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RELATIONSHIPS BETWEEN POSSIBILITY MEASURES AND NESTED RANDOM SETS

ENRIQUE MIRANDA, INÉS COUSO, and PEDRO GIL
Department of Statistics and Operations Research
University of Oviedo
C/ Calvo Sotelo, s/n 33007 Oviedo, Spain
e-mail: (alu426, couso, pedro)@pinon.ccu.uniovi.es

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Different authors have observed some relationships between consonant random sets and possibility measures, specially for finite universes. In this paper, we go deeply into this matter and propose several possible definitions for the concept of consonant random set. Three of these conditions are equivalent for finite universes. In that case, the random set considered is associated to a possibility measure if and only if any of them is satisfied. However, in a general context, none of the six definitions here proposed is sufficient for a random set to induce a possibility measure. Moreover, only one of them seems to be necessary.

Keywords: Possibility measure, Dempster-Shafer theory of evidence, random set, upper probability.

1. Introduction

Since they were defined by Zadeh [27] in 1978, possibility measures have been studied in a variety of contexts. On the one hand, they are very related to fuzzy set theory, as we observe in [3,8,10,15] and [27]. In fact, possibility measures provide us a mathematical tool to handle natural language (see for instance [4,23,25,27]). On the other hand, they also constitute a special class of upper probabilities in the theory of imprecise probabilities [22], as it is argued in [5,23,24] and [26].

In this paper, we do not deal with the discussion about the interpretation of possibility measures, but we treat some mathematical aspects concerning random set theory [14,18] and Dempster-Shafer's theory of evidence [7,21]. When a finite referential set is considered, a possibility measure is equivalent, in some sense, to a class of nested random sets (see [16,20,21]). In some recent works, (see [2,3,6,11]) some relationships between possibility measures and a particular class of nested random sets are shown for arbitrary referential sets, not necessarily finite. We ask ourselves if the equivalence relation between possibility measures and nested random sets that is fulfilled on finite referential sets also stands in the more general case of arbitrary final spaces, and without any assumption on the images of the random set. To our surprise, the answer to the previous question is negative: we will give examples in this paper where the upper probability induced by a nested random set is not necessarily a possibility measure. Conversely, we are also going to show that a random set associated to a possibility measure does not need to be nested; neither it must satisfy other weaker "nesting" conditions here considered.

The paper is organized as follows. In section 2 we give a brief introduction to random set theory, Dempster-Shafer's theory of evidence and possibility measures, which is necessary for understanding the rest of the paper. The concept of "consonant random set" is studied in detail in section 3, where some different definitions are proposed. Then we deal with the relationships between this concept and possibility measures, firstly for finite referential sets (section 4) and secondly for the general case (section 5). Finally, a few comments and concluding remarks on the studies in this paper are given.

2. Preliminary concepts and notation

Given a probabilistic space $(\Omega, \mathcal{A}, P)$ and some referential Ω', a *random set* is a mapping defined on Ω with values on $\mathcal{P}(\Omega')$, $\Gamma : \Omega \to \mathcal{P}(\Omega')$, which is measurable for some σ-algebra defined on some subset of $\mathcal{P}(\Omega')$. In this paper, we consider the σ-algebra generated by the class $\boldsymbol{\mathcal{C}}(\mathcal{A}') = \{\mathcal{C}_B \mid B \in \mathcal{A}'\}$, where $\mathcal{A}'$ is some σ-algebra defined on Ω' and $\mathcal{C}_B = \{C \subseteq \Omega' \mid C \cap B \neq \emptyset\}$, for all $B \in \mathcal{A}'$. For any measurable subset of Ω', $B \in \mathcal{A}'$, the set $\Gamma^{-1}(\mathcal{C}_B)$ will be called the *upper inverse* [19] of B. From now on we will use the simpler notation $B^* = \Gamma^{-1}(\mathcal{C}_B)$ and $B_* = [\Gamma^{-1}(\mathcal{C}_{B^c})]^c$. Under this measurability condition, the induced *upper* and *lower probabilities* [7] of any measurable set B are well defined by the formulae:

$$\begin{aligned} \mathrm{P_\Gamma}^*(B) = P(\{\omega \in \Omega \mid \Gamma(\omega) \cap B \neq \emptyset\}) / P(\{\omega \in \Omega \mid \Gamma(\omega) \neq \emptyset\}) = \\ P(\Gamma^{-1}(\mathcal{C}_B) \mid \Gamma^{-1}(\mathcal{C}_{\Omega'})) = P(B^* \mid \Omega'^*),\ \forall\, B \in \mathcal{A}' \end{aligned} \tag{1}$$

$$\begin{aligned} \mathrm{P}_{\Gamma *}(B) = P(\{\omega \in \Omega \mid \Gamma(\omega) \subseteq B,\, \Gamma(\omega) \neq \emptyset\}) / P(\{\omega \in \Omega \mid \Gamma(\omega) \neq \emptyset\}) = \\ P([\Gamma^{-1}(\mathcal{C}_{B^c})]^c \mid \Gamma^{-1}(\mathcal{C}_{\Omega'})) = P(B_* \mid \Omega'^*),\ \forall\, B \in \mathcal{A}'. \end{aligned} \tag{2}$$

In [21], Shafer offers a reinterpretation of Dempster's work on upper and lower probabilities for the particular case of a finite referential space. He defines a *basic probability assignment* as a function $m : \mathcal{P}(\Omega') \to [0,1]$ satisfying the properties $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega'} m(A) = 1$. The (finite) class of sets $\mathcal{F} = \{A \subseteq \Omega' \mid m(A) > 0\}$ is called the class of *focal elements* of m. Any function Bel: $\mathcal{P}(\Omega') \to [0,1]$ obtained from a basic probability assignment by the formula:

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B), \forall\, A \subseteq \Omega'$$

is called a *belief function*. The author also defines the *dual plausibility measure*, Pl:$\mathcal{P}(\Omega') \to [0,1]$, of a belief function Bel by

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(A^c),\ \forall\, A \subseteq \Omega'.$$

It is related to the corresponding basic probability assignment by the formula

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B),\ \forall\, A \in \mathcal{P}(\Omega').$$

A belief function Bel and a plausibility Pl are said to be *dual* when they are obtained from the same basic probability assignment m. In that case, it can be shown that this assignment is unique and satisfies

$$m(A) = \sum_{B \subset A} (-1)^{|A-B|}\mathrm{Bel}(B),\ \forall A \subset \Omega'.$$

The reader can find in [21] further explanation about the semantic interpretation of these three concepts.

When a random set takes values on a finite space, Dempster's lower and upper probabilities constitute a pair of dual belief and plausibility functions. Furthermore, the basic probability assignment associated to them coincides with the probability mass of the particular random set considered.

When the focal elements of Bel, Pl are nested, i.e., they can be arranged in an order such that each one is contained in the following one, the associated pair of belief and plausibility functions are called *consonant*. In that case, they satisfy the following equations [21] :

1. $\mathrm{Bel}(A \cap B) = \min\{\mathrm{Bel}(A), \mathrm{Bel}(B)\}, \forall\, A, B \in \mathcal{P}(\Omega')$
2. $\mathrm{Pl}(A \cup B) = \max\{\mathrm{Pl}(A), \mathrm{Pl}(B)\}, \forall\, A, B \in \mathcal{P}(\Omega')$
3. $\mathrm{Pl}(A) = \max_{x \in A} \mathrm{Pl}(\{x\}), \forall\, A \in \mathcal{P}(\Omega')$.

Since Bel and Pl are dual set functions, these three conditions are equivalent. The converse of the above result above mentioned is also true. That is, when a pair of dual belief and plausibility functions satisfies any of these three equations, then the focal elements must be nested.

Our previous remark about the relation between belief/plausibility functions and lower/upper probabilities induced by a random set allows us to think of *consonant* random sets in terms of the consonance of their induced upper and lower probabilities. This concept will be detailed in the following sections.

Consonant belief and plausibility measures are usually referred to as *necessity* and *possibility measures*, respectively. The definition of possibility measure can be extended to the case of arbitrary referential sets as follows.

Definition 1 [17] *Given a measurable space* $(\Omega', \mathcal{A}')$, *a* **possibility measure** *is a function* $\Pi : \mathcal{A}' \to [0,1]$ *satisfying:*

P1 $\Pi(\emptyset) = 0$

P2 $\Pi(\Omega') = 1$, *and*

P3 *The grade of possibility of an arbitrary union of sets coincides with the supremum of their possibility grades, i.e., for any family* $\{A_j \mid j \in J\}$ *of measurable subsets of* Ω',

$$\Pi\Big(\bigcup_{j \in J} A_j \Big) = \sup_{j \in J} \Pi(A_j).$$

A possibility measure is usually defined on $\mathcal{P}(\Omega')$ (see [15], for instance). Nevertheless, the possibility measures here considered will be induced by an $\mathcal{A}$-$\sigma\langle \boldsymbol{C}(\mathcal{A}')\rangle$ measurable set-valued function, and hence we need this more general definition. Some authors (see [6], for instance) do not require Π to fulfill property **P2** and they call *normal* to any possibility measure satisfying it. This consideration is not relevant to the results in this paper.

The existence of random sets with nested images associated with any possibility measure is shown in [6,11] and [12]. The following result can be found in [11]:

Theorem 1 *Let* $\Pi : \mathcal{P}(\Omega') \to [0,1]$ *be a possibility measure. Consider the multi-valued mapping* $\Gamma : [0,1] \to \mathcal{P}(\Omega')$ *given by* $\Gamma(\alpha) = \{\omega \in \Omega \mid \Pi(\omega) \geq \alpha\}$. *Then,* Γ *is a random set (it is measurable) and* P^*_Γ *coincides with* Π.

This theorem establishes that, for an arbitrary possibility measure, there exists at least one random set with the same upper probability. The author calls it the *random level set* associated with Π. We also observe that the images of the random level set Γ are nested, since $\alpha_1 \leq \alpha_2$ implies that $\Gamma(\alpha_1) \supseteq \Gamma(\alpha_2)$. We can also find in [6] this other result:

Theorem 2 *Consider the the Borel measurable space* $([0,1], \beta_{[0,1]})$ *and an arbitrary probability measure defined on it,* P. *Let us also consider another referential* Ω'. *If the multi-valued mapping* $\Gamma : [0,1] \to \mathcal{P}(\Omega')$ *is antitone, i.e., it satisfies the following condition:*

$$\left[\forall\, (x,y) \in [0,1]^2\right] \; [x \geq y \Rightarrow \Gamma(x) \subseteq \Gamma(y)]\,, \tag{3}$$

then it is $\beta_{[0,1]}$-$\sigma\langle \boldsymbol{C}(\mathcal{P}(\Omega'))\rangle$ *measurable and it induces a possibility measure on* $\mathcal{P}(\Omega')$.

We wonder if this relationship between antitone random sets and possibility measures is simply a coincidence or, on the contrary, the concept of possibility measure is somehow related to some nesting property on random sets. To answer this question, we need first to specify what we will mean by “nested” or“consonant” random set. We will propose some different definitions.

3. Consonant random sets

In this section, we introduce several possible definitions for the concept of “consonant random set”. Three of them are equivalent in the case of a finite referential set, as we show in section 4. However, this is not true for an arbitrary referential. The interest of the three remaining definitions is justified in sections 4 and 5. From now on, we will assume that the singletons in the final space, Ω', are measurable subsets.

Definition 2 *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $(\Omega', \mathcal{A}')$ a measurable space. We say that the random set $\Gamma : \Omega \to \mathcal{P}(\Omega')$ is* **consonant C1** *when the family of sets $\{\Gamma(\omega) \mid \omega \in \Omega\}$ is totally ordered for the inclusion relation, i.e., if for arbitrary $\omega_1, \omega_2 \in \Omega$, we have either $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$ or $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$.*

Condition C1 is equivalent to the following one.

Lemma 3 *Given a probability space $(\Omega, \mathcal{A}, P)$, and $(\Omega', \mathcal{A}')$ a measurable space, the random set $\Gamma : \Omega \to \mathcal{P}(\Omega')$ is consonant C1 if and only if the class of sets $\{\{\omega'\}^* \mid \omega' \in \Omega'\}$ is totally ordered for the inclusion relation, in other words, when for arbitrary $x_1, x_2 \in \Omega'$, at least one of these conditions is satisfied: $\{x_1\}^* \subseteq \{x_2\}^*$ or $\{x_2\}^* \subseteq \{x_1\}^*$.*

Proof *Taking into account the equivalence relation*

$$x \in \Gamma(\omega) \Longleftrightarrow \omega \in \{x\}^*, \ \forall\, \omega \in \Omega,\, x \in \Omega',$$

the result is immediately derived.$\square$

Remark 1 *Note that the existence of a possibility measure associated to the random set Γ will not guarantee in general the nesting property for its images on every point of Ω: the behaviour of Γ on a null subset of Ω does not affect its upper probability. Hence, condition C1 seems to be too restrictive, and we must consider weaker conditions.*

Definition 3 *Under the general hypotheses of the last definition, we say that the random set Γ is* **consonant C2** *if there exists a null set of $N \in \mathcal{N}_P$, such that $\forall \omega_1, \omega_2 \in \Omega \setminus N$, at least one of these two conditions is fulfilled: $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$ or $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$, i.e., when condition C1 is satisfied except on a null subset of Ω.*

The following is a slight variation of definition 2.

Definition 4 *Under the preceding general hypothesis, we say that* Γ *is* **consonant C3** *when* $\forall\,\omega_1,\omega_2 \in \Omega, \Gamma(\omega_1)\backslash\Gamma(\omega_2)$ *or* $\Gamma(\omega_2)\backslash\Gamma(\omega_1)$ *is null, for the upper probability* P^*_Γ.

Remark 2 *We show in* [2] *how the concepts of* null set *and* completed σ-algebra *can be successfully extended to the case of non-additive measures, when they are monotone, sub-additive and lower continuous. The upper probability associated to an arbitrary random set satisfies all these properties.*

The final space is "probabilized" by the set function P^*_Γ associated to Γ. Condition C3 means that the set difference between the images of two elements is not necessarily the empty set as in condition C1, but it is included in a set of null upper probability.

Definition 5 *Under the hypotheses of previous definitions, we will say that* Γ *is* **consonant C4** *if* $\forall\,\omega_1,\omega_2 \in \Omega\setminus N$, $\Gamma(\omega_1)\setminus\Gamma(\omega_2)$ *or* $\Gamma(\omega_2)\setminus\Gamma(\omega_1)$ *is* P^*_Γ*-null for some null set* $N \in \mathcal{N}_P$. *In other words, when condition C3 is satisfied on a subset with probability 1, but not necessarily on the whole space* Ω.

The following condition is weaker than all the previous ones.

Definition 6 *Under the general hypotheses above considered, we say that* Γ *is* **consonant C5** *if one of the two following conditions is satisfied:*

- *The initial set* Ω *has only one element.*
- *There exists* $N \in \mathcal{N}_P$ *a null subset of* Ω *such that* $\forall\,\omega_1 \in \Omega\setminus N$, $\exists\,\omega_2 \neq \omega_1$ *s.t.* $P^*_\Gamma(\Gamma(\omega_1)\setminus\Gamma(\omega_2)) = 0$ *or* $P^*_\Gamma(\Gamma(\omega_2)\setminus\Gamma(\omega_1)) = 0$.

The interest of condition C5 comes from its weakness. When it is not fulfilled, the set of elements whose images are not nested with the images of any other is not null. This is incompatible with the intuitive notion of consonance. We will show in section 5 that a random set associated to a possibility measure does not need to satisfy condition C5. Now we will introduce the last definition of this section.

Definition 7 *Under the general hypotheses above considered, we say that* Γ *is* **consonant C6** *if* $\forall\,x_1,\,x_2 \in \Omega'$ *it is* $P(\{x_1\}^*\setminus\{x_2\}^*) = 0$ *or* $P(\{x_2\}^*\setminus\{x_1\}^*) = 0$.

We observe that, when this condition is satisfied, each pair of elements in Ω' cannot be "separated" by images of Γ, since for any $x_1, x_2 \in \Omega'$ it is not possible to find two measurable sets $A, B \in \mathcal{A}$ with positive probabilities such that $x_1 \in \Gamma(\omega), x_2 \notin \Gamma(\omega), \forall\,\omega \in A$ and $x_1 \notin \Gamma(\omega), x_2 \in \Gamma(\omega), \forall\,\omega \in B$.

There exist some implication relationships among the conditions we have introduced in this section.

Theorem 4 *Consider* $(\Omega, \mathcal{A}, P)$ *a probability space,* $(\Omega', \mathcal{A}')$ *a measurable space and* $\Gamma : \Omega \longrightarrow \mathcal{P}(\Omega')$ *a random set.*

1. *If* Γ *is consonant C1, then it satisfies conditions C2 and C3.*
2. *If* Γ *is consonant C2, then it satisfies condition C4.*
3. *If* Γ *is consonant C3, then it satisfies condition C4.*
4. *If* Γ *is consonant C4, then it satisfies conditions C5 and C6.*

Proof

- *We deduce 1, 2 and 3 immediately.*
- *To prove the first implication of 4, assume that* Γ *satisfies condition C4 for some null set N, but for some* $\{\omega_1\} \notin N$ *there does not exist* $\omega_2 \neq \omega_1$ *such that* $P^*_\Gamma(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = 0$ *or* $P^*_\Gamma(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) = 0$. *In that case, the complement* $\Omega \setminus N$ *must be the singleton* $\{\omega_1\}$. *Hence, if* Ω *had at least two elements,* $P^*_\Gamma(\Gamma(\omega_2) \setminus \Gamma(\omega_1))$ *should be zero, for all* $\omega_2 \neq \omega_1$, *since, in that case,* $(\Gamma(\omega_2) \setminus \Gamma(\omega_1))^* \subset \{w_1\}^c = N$. *Thus,* Ω *has to be a singleton, and condition C5 holds.*

  Let us now prove by contradiction that condition C4 implies C6. Suppose that we can find two elements in the final space $x_1\, x_2 \in \Omega'$ *satisfying* $P(\{x_1\}^* \setminus \{x_2\}^*) > 0$ *and* $P(\{x_2\}^* \setminus \{x_1\}^*) > 0$. *Then, for an arbitrary null set* $N \in \mathcal{N}_P$, *we know that there exist at least two elements* $\omega_1 \in (\{x_1\}^* \setminus \{x_2\}^*) \cap (\Omega \setminus N)$ *and* $\omega_2 \in (\{x_2\}^* \setminus \{x_1\}^*) \cap (\Omega \setminus N)$. *We can say, equivalently, that* $x_1 \in \Gamma(\omega_1) \setminus \Gamma(\omega_2)$ *and* $x_2 \in \Gamma(\omega_2) \setminus \Gamma(\omega_1)$. *Hence, we obtain:*

  - $P^*_\Gamma(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) \geq P^*_\Gamma(\{x_1\}) \geq P(\{x_1\}^* \setminus \{x_2\}^*)/P(\Omega^*) > 0.$
  - $P^*_\Gamma(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) \geq P^*_\Gamma(\{x_2\}) \geq P(\{x_2\}^* \setminus \{x_1\}^*)/P(\Omega^*) > 0.$

  This contradicts condition C4, and the proof is completed.$\square$

Condition C2 does not imply C3, as we will show in the following section. Conversely, C3 does not imply C2 in general (it suffices to consider $\Gamma : [0,1] \to \mathcal{P}([0,1])$ given by $\Gamma(x) = \{x\}$). Neither C5 implies C6, as we show in the following example:

Example 1 *Consider the initial probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$, *where* $\beta_{[0,1]}$ *is the Borel* σ*-algebra induced on* $[0,1]$ *by the usual metric and* $\lambda_{[0,1]}$ *is the restriction of Lebesgue measure to the unit interval* $[0,1]$. *Let* $\Gamma : [0,1] \to \mathcal{P}(\{1,2\})$ *be defined as* $\Gamma(\omega) = \{1\}$ *if* $\omega \in [0, 0.5]$, $\Gamma(\omega) = \{2\}$ *if* $\omega \in (0.5, 1]$. *This random set* *satisfies condition C5, since* $\forall\, \omega \in \Omega,\ \exists\, \omega' \in \Omega$ *such that* $\Gamma(\omega) = \Gamma(\omega')$ *whence* $P^*_\Gamma(\Gamma(\omega) \setminus \Gamma(\omega')) = 0$. *However,* $P(\{1\}^* \setminus \{2\}^*) > 0$, $P(\{2\}^* \setminus \{1\}^*) > 0$, *and C6 is not satisfied.*

Condition C6 does not imply any other previous nesting condition, as we will justify later in section 5.

Now we will investigate the relationships between the concepts introduced in this section and the properties of the upper probability associated to a random set. First we will limit our attention to the case where the final space Ω' is finite and secondly we will deal with the general case.

4. The finite case

In this section we will find necessary and sufficient conditions for a random set to be associated with a possibility measure in the particular case where the referential Ω' is finite. Recall from section 2 that P^*_Γ is a possibility measure if and only if its focal elements are nested sets. On the other hand, we can easily prove the following lemma:

Lemma 5 *Consider* $(\Omega, \mathcal{A}, P)$ *a probability space and* $(\Omega', \mathcal{P}(\Omega'))$ *a measurable space, with* Ω' *finite. Let* $\Gamma : \Omega \to \mathcal{P}(\Omega')$ *be a random set and consider the induced belief measure given by the lower probability, Bel:*$\mathcal{P}(\Omega') \to [0, 1]$. *Then we have:*

1. $\Gamma^{-1}(\{A\}) = \{\omega \in \Omega \mid \Gamma(\omega) = A\} \subseteq \Omega$ *is* $\mathcal{A}$*-measurable,* $\forall\, A \subseteq \Omega'$.
2. *Bel has focal elements* $\mathcal{F} = \{A_1, \ldots, A_m\}$ *with mass* $m(A_i) = m_i,\ \forall\, i = 1, \ldots, m$ *if and only if* $P(\{\omega \in \Omega \mid \Gamma(\omega) = A_i\}) = m_i,\ \forall\, i = 1, \ldots, m$. *Under these conditions,* $P(\{\omega \in \Omega \mid \Gamma(\omega) \notin \mathcal{F}\}) = 0$.

First we will look for necessary conditions for Γ to induce on Ω' a possibility measure.

Proposition 6 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$ *and the measurable space* $(\Omega', \mathcal{P}(\Omega'))$, *where* Ω' *is finite. Consider a random set* $\Gamma : \Omega \to \mathcal{P}(\Omega')$. *If* Γ *is consonant C6 then its upper probability is a possibility measure.*

Proof *Since* Ω' *is finite, it suffices to check the maximization property of possibility measures for pairs of elements of* Ω'. *Let us make a proof by contradiction. Suppose there exist two elements* $x_1, x_2 \in \Omega'$ *such that* $P^*_\Gamma(\{x_1, x_2\}) > \max\{P^*_\Gamma(\{x_1\}), P^*_\Gamma(\{x_2\})\}$. *Then, since* $\{x_1\}^* \cup \{x_2\}^* = \{x_1, x_2\}^*$, *we obtain* $P(\{x_1\}^* \setminus \{x_2\}^*) > 0$ *and* $P(\{x_2\}^* \setminus \{x_1\}^*) > 0$. *Hence, condition C6 is not satisfied. This completes the proof.*$\Box$

Remark 3 *A similar proof could be used to show that the converse of this last implication is also true.*

It is derived that conditions C1 to C4 are sufficient for P^*_Γ to be a possibility measure. As we could expect, condition C5 does not imply that P^*_Γ satisfies the maximization property of possibility measures:

Example 2 *Let us consider the same random set as in example 1,* $\Gamma : [0,1] \to \mathcal{P}(\{1,2\})$ *given by* $\Gamma(\omega) = \{1\}$ *if* $\omega \in [0,0.5]$*, and* $\Gamma(\omega) = \{2\}$ *if* $\omega \in (0.5,1]$*. Then,* $\forall\, \omega \in \Omega$*, there exists* $\omega' \in \Omega$ *such that* $\Gamma(\omega) = \Gamma(\omega') \Rightarrow P^*_\Gamma(\Gamma(\omega) \setminus \Gamma(\omega')) = 0$*, and C5 holds. However,* $P^*_\Gamma(\{1,2\}) = 1 > \max\{P^*_\Gamma(\{1\}), P^*_\Gamma(\{2\})\} = 0.5$ *and thus* P^*_Γ *is not a possibility measure.*

Now we will examine the converse problem. We will investigate whether a random set associated to a possibility measure needs to be consonant or not. We obtain the following result:

Proposition 7 *Let consider a probability space* $(\Omega, \mathcal{A}, P)$ *and another measurable space* $(\Omega', \mathcal{P}(\Omega'))$*, where* $\Omega' = \{x_1, \ldots, x_n\}$ *is finite. Let also consider a random set* $\Gamma : \Omega \to \mathcal{P}(\Omega')$ *inducing a possibility measure* P^*_Γ *on* $\mathcal{P}(\Omega')$*. Then* Γ *is consonant C2.*

Proof *Suppose that* P^*_Γ *is a possibility measure. Then, as we have noted in remark 3, for every pair* $x_i, x_j \in \Omega'$*, at least one of the difference sets* $\{x_i\}^* \setminus \{x_j\}^*$ *or* $\{x_j\}^* \setminus \{x_i\}^*$ *must be null for the probability measure* P*. Hence, we can order the indices so that* $\{x_i\}^* \setminus \{x_{i-1}\}^*$ *is* P*-null, for all* $i \in 2, \ldots, n$*. Thus, the finite union* $N = \cup_{i=2}^n \{x_i\}^* \setminus \{x_{i-1}\}^*$ *is also null. Then, for every* $\omega \in \Omega \setminus N$ *and* $i \in \{2, \ldots, n\}$*,* $x_i \in \Gamma(\omega) \Rightarrow x_{i-1} \in \Gamma(\omega)$*. Thus,* $\forall\, \omega_1, \omega_2 \in \Omega \setminus N$*, we have either* $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$ *or* $\Gamma(\omega_2) \subseteq \Gamma(\omega_1)$*.*$\square$

We deduce from theorem 4 that any random set associated to a possibility measure must satisfy conditions C4 and C5. Of course, conditions C1 and C3 do not need to be fulfilled, since the behaviour of Γ on a null subset of Ω does not affect its upper probability, as we observed in lemma 5. Let us see the following counterexample.

Example 3 *Consider the initial probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$*, as in example 1. Let* $\Gamma : [0,1] \to \mathcal{P}(\{1,2\})$ *be defined by* $\Gamma(0) = \{1\}, \Gamma(1) = \{2\}$*, and* $\Gamma(\omega) = \{1,2\}\ \forall\, \omega \notin \{0,1\}$*. Clearly* Γ *induces a possibility measure on* $\mathcal{P}(\{1,2\})$*. On the other hand, we have* $P^*_\Gamma(\Gamma(0) \setminus \Gamma(1)) = P^*_\Gamma(\{1\}) = 1$*, and* $P^*_\Gamma(\Gamma(1) \setminus \Gamma(0)) = P^*_\Gamma(\{2\}) = 1$*, and hence conditions C1 and C3 are not fulfilled.*

The main results in this section may be summarized in the following theorem:

Theorem 8 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$ and another measurable space $(\Omega', \mathcal{P}(\Omega'))$, where Ω' is finite. Consider also a random set $\Gamma : \Omega \to \mathcal{P}(\Omega')$ with upper probability P^*_Γ. Then the following conditions are equivalent:*

1. *P^*_Γ is a possibility measure*
2. *Γ is consonant C2*
3. *Γ is consonant C4*
4. *Γ is consonant C6*

5. The general case

In the previous section, we have shown that, for a finite referential, the upper probability induced by a random set is a possibility measure if and only if its images are nested except on a null set. Now we will study the relationship between consonant random sets and possibility measures for the more general case of arbitrary referential sets. First we will investigate which conditions of those introduced in section 3 guarantee that the induced upper probability is a possibility measure. Assuming the truth of Zermelo theorem, we will conclude that condition C1 (which is the strongest nesting condition we have introduced) does not suffice for Γ to be associated to a possibility measure.

We have shown in lemma 3 that a random set fulfills condition C1 if and only if the class of sets $\{\{x\}^* \mid x \in \Omega'\}$ is totally ordered for the inclusion relation. Now we wonder if this last condition implies the equality $P^*_\Gamma(A) = \sup_{x\in A} P^*_\Gamma(\{x\})$ for all A in $\mathcal{A}'$. To answer this query, we will make use of the following auxiliary result.

Lemma 9 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$ and another measurable space $(\Omega', \mathcal{A}')$. Consider also a random set $\Gamma : \Omega \to \mathcal{P}(\Omega')$. Suppose that the class of sets $\{\{x\}^* \mid x \in \Omega'\}$ is totally ordered for the inclusion relation. Then, for an arbitrary measurable set $A \in \mathcal{A}'$, the two following conditions are equivalent:*

1. *$P^*_\Gamma(A) = \sup_{x\in A} P^*_\Gamma(\{x\})$*
2. *There exists a countable set $B \subseteq A$ such that $P^*_\Gamma(B) = P^*_\Gamma(A)$.*

Proof *Consider an arbitrary measurable set $A \in \mathcal{A}'$. First, assume that the equality $P^*_\Gamma(A) = \sup_{x\in A} P^*_\Gamma(\{x\})$ holds; then, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ such that $\lim_{n\to\infty} P^*_\Gamma(\{x_n\}) = \sup_{x\in A} P^*_\Gamma(\{x\}) = P^*_\Gamma(A)$. Hence, by monotonicity of P^*_Γ, the measurable set $B = \cup_{n=1}^\infty \{x_n\}$ satisfies the requirement $P^*_\Gamma(B) = P^*_\Gamma(A)$.*

Conversely, suppose that there exists a countable set $B = \cup_{n=1}^\infty \{x_n\}$ such that $P^*_\Gamma(B) = P^*_\Gamma(A)$. Under the hypotheses of total-ordering this lemma, $P^*_\Gamma(B) =$

$\lim_n P^*_\Gamma(\{x_1,\ldots,x_n\})$ coincides with $\sup_{n\in\mathbb{N}} P^*_\Gamma(\{x_n\})$. Furthermore, by monotonicity of P^*_Γ we know that the inequality $\sup_{x\in A} P^*_\Gamma(\{x\}) \leq P^*_\Gamma(A)$ holds. This implies the equality $P^*_\Gamma(A) = \sup_{x\in A} P^*_\Gamma(\{x\})$. □

In other words, if we assume that Γ satisfies condition C1, then it will induce a possibility measure on $\mathcal{A}'$ if and only if, for all $A \in \mathcal{A}'$ there exists a countable subset $B \subseteq A$ with the same upper probability. To solve our problem, we will first examine if there exists, for an arbitrary chain of sets a countable sub-chain with the same union. The result is false, as we show in the following counterexample.

Example 4 *Let us consider a non-countable set. Let it be, for instance, the set of real numbers, $\mathbb{R}$. By Zermelo theorem [13], there exists a well-ordering "$\leq$" on it, i.e., a total-ordering such that any subset of $\mathbb{R}$ has a first element. Let us use the notation $P_x = \{y \in \mathbb{R} \mid y < x\}$. We are going to prove the existence an element x_0 with an uncountable number of predecessors, but such that every preceding element has a countable number of predecessors.*

Let x be an element such that P_x is non countable. If P_y is countable for all $y \in P_x$, then x is the element we are looking for, x_0. Let us consider the set $H = \{y \in P_x \mid P_y \text{ non countable}\}$ in the opposite case. Since "$\leq$" is a well-ordering, H has a first element, which is the element we wanted to find. Once the element x_0 is determined, we can observe that the equality $P_{x_0} = \cup_{y<x_0} P_y$ holds. In fact, for any $y \in P_{x_0}$, there exists $z \in P_{x_0}$ such that $y \in P_z$; if this was not the case, the equality $P_{x_0} = P_y \cup \{y\}$ should hold, and P_{x_0} should be countable. The total-ordering "$\leq$" allows us to establish a total-ordering on $\{P_y \mid y \in P_{x_0}\}$ for the inclusion relation. Hence, we have obtained a chain with no countable sub-chains with the same union, since all the elements of the chain are countable.

Next we will make use of these considerations to find a consonant C1 random set whose induced upper probability is not a possibility measure. First we will construct a suitable probability space:

Let $\mathcal{C}$ denote the class $\{P_x \mid x \in \mathbb{R}\}$. First we will describe the algebra generated by $\mathcal{C}$. To this purpose, we will consider the classes:

$\mathcal{C}_1 := \{\mathbb{R}, \emptyset, P_x, P_x^c \mid x \in \mathbb{R}\}$

$\mathcal{C}_2 := \{\cap_{j=1}^n B_j \mid B_j \in C_1\} = \{\mathbb{R}, \emptyset, P_x, P_y, P_x \setminus P_y \mid x, y \in \mathbb{R}\}$

$\mathcal{Q} := \{D_1 \cup \ldots \cup D_l \mid D_j \in C_2, D_i \cap D_j = \emptyset \ \forall i \neq j, l \in \mathbb{N}\}$.

It can be proven (see [1]) that $\mathcal{Q}$ coincides with the algebra generated by $\mathcal{C}$. Note also that the singletons belong to this algebra (as we are requiring in this paper), for given $x \in \mathbb{R}$ we can determine x' the following element through $<$, and then it is $\{x\} = P_{x'} \setminus P_x$.

To construct a suitable counterexample for our general problem, we look for a σ-additive set function P' defined on $\mathcal{Q}$ whose restriction to $\mathcal{C}$ is given by:

$$P'\rceil_{\mathcal{C}}(P_x) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geq x_0 \end{cases}$$

The set function $P' : \mathcal{Q} \to [0,1]$ given by:

$$P'(A) = \begin{cases} 1 & \text{if } \exists\, z < x_0 \text{ s.t. } P_{x_0} \setminus P_z \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

satisfies the conditions required: we can easily check that it is finitely additive, $P'(\mathbb{R}) = 1$ and $P'(\emptyset) = 0$. The construction made above makes $\lim_n P'(A_n) = 0$ when $(A_n)_n \downarrow \emptyset$.

Hence, by Carathéodory theorem, it can be uniquely σ-additively extended to $\sigma(\mathcal{Q})$. Let us denote this extension by P. Once we have found a particular probability space satisfying the properties we were looking for, we construct the counterexample.

Example 5 *Consider the probability space $(\mathbb{R}, \sigma(\mathcal{Q}), P)$ described above. Define the multi-valued mapping $\Gamma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by $\Gamma(x) = \{y \in \mathbb{R} \mid y > x\}$ for the well-ordering previously described. For an arbitrary $A \in \beta_{\mathbb{R}}$, the set $A^* = \{x \in \mathbb{R} \mid \exists\, y \in A,\ y > x\} = P_{\sup A}$ belongs to $\sigma(\mathcal{Q})$, and hence, Γ is $\sigma(\mathcal{Q})$-$\sigma\langle \mathbf{C}(\beta_{\mathbb{R}})\rangle$ measurable (remark that the supremum is taken in the well-ordering we are considering). Let x_0 be the element with the properties described in the previous example. Then we observe that $(P_{x_0})^* = P_{x_0} \Rightarrow P^*_\Gamma(P_{x_0}) = P(P_{x_0})$. This does not coincide with $P^*_\Gamma(B)$ for any countable subset of P_{x_0}, B, since we have $B^* = \cup_{x \in B}\{x\}^* = \cup_{x\in B} P_x$ and $P(\cup_{x \in B} P_x) = 0$, by definition of P. Thus, P^*_Γ is not a possibility measure. However the random set Γ is consonant C1 because of the properties of total ordering of "$\leq$".*

Note that this counterexample does not contradict theorem 2, because the σ-albegra considered here does not coincide with the Borel σ-algebra on $\mathbb{R}$.

It is obvious that none of the nesting conditions introduced in section 3 is sufficient for Γ to be associated to a possibility measure, since C1 is the strongest one. In relation with the converse problem, we deduce from our results in section 4 that C1 and C3 are not necessarily satisfied. In fact, in this more general case, conditions C2 and C4 do not necessarily hold, and neither does C5. Let us show an example.

Example 6 *Consider $((0.5, 0.75), \beta_{(0.5,0.75)}, 4\lambda\rceil_{\beta_{(0.5,0.75)}})$ a probability space, with $\beta_{(0.5,0.75)}$ the restriction to $(0.5, 0.75)$ of the Borel σ-algebra induced by the usual distance in $\mathbb{R}$ and λ the Lebesgue measure. Let us define the random set $\Gamma : (0.5, 0.75) \to \mathcal{P}([0,1])$ by $\Gamma(\omega) = [0,\omega] \setminus \{2\omega - 1\}$, $\forall\, \omega \in (0.5, 0.75)$.*

Let us now consider two arbitrary elements, $\omega_1, \omega_2 \in (0.5, 0.75)$ and suppose that $\omega_1 < \omega_2$. Then we have $\omega_2 \in \Gamma(\omega_2) \setminus \Gamma(\omega_1)$, $P^*_\Gamma(\{\omega_2\}) = 4\,\lambda([\omega_2, 0.75)) > 0$; on the other hand, $P^*_\Gamma(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) = P^*_\Gamma(\{2\omega_2 - 1\}) > 0$. Hence, for any pair of elements $\omega_1, \omega_2 \in (0.5, 0.75)$ we see that $P^*_\Gamma(\Gamma(\omega_2) \setminus \Gamma(\omega_1)) > 0$, and $P^*_\Gamma(\Gamma(\omega_1) \setminus \Gamma(\omega_2)) > 0$, whence Γ does not fulfill condition P5. However, we can verify that P^*_Γ is a possibility measure: for an arbitrary $A \in \beta_{[0,1]}$, $P^*_\Gamma(A) = 4\,\lambda(A^*) = 4\,\lambda(\cup_{x\in A}\{x\}^*) = 4\,\lambda(\cup_{x \in A}[x, 0.75) \setminus \{\frac{x+1}{2}\}) = 3(0.75 - \inf_{x \in A} x) = \sup_{x\in A} P^*_\Gamma(\{x\})$.

Thus, the equivalence we have found in section 2 for the finite case cannot be extended for the general case. We can only say that Γ must necessarily satisfy condition C6 in order to induce a possibility measure. But this nesting condition is not sufficient, since it is weaker than C4, and this last condition does not suffice, as we have explained above.

Remark 4 *Some authors have paid attention to* maxitive *set functions, that is, functions* Π *satisfying* $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}\ \forall A, B$, *but not necessarily supremum-preserving. It is clear that these functions are equivalent to possibility measures in the finite case, but not in general. We can deduce from example 6 that if the upper probability induced by the random set is maxitive, it does not necessarily hold that* Γ *is consonant C5. Moreover, we can see that condition C6 is necessary in order for* P^*_Γ *to be maxitive. Conversely, it can be checked that conditions C1 and C2 are sufficient for the maxitivity of* P^*_Γ *(in contradistinction with possibility measures!), but conditions C3 to C6 are not sufficient (take for instance* $\Gamma : [0,1] \to \mathcal{P}([0,1])$ *given by* $\Gamma(x) = \{x\}$*).*

6. Concluding remarks

We have examined the problem of finding sufficient and necessary conditions that the images of a random set should satisfy in order to induce a possibility measure. To this purpose, we have proposed six different possible definitions of the concept of "consonant random set". For the particular case of finite universes, we have obtained an equivalence relation between three of these definitions and possibility measures, which is easily derived from a well-known result in Shafer's theory of evidence. As the main contribution in the paper, we have constructed suitable examples to show that this equivalence relation does not extend to arbitrary non finite universes.

Nevertheless, the initial probability space and the images of the random set considered in example 5 have very particular properties. We think that perhaps some additional topological requirements would produce interesting results about the relationship between consonant random sets and possibility measures. The most interesting case would be the one where $\Omega' = \mathbb{R}^n$, and Γ being closed-valued, because some authors consider mainly this type of random sets (see for instance [18]). We intend to work on this particular case in the future. On the other hand, in counterexample 6, we observe that none of the first five nesting conditions here studied is fulfilled. However, the images of different elements in the initial space are somehow related. Hence, we do not reject the possibility of finding successful results for some other nesting condition not considered here.

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