(\omega) \subseteq \Gamma_{n+1}(\omega) \subseteq \Gamma_n(\omega) \forall n$ , whence  $\Gamma(\omega) \subseteq \cap_n \Gamma_n(\omega)$ . Conversely,  $x \in \Gamma_n(\omega) \Rightarrow d(x, \Gamma(\omega)) \leq \frac{1}{2^n}$ , and then  $x \in \cap_n \Gamma_n(\omega)$  implies  $x \in \Gamma(\omega)$ , because this set is closed.
- $P_{\Gamma_n}^*(A) = \frac{P(\Gamma_n^*(A))}{P(\Gamma_n^*(X))} = \frac{P(\Gamma^*(\cup_{A \cap A_i^n \neq \emptyset} A_i^n))}{P(\Gamma^*(X))} = P_{\Gamma^*}^*(\cup_{A \cap A_i^n \neq \emptyset} A_i^n) \forall A \in \beta(d)$ ; the maxitivity of  $P^*$  implies then that  $P_{\Gamma_n}^*$  is maxitive.

Reasoning in the same way as in the finite case, we can deduce that  $\Gamma_n$  is  $C2$ : we can assume without loss of generality  $P_{\Gamma_n}^*(A_1^n) \leq P_{\Gamma_n}^*(A_2^n) \leq \dots \leq P_{\Gamma_n}^*(A_{k_n}^n)$ . Take  $M_i = (A_i^n)^* \setminus (A_{i+1}^n)^*, i = 1, \dots, k_n - 1$ . Then,  $N_n = \cup_{i=1}^{k_n-1} M_i$  is null and we can check that  $\forall \omega_1, \omega_2 \in \Omega \setminus N_n$ , it is  $\Gamma_n(\omega_1) \subseteq \Gamma_n(\omega_2)$  or  $\Gamma_n(\omega_2) \subseteq \Gamma_n(\omega_1)$ .

Take now  $N = \cup_n N_n$  null,  $\omega_1, \omega_2 \in \Omega \setminus N$ . Then,  $\Gamma(\omega_1) = \cap_n \Gamma_n(\omega_1)$ ,  $\Gamma(\omega_2) = \cap_n \Gamma_n(\omega_2)$ . If  $\Gamma(\omega_1) \setminus \Gamma(\omega_2) \neq \emptyset$ , there exists some  $n_1 \in \mathbb{N}$  s.t.  $\Gamma(\omega_1) \setminus \Gamma_n(\omega_2) \neq \emptyset \forall n \geq n_1$ , whence  $\Gamma_m(\omega_1) \setminus \Gamma_n(\omega_2) \neq \emptyset \forall n \geq n_1, \forall m$ . Similarly,  $\Gamma(\omega_2) \setminus \Gamma(\omega_1) \neq \emptyset \Rightarrow \exists n_2 \in \mathbb{N}$  s.t.  $\Gamma(\omega_2) \setminus \Gamma_m(\omega_1) \neq \emptyset \forall m \geq n_2 \Rightarrow \Gamma_n(\omega_2) \setminus \Gamma_m(\omega_1) \neq \emptyset \forall m \geq n_2, \forall n$ . Given  $n_3 = \max\{n_1, n_2\}$ , it is  $\Gamma_{n_3}(\omega_1) \setminus \Gamma_{n_3}(\omega_2) \neq \emptyset, \Gamma_{n_3}(\omega_2) \setminus \Gamma_{n_3}(\omega_1) \neq \emptyset$ , a contradiction. Therefore,  $\Gamma$  is  $C2$ . ■

Next we extend this proposition for more general cases. Let us previously recall a property proven by Arstein<sup>4</sup>:

**Proposition 4.3.** *[1, lemma 3-3] Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \tau)$  a Polish space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a compact-valued random set. Then, there exists a  $\sigma$ -compact set  $E$  s.t.  $P_*(E) = 1$ .*

<sup>4</sup>Although his result comes in a context where  $(\Omega, \mathcal{A}, P)$  is assumed to be a Polish space, this hypothesis is not necessary for the proposition.

**Theorem 4.4.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, d)$  a metric space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. If  $P^*$  is maxitive, then  $\Gamma$  is C2 if we are under any of the following conditions:*

- $\Gamma$  is compact-valued and  $X$  is Polish.
- $\Gamma$  is closed-valued and  $X$  is  $\sigma$ -compact.

**Proof:**

- From proposition 4.3, there is a  $\sigma$ -compact set  $E = \cup_n K_n$  s.t.  $P_*(E) = 1$ . We can assume that the sequence of compact sets  $(K_n)_n$  is non-decreasing. Fixed  $n$ , the multi-valued mapping  $\Gamma_n : \Omega \rightarrow \mathcal{P}(X)$  given by  $\Gamma_n(\omega) = \Gamma(\omega) \cap K_n$  is strongly measurable and closed-valued on the compact space  $(K_n, d|_{K_n})$ . Moreover, it satisfies  $P_{\Gamma_n}^*(A) = \frac{P(\Gamma^*(A \cap K_n))}{P(\Gamma^*(K_n))} \forall A \in \beta(\tau)$ . Therefore,  $P_{\Gamma_n}^*$  is maxitive, and from proposition 4.2  $\Gamma_n$  is C2. That is, there exists  $N_n \subset \Omega$  null s.t. for any  $\omega_1, \omega_2 \in \Omega \setminus N_n$ , it is either  $\Gamma_n(\omega_1) \subseteq \Gamma_n(\omega_2)$  or  $\Gamma_n(\omega_2) \subseteq \Gamma_n(\omega_1)$ .

Take  $N = (\cup_n N_n) \cup ((E_*)^c \cap X^*)$  null, and consider  $\omega_1, \omega_2 \in \Omega \setminus N$ . Then, it is  $\Gamma(\omega_1), \Gamma(\omega_2) \subseteq E$ , whence  $\Gamma(\omega_1) = \cup_n \Gamma_n(\omega_1)$ ,  $\Gamma(\omega_2) = \cup_n \Gamma_n(\omega_2)$ .

If  $\Gamma(\omega_1) \not\subseteq \Gamma(\omega_2)$ , there is some  $n_1 \in \mathbb{N}$  s.t.  $\Gamma_{n_1}(\omega_1) \not\subseteq \Gamma(\omega_2)$ , whence  $\Gamma_m(\omega_1) \not\subseteq \Gamma(\omega_2) \forall m \geq n_1$ . Conversely,  $\Gamma(\omega_2) \not\subseteq \Gamma(\omega_1)$  implies the existence of some  $n_2 \in \mathbb{N}$  s.t.  $\Gamma_m(\omega_2) \not\subseteq \Gamma(\omega_1) \forall m \geq n_2$ . Take  $n_3 = \max\{n_1, n_2\}$ . Then, since  $\Gamma_{n_3}(\omega_2) \subseteq \Gamma(\omega_2)$ , we have  $\Gamma_{n_3}(\omega_1) \not\subseteq \Gamma_{n_3}(\omega_2)$  and similarly  $\Gamma_{n_3}(\omega_2) \not\subseteq \Gamma_{n_3}(\omega_1)$ . This is a contradiction. Hence, it is  $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$  or  $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$  for any  $\omega_1, \omega_2 \in \Omega \setminus N$ , and  $\Gamma$  is C2.

- The proof is similar to that of the first point, now with  $X$  playing the role of  $E$ . ■

This theorem is interesting because most authors (see for instance [18]) only consider closed and compact-valued random sets on  $(\mathbb{R}^n, \beta)$ , and we have proven that in this case the supremum-preserving property of  $P^*$  guarantees the consonance C2 of  $\Gamma$ .

Note also that, from the relationships among the conditions (see figure 1), we can deduce that when  $P^*$  is maxitive on a metric space and we are under

any of the hypotheses of theorem 4.4,  $\Gamma$  is also consonant  $C4, C5$  and  $C6$ . As we have already remarked, it will not imply that  $\Gamma$  is  $C1$  (nor  $C3$ ).

Even though we have clarified the situation for the most important classes of random sets, it is interesting to study the behaviour of random sets not necessarily closed or compact-valued and whose upper probability is a possibility measure. In this sense, we have proven a connection between these type of random sets and their closure:

**Proposition 4.5.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \tau)$  a Polish space and  $\Gamma$  a random set s.t.  $P_\Gamma^*$  is maxitive and  $\bar{\Gamma}$  is strongly measurable. If either of the following conditions holds:*

- $\bar{\Gamma}(\omega)$  is compact for every  $\omega$ ,
- $X$  is locally compact,

then  $\bar{\Gamma}$  is consonant  $C2$ .

**Proof:**

- Assume first that  $\bar{\Gamma}$  is compact, and let us show that then  $P_\Gamma^*$  maxitive implies  $P_{\bar{\Gamma}}^*$  maxitive:

Note that given  $G$  open,  $\bar{\Gamma}(\omega) \cap G \neq \emptyset$  if and only if  $\Gamma(\omega) \cap G \neq \emptyset$ , by the definition of closure. Hence,  $\bar{\Gamma}^*(G) = \{\omega : \bar{\Gamma}(\omega) \cap G \neq \emptyset\} = \{\omega : \Gamma(\omega) \cap G \neq \emptyset\} = \Gamma^*(G)$ . From proposition 2.5 and the compactness of  $\bar{\Gamma}(\omega) \forall \omega$ , it is  $P_{\bar{\Gamma}}^*(A) = \inf_{A \subseteq G_{\text{open}}} P_\Gamma^*(G) = \inf_{A \subseteq G_{\text{open}}} P_\Gamma^*(G) \forall A \in \beta(\tau)$ .

Consider  $A, B \in \beta(\tau)$ , and assume for instance  $\max\{P_\Gamma^*(A), P_\Gamma^*(B)\} = P_\Gamma^*(A)$ . For every  $n$ , there are  $C_n, D_n \in \tau$  s.t.  $A \subseteq C_n, B \subseteq D_n, P_\Gamma^*(C_n) - P_\Gamma^*(A) < \frac{1}{n}, P_\Gamma^*(D_n) - P_\Gamma^*(B) < \frac{1}{n}$ . Now,

$$\begin{aligned} P_{\bar{\Gamma}}^*(A \cup B) &= \inf_{(A \cup B) \subseteq G_{\text{open}}} P_\Gamma^*(G) \leq \inf_n P_\Gamma^*(C_n \cup D_n) = \inf_n P_\Gamma^*(C_n \cup D_n) = \\ &= \inf_n \max\{P_\Gamma^*(C_n), P_\Gamma^*(D_n)\} \leq \inf_n (P_\Gamma^*(A) + \frac{1}{n}) = P_\Gamma^*(A), \end{aligned}$$

whence  $P_{\bar{\Gamma}}^*(A \cup B) = P_\Gamma^*(A)$  and  $P_{\bar{\Gamma}}^*$  is maxitive.

Now, applying theorem 4.4, we deduce that  $\bar{\Gamma}$  is  $C2$ .

- Taking into account part (4) of theorem 2.7, we can prove, proceeding as in the previous point that  $P_{\bar{\Gamma}}^*$  is maxitive. Now, locally compact Polish spaces are in particular  $\sigma$ -compact metric spaces. Applying theorem 4.4, we deduce that  $\bar{\Gamma}$  is  $C2$ . ■

Let us give a few comments on this result:

**Example 4.2.** *The converse of the proposition is not true in general: consider  $\Gamma : [0, 1] \rightarrow \mathcal{P}([0, 1])$  given by  $\Gamma(\omega) = \mathbb{Q} \cap [0, 1]$  if  $\omega \leq \frac{1}{2}$ ,  $\Gamma(\omega) = \mathbb{I} \cap [0, 1]$  otherwise. Then,  $P_{\bar{\Gamma}}^*$  is not maxitive, because  $P^*({0.1, \frac{\pi}{4}}) = 1 > \frac{1}{2} = P^*({0.1}) = P^*({\frac{\pi}{4}})$ . However,  $\overline{\Gamma(\omega)} = [0, 1] \forall \omega$ , whence  $\bar{\Gamma}$  is  $C2$ .*

**Remark 4.2.** *The result can be given more generally in the following way: if  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  is a random set such that  $\bar{\Gamma}$  is strongly measurable and  $P_{\bar{\Gamma}}^*$  satisfies the approximation  $P_{\bar{\Gamma}}^*(A) = \inf_{A \subseteq G_{open}} P_{\bar{\Gamma}}^*(G)$ , then  $P_{\bar{\Gamma}}^*$  maxitive implies  $P_{\bar{\Gamma}}^*$  maxitive.*

**Remark 4.3.** *It is not clear to us whether this proposition holds for random sets on a  $\sigma$ -compact metric space; i.e., we would like to know if the closure of a random set on a  $\sigma$ -compact metric space whose upper probability is maxitive is a  $C2$  random set. The property  $P^*(A) = \inf_{A \subseteq G_{open}} P^*(G)$ , which seems to us essential for the proof, has only been proven for the particular case of locally compact Polish spaces, which are in fact locally compact  $\sigma$ -compact metric spaces. This is the only proposition in this paper where compact random sets on Polish spaces and closed random sets on  $\sigma$ -compact metric spaces behave differently. Hence, it would be interesting to study whether the local compactness can be removed.*

In particular, we infer that if  $\Gamma$  is a random set on  $(\mathbb{R}^n, \beta)$  and  $P_{\bar{\Gamma}}^*$  is a possibility measure, the closure random set  $\bar{\Gamma}$  will be consonant  $C2$ .

Finally, we are going to give an additional property of random sets inducing a possibility measure. Let us recall a result from Goodman ([13]):

**Proposition 4.6.** *Let  $\Pi : \mathcal{P}(X) \rightarrow [0, 1]$  be a possibility measure. Consider  $f : [0, 1] \rightarrow [0, 1]$  a uniformly distributed random variable, and define  $\Gamma : [0, 1] \rightarrow \mathcal{P}(X)$  by  $\Gamma(\alpha) = \{\omega \in X \mid \Pi(\omega) \geq f(\alpha)\}$ . Then,  $\Gamma$  is a random set (it is strongly measurable) and  $P_{\bar{\Gamma}}^*$  coincides with  $\Pi$ .*

With this proposition, Goodman gives a Choquet representation theorem for possibility measures: he shows that given a possibility measure, there is always a random set (which is consonant) inducing it. This result has also been commented by De Cooman and Aeyels [4], who have shown that we can also consider the strict  $\alpha$ -cut  $\Gamma(\alpha) = \{\omega \in \Omega : \Pi(\omega) > \alpha\}$ .

We have proven something more important in our context: if the upper probability induced by a random set is a possibility measure, there is a consonant random set *defined between the same spaces* inducing this possibility measure. Hence, the class of random sets inducing a possibility measure could be reduced, for practical purposes, to the class of consonant random sets inducing a possibility measure. This shows the importance of these particular type of multi-valued mappings.

**Theorem 4.7.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \mathcal{P}(X))$  a measurable space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set s.t.  $P_\Gamma^*$  is a possibility measure. Then, there exists a C1 random set  $\Gamma' : \Omega \rightarrow \mathcal{P}(X)$  satisfying  $P_{\Gamma'}^* = P_\Gamma^*$ .*

**Proof:** Let us denote  $A_x = \{y : P_\Gamma^*(\{y\}) \geq P_\Gamma^*(\{x\})\}$ ,  $B_x = \{y : P_\Gamma^*(\{y\}) \leq P_\Gamma^*(\{x\})\}$ . Note that  $P_\Gamma^*(B_x) = P_\Gamma^*(\{x\})$ , for  $P_\Gamma^*$  is a possibility measure. Define  $\Gamma' : \Omega \rightarrow \mathcal{P}(X)$  by  $\Gamma'(\omega) = \cup_{x \in \Gamma(\omega)} A_x$ .

- $\Gamma'$  is strongly measurable: given  $x \in X$ , it is  $\Gamma'^*(\{x\}) = \{\omega : x \in \Gamma'(\omega)\} = \{\omega : \exists y \in \Gamma(\omega), x \in A_y\} = \{\omega : \exists y \in \Gamma(\omega), y \in B_x\} = \Gamma^*(B_x) \in \mathcal{A}$ , because  $\Gamma$  is strongly measurable. Now, given  $C \subseteq X$ ,  $\Gamma'^*(C) = \cup_{x \in C} \Gamma'^*(\{x\}) = \cup_{x \in C} \Gamma^*(B_x) = \Gamma^*(\cup_{x \in C} B_x) \in \mathcal{A}$ .
- $\Gamma'$  is C1: consider  $\omega_1, \omega_2 \in \Omega$ , and let us define  $z_{\omega_1} := \inf_{x \in \Gamma(\omega_1)} P^*(\{x\})$ ,  $z_{\omega_2} := \inf_{x \in \Gamma(\omega_2)} P^*(\{x\})$ . If  $z_{\omega_1}$  is a minimum, it is  $\Gamma'(\omega_1) = \{y : P^*(\{y\}) \geq z_{\omega_1}\}$  and otherwise it is  $\Gamma'(\omega_1) = \{y : P^*(\{y\}) > z_{\omega_1}\}$ ; the same happens with  $\Gamma'(\omega_2)$  respect to  $z_{\omega_2}$ . Hence, either  $\Gamma'(\omega_1) \subseteq \Gamma'(\omega_2)$  or  $\Gamma'(\omega_2) \subseteq \Gamma'(\omega_1)$ .
- Given  $x \in X$ ,  $P_{\Gamma'}^*(\{x\}) = P_\Gamma^*(B_x) = P_\Gamma^*(\{x\})$ . Therefore, the one-point coverage functions of  $P_\Gamma^*$  and  $P_{\Gamma'}^*$  coincide.
- Given  $C \subset X$ , it is  $P_{\Gamma'}^*(C) = P_\Gamma^*(\cup_{x \in C} B_x) = \sup_{x \in C} P_\Gamma^*(B_x) = \sup_{x \in C} P_\Gamma^*(\{x\}) = \sup_{x \in C} P_{\Gamma'}^*(\{x\})$ . Hence,  $P_{\Gamma'}^*$  coincides with  $P_\Gamma^*$  and it

is a possibility measure. ■

Note that we are considering the  $\sigma$ -field  $\mathcal{P}(X)$  on the final space in order to guarantee the strong measurability of  $\Gamma'$ , for which  $\Gamma^*(B_x)$  must belong to  $\mathcal{A}$  for every  $x$ .

The random set we are constructing in this theorem is not always closed-valued, as we can see if we consider  $\Gamma : [0, 1] \rightarrow \mathcal{P}([0, 1])$  given by  $\Gamma(\omega) = \mathbb{Q} \cap [0, 1] \forall \omega$ : in that case,  $A_x = \mathbb{Q} \cap [0, 1]$  if  $x \in \mathbb{Q} \cap [0, 1]$ , and  $A_x = [0, 1]$  otherwise. Hence,  $\Gamma'(\omega) = \Gamma(\omega) \forall \omega \in [0, 1]$ .

## 5 $\Gamma$ nested implies $P^*$ possibility?

We aim our attention now towards the inverse problem; that is, we study whether a consonant random set induces a possibility measure or not. Concerning this problem we must remark that condition  $C1$  will imply that  $P^*$  is a possibility measure if and only if  $C2$  does so, and that the same applies to conditions  $C3$  and  $C4$ . This is because the behaviour of  $\Gamma$  on a null subset of  $\Omega$  does not alter the upper probability. One of the main results on the subject is the following:

**Proposition 5.1.** [5] *Consider the measurable space  $([0, 1], \beta_{[0,1]})$  and a probability measure  $P$  on it. Take a measurable space  $(X, \mathcal{P}(X))$ , and a multi-valued mapping  $\Gamma : [0, 1] \rightarrow \mathcal{P}(X)$ . If*

$$[\forall (x, y) \in [0, 1]^2] [x \geq y \Rightarrow \Gamma(x) \subseteq \Gamma(y)],$$

*then  $\Gamma$  is strongly measurable and induces a possibility measure.*

This proposition extends a result from [9], and also proposition 4.6 can be derived as a corollary. On the other hand, it also appears in [4], where it is proven for the case of an arbitrary lattice structure on the initial space. However, in both cases the argument identifies the nesting order on the images with the order on the initial lattice, in the sense of  $s \prec t \Rightarrow \Gamma(s) \supseteq \Gamma(t)$ . Moreover, in [4] some additional requirements are made on the images of the random set. However, we are looking here at the general case where the random set goes from an *arbitrary* probability space to an *arbitrary* measurable space, and where no

other assumption is made on the images apart from the nesting. This has been one of the main motivations for our more general definitions of consonance.

Our goal in this section is to see if we can loosen the prerequisites that appear in proposition 5.1. The following result gives a positive answer for maxitive measures:

**Proposition 5.2.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, \mathcal{A}')$  a measurable space, and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a  $C2$  random set. Then,  $P^*$  is maxitive.*

**Proof:** If  $\Gamma$  is  $C2$ , there is some  $N \subseteq \Omega$  null s.t.  $\forall \omega_1, \omega_2 \in \Omega \setminus N$ ,  $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$  or  $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$ . Suppose that  $P^*$  is not maxitive. Then, there exist  $A, B \in \mathcal{A}'$  s.t.  $P^*(A \cup B) > \max\{P^*(A), P^*(B)\}$ . Hence,  $P(A^* \setminus B^*) > 0$ ,  $P(B^* \setminus A^*) > 0$ , and we can select in particular  $x_1 \in (\Omega \setminus N) \cap (A^* \setminus B^*)$ ,  $x_2 \in (\Omega \setminus N) \cap (B^* \setminus A^*)$ . Because of the  $C2$  condition it should be either  $\Gamma(x_1) \subseteq \Gamma(x_2)$  or  $\Gamma(x_2) \subseteq \Gamma(x_1)$ . Let us assume for instance  $\Gamma(x_1) \subseteq \Gamma(x_2)$ . Then,  $x_1 \in A^* \Rightarrow \Gamma(x_1) \cap A \neq \emptyset$ , whence  $\Gamma(x_2) \cap A \neq \emptyset$  and  $x_2 \in A^*$ . This contradicts  $x_2 \in B^* \setminus A^*$ . We get an analogous contradiction if  $\Gamma(x_2) \subseteq \Gamma(x_1)$ . Therefore,  $P^*$  is maxitive. ■

In particular, this proposition generalizes the result for the finite case. Unfortunately, in general the consonance of  $\Gamma$  does not imply that  $P^*$  is supremum-preserving, as we showed in a counterexample in [20]. Let us briefly outline the main ideas:

**Example 5.1.** *Consider a well-order in  $[0, 1]$  (existing because of Zermelo's theorem), and denote  $P_x$  the set of predecessors of  $x$ . Let  $x_0$  be the first element with an uncountable number of predecessors, and define a probability measure  $Q$  on  $\mathcal{A} = \sigma(\{P_x : x \in [0, 1]\})$  satisfying  $Q(A) = 1$  if there exists  $x \in P_{x_0}$  such that  $P_{x_0} \setminus P_x \subset A$ , and  $Q(A) = 0$  otherwise. The multi-valued mapping  $\Gamma : ([0, 1], \mathcal{A}, Q) \rightarrow \mathcal{P}([0, 1])$  given by  $\Gamma(\omega) = (P_\omega)^c$  is  $C1$  but its upper probability is not a possibility measure.*

The key in this example is to construct a non-condensable upper probability. For this, it suffices to have some set  $A$  with  $P^*(A) > P^*(B)$  for every  $B \subseteq A$  countable.

This example and the previous proposition point out an advantage of maxitive set functions over possibility measures: they capture the behaviour of

consonant random sets without any additional assumption, whereas a consonant random set must satisfy some extra requirements in order to induce a possibility measure, as we show next.

**Theorem 5.3.** *Let  $(\Omega, \mathcal{A}, P)$  be a probability space,  $(X, d)$  a metric space and  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$  a random set. Then, if  $\Gamma$  is consonant C2,  $P^*$  is a possibility measure when any of the following conditions hold:*

- $\Gamma$  is compact-valued and  $X$  is Polish.
- $\Gamma$  is closed-valued and  $X$  is  $\sigma$ -compact.

**Proof:** It is a consequence of theorem 2.8 and proposition 5.2. ■

**Remark 5.1.** *We can see that even if  $\Gamma$  satisfies the previous two hypotheses, conditions C3 to C6 do not imply that  $P^*$  is a possibility measure (nor that it is maxitive): take for instance  $(\Omega, \mathcal{A}, P) = ([0, 1], \beta_{[0,1]}, \lambda)$ , and let  $\Gamma : [0, 1] \rightarrow \mathcal{P}([0, 1])$  be given by  $\Gamma(x) = \{x\}$ ; then, for any  $\omega_1, \omega_2 \in [0, 1]$ , it is  $P^*(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = \lambda(\{\omega_1\}) = 0$ ,  $P^*(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) = \lambda(\{\omega_2\}) = 0$ . Hence,  $\Gamma$  is C3 (and consequently C4, C5, C6). However,  $P^* = \lambda$ , which is not maxitive.*

If we join these results with the ones from the previous section, we see that even if neither of the implications holds in general for possibility measures (i.e.,  $P^*$  possibility does not imply  $\Gamma$  C1, ..., C5 and  $\Gamma$  C1, ..., C6 does not imply  $P^*$  possibility), we have the equivalence for some particular cases:

**Corollary 5.4.** *Consider  $(\Omega, \mathcal{A}, P)$  a probability space,  $(X, d)$  a metric space and a random set  $\Gamma : \Omega \rightarrow \mathcal{P}(X)$ .*

1. *If  $\Gamma$  is closed and  $X$  is  $\sigma$ -compact, or if  $\Gamma$  is compact and  $X$  Polish, then the following are equivalent:*

- $\Gamma$  is C2.
- $P^*$  is a possibility measure
- $P^*$  is maxitive.
- $P^*(K) = \max_{x \in K} P^*(\{x\})$  for every  $K$  compact.



2. Assume now  $X$  Polish and  $\bar{\Gamma}$  strongly measurable. If  $X$  is locally compact or  $\bar{\Gamma}$  is compact, then  $P_{\Gamma}^*$  maxitive implies  $\bar{\Gamma}$  is  $C2$ .

**Proof:**

1. From theorem 2.8, we have the equivalence among  $P^*$  possibility,  $P^*$  maxitive and  $P^*(K) = \max_{x \in K} P^*({x}) \forall K$  compact under these conditions. The equivalence with condition  $C2$  follows from theorems 4.4 and 5.3.
2. It is proposition 4.5. ■

In particular we have the equivalence between possibility measures and consonant random sets when  $\Gamma$  is closed-valued on  $\mathbb{R}^n$ . Moreover, if the upper probability of a random set  $\Gamma$  on  $\mathbb{R}^n$  is maxitive, then the closure random set  $\bar{\Gamma}$  is  $C2$ . This is interesting because closed-random sets on  $\mathbb{R}^n$  are mostly used in practice.

The results obtained in this paper are drawn in figures 3 and 4. The first one represents the relationships between consonant random sets and possibility measures that hold in general, and the second one shows the situation for closed random sets on  $\sigma$ -compact metric spaces.

Figure 3: Relationships among the nesting conditions,  $P^*$  possibility and  $P^*$  maxitive in general.

Figure 4: Relationships among the nesting conditions,  $P^*$  possibility and  $P^*$  maxitive when  $\Gamma$  is closed-valued on  $(X, d)$   $\sigma$ -compact.

## 6 Concluding remarks

In this paper, we have completed the studies we initiated in [20] about consonant random sets. We have characterized these mappings through the upper probability they induce. This upper probability is uniquely determined in the case of a finite referential by its focal elements, which are in correspondence with the images of the random set.

In the general framework we have studied here, there are no focal elements that we can take advantage of. Another difficulty is the lack of a general definition of consonance: most of the authors who have given results on the subject have considered multi-valued mappings defined on  $([0, 1], \beta_{[0,1]})$  or, more generally, on a lattice, and have established a correspondence between the order on the lattice and the one we have on the images of the random set. For general probability spaces, it becomes necessary to compare all pairs of elements in the initial space to check the nesting of their images in the final space. This has provided us six definitions of consonance. Conditions  $C1$  and  $C2$  are arguably the most intuitive ones, and they also turn out to capture the essence of possibility measures. With definitions  $C3$  to  $C6$  we have made a study of nesting by means of the upper inverses and probabilities, which are the elements extending the concept of distribution from random variables to random sets. These conditions are in fact too weak to deal with the consonance in a proper way, mainly because in the infinite case we can partition the final space by an uncountable number of sets of null upper probability.

We have also paid attention to maxitive set functions, as an alternative to possibility measures. These two types of set functions are not equivalent in general, and we have studied in this paper the properties that separate them. These properties make maxitive set functions more related to consonance than possibility measures, contrary to what might have been expected: we have checked that in general the upper probability of a  $C2$  random set is maxitive and is not necessarily a possibility measure. Nevertheless, we have the equivalence among  $C2$  consonance, possibility and maxitive measures for fairly general cases, such as compact random sets on Polish spaces or closed random sets on a  $\sigma$ -compact metric spaces. These are the types of random sets mostly used in practice. In those cases, possibility and maxitive measures are also equivalent to another condition, namely being maximum-preserving on compact sets.

It is interesting to remark that although compact random sets on Polish spaces and closed random sets on  $\sigma$ -compact metric spaces do not have the same implications on the upper probability in general (for instance, in the first case  $P^*$  is always a Choquet capacity, and this does not hold for the second one), we have shown that they share the same behaviour in the context of possibility

measures and consonant random sets.

Note also that although our attention has been centered on the behaviour of closed or compact random sets, we have obtained some results on multi-valued mappings where no assumption is made on the images. In this sense, given a random set on a locally compact Polish space (like  $\mathbb{R}^n$ ), we have obtained a necessary condition for its upper probability to be maxitive in terms of the consonance (namely that  $\bar{\Gamma}$  is  $C2$ ), and also a different, sufficient one (that  $\Gamma$  is  $C2$ ). Moreover, we have showed by different counterexamples that these conditions are not sufficient and necessary, respectively.

We conclude outlining some of the open problems derived from this research: we intend to make a deeper study of the relationship between maxitive and possibility measures. Also, other particular cases of random sets could be studied. Nevertheless, we are not very optimistic about the existence of positive results for those cases, because we have used some specific properties of the classes of random sets considered in this paper, such as the capacitability of the Borel sets.

Finally, let us remark that one of the reasons why a consonant random set always induces a maxitive measure but not necessarily a possibility measure might be that in our definition of consonance we are comparing the images of pairs of elements of the initial space. We could then wonder if a condition of consonance where we compared the images of all elements on an infinite set would be more related with possibility measures. It is not clear to us at this point whether this is the case or if a consonance condition of this type would be equivalent to some of those considered here.

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