LEXICOGRAPHIC CHOICE FUNCTIONS

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ABSTRACT. We investigate a generalisation of the coherent choice functions considered by Seidenfeld et al. (2010), by sticking to the convexity axiom but imposing no Archimedeanity condition. We define our choice functions on vector spaces of options, which allows us to incorporate as special cases both Seidenfeld et al.’s (2010) choice functions on horse lotteries and also pairwise choice—which is equivalent to sets of desirable gambles (Quaeghebeur, 2014)—, and to investigate their connections.

We show that choice functions based on sets of desirable options (gambles) satisfy Seidenfeld’s convexity axiom only for very particular types of sets of desirable options, which are exactly those that are representable by lexicographic probability systems that have no non-trivial Savage-null events. We call them lexicographic choice functions. Finally, we prove that these choice functions can be used to determine the most conservative convex choice function associated with a given binary relation.

1. INTRODUCTION

Since the publication of the seminal work of Arrow (1951) and Uzawa (1956), coherent choice functions have been used widely as a model of the rational behaviour of an individual or a group. In particular, Seidenfeld et al. (2010) established an axiomatisation of coherent choice functions, generalising Rubin’s (1987) axioms to allow for incomparability. Under this axiomatisation, they proved a representation theorem for coherent choice functions in terms of probability-utility pairs: a choice function \( C \) satisfies their coherence axioms if and only if there is some non-empty set \( S \) of probability-utility pairs such that \( f \in C(A) \) whenever the option \( f \) maximises \( p \)-expected \( u \)-utility over the set of options \( A \) for some \((p,u)\) in \( S \).

Allowing for incomparability between options may often be of crucial importance. Faced with a choice between two options, a subject may not have enough information to establish a (strict or weak) preference of one over the other: the two options may be incomparable. This will indeed typically be the case when the available information is too vague or limited. It arises quite intuitively for group decisions, but also for decisions made by a single subject, as was discussed quite thoroughly by Williams (1975), Levi (1980), and Walley (1991), amongst many others. Allowing for incomparability lies at the basis of a generalising approach to probability theory that is often referred to by the term imprecise probabilities. It unifies a diversity of well-known uncertainty models, including typically non-linear (or non-additive) functionals, credal sets, and sets of desirable gambles; see the introductory book by Augustin et al. (2014) for a recent overview. Among these, coherent sets of desirable gambles, as discussed by Quaeghebeur (2014), are usually considered to constitute the most general and powerful type of model. Such sets collect the gambles that a given subject considers strictly preferable to the status quo.

Nevertheless, choice functions clearly lead to a still more general model than sets of desirable gambles, because the former’s preferences are not necessarily completely determined by the pair-wise comparisons between options that essentially constitute the latter. This was of course already implicit in Seidenfeld et al.’s (2010) work, but was investigated in detail in one of our recent papers (Van Camp et al., 2017), where we

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zoomed in on the connections between choice functions, sets of desirable gambles, and indifference.

In order to explore the connection between indifference and the strict preference expressed by choice functions, we extended the above-mentioned axiomatisation by Seidenfeld et al. (2010) to choice functions defined on vector spaces of options, rather than convex sets of horse lotteries, and also let go of two of their axioms: (i) the Archimedean one, because it prevents choice functions from modelling the typically non-Archimedean preferences captured by coherent sets of desirable gambles; and (ii) the convexity axiom, because it turns out to be hard to reconcile with Walley–Sen maximality as a decision rule, something that is closely tied in with coherent sets of desirable options (Troffaes, 2007). Although our alternative axiomatisation allows for more leeway, and for an easy comparison with the existing theory of sets of desirable gambles, it also has the drawback of no longer forcing a representation theorem, or in other words, of not leading to a strong belief structure (we refer to De Cooman (2005) for a more detailed discussion of belief models that constitute a strong belief structure). Such a representation is nevertheless interesting because strong belief structures have the advantage that their coherent models are infima (under a partial order implicit in the structure) of their dominating maximally informative models. This allows for reasoning with the (typically simpler) maximally informative dominating models, instead of the (possibly more complex) models themselves. In an earlier paper (Van Camp et al., 2017), we discussed a few interesting examples of special ‘representable’ choice functions, such as the ones from a coherent set of desirable gambles via maximality, or those determined by a set of probability measures via E-admissibility.

The goal of the present paper is twofold: to (i) further explore the connection of our definition of choice functions with Seidenfeld et al.’s (2010); and to (ii) investigate in detail the implications of Seidenfeld et al.’s (2010) convexity axiom in our context. We will prove that, perhaps somewhat surprisingly, for those choice functions that are uniquely determined by binary comparisons, convexity is equivalent to being representable by means of a lexicographic probability measure. This is done by first establishing the implications of convexity in terms of the binary comparisons associated with a choice function, giving rise to what we will call lexicographic sets of desirable gambles. These sets include as particular cases the so-called maximal and strictly desirable sets of desirable gambles. Although in the particular case of binary possibility spaces these are the only two possibilities, for more general spaces lexicographic sets of gambles allow for a greater level of generality, as one would expect considering the above-mentioned equivalence.

A consequence of our equivalence result is that we can consider infima of choice functions associated with lexicographic probability measures, and in this manner subsume the examples of E-admissibility and M-admissibility discussed by Van Camp et al. (2017). It will follow from the discussion that these infima also satisfy the convexity axiom. As one particularly relevant application of these ideas, we prove that the most conservative convex choice function associated with a binary preference relation can be obtained as the infimum of its dominating lexicographic choice functions.

The paper is organised as follows. In Section 2, we recall the basics of coherent choice functions on vector spaces of options as introduced in our earlier work (Van Camp et al., 2015). We motivate our definitions by showing in Section 3 that they include in particular coherent choice functions on horse lotteries, considered by Seidenfeld et al.’s (2010), and we discuss in some detail the connection between the rationality axioms considered by Seidenfeld et al.’s (2010) and ours.

As a particularly useful example, we discuss in Section 4 those choice functions that are determined by binary comparisons. We have already shown before (Van Camp et al., 2017) that this leads to the model of coherent sets of desirable gambles; here we study the implications of including convexity as a rationality axiom.
In Section 2, we motivate our definition of lexicographic choice functions and study the properties of their associated binary preferences. We prove the connection with lexicographic probability systems and show that the infima of such choice functions can be used when we want to determine the implications of imposing convexity and maximality. We conclude with some additional discussion in Section 6.

2. Coherent choice functions on vector spaces

Consider a real vector space \( V \) provided with the vector addition + and scalar multiplication. We denote the additive identity by 0. For any subsets \( A_1 \) and \( A_2 \) of \( V \) and any \( \lambda \) in \( \mathbb{R} \), we let \( \lambda A_1 := \{ \lambda u : u \in A_1 \} \) and \( A_1 + A_2 := \{ u + v : u \in A_1 \text{ and } v \in A_2 \} \).

Elements of \( V \) are intended as abstract representations of options amongst which a subject can express his preferences, by specifying choice functions. Often, options will be real-valued maps on some possibility space, interpreted as uncertain rewards—and therefore also called gambles. More generally, they can be vector-valued gambles: vector-valued maps on the possibility space. We will see further on that by using such vector-valued gambles, we are able to include as a special case horse lotteries, the options considered for instance by Seidenfeld et al. (2010). Also, we have shown (Van Camp et al., 2017) that indifference for choice functions can be studied efficiently by also allowing equivalence classes of indifferent gambles as options; these yet again constitute a vector space, where now the vectors cannot always be identified easily with maps on some possibility space, or gambles. For these reasons, we allow in general any real vector space to serve as our set of (abstract) possible options. We will call such a real vector space an option space.

We denote by \( Q(V) \) the set of all non-empty finite subsets of \( V \), a strict subset of the power set of \( V \). When it is clear what option space \( V \) we are considering, we will also use the simpler notation \( Q \), and use \( Q_0 \) to denote those option sets that include the option that is constant on 0. Elements \( A \) of \( Q \) are the option sets amongst which a subject can choose his preferred options.

**Definition 1.** A choice function \( C \) on an option space \( V \) is a map

\[
C : Q \to Q \cup \{ \emptyset \} : A \mapsto C(A) \text{ such that } C(A) \subseteq A.
\]

We collect all the choice functions on \( V \) in \( \mathcal{C}(V) \), often denoted as \( \mathcal{C} \) when it is clear from the context what the option space is.

The idea underlying this simple definition is that a choice function \( C \) selects the set \( C(A) \) of ‘best’ options in the option set \( A \). Our definition resembles the one commonly used in the literature (Aizerman, 1985; Seidenfeld et al., 2010; Sen, 1977), except perhaps for an also not entirely unusual restriction to finite option sets (He, 2012; Schwartz, 1972; Sen, 1971).

Equivalently to a choice function \( C \), we may consider its associated rejection function \( R \), defined by \( R(A) := A \setminus C(A) \) for all \( A \) in \( Q \). It returns the options \( R(A) \) that are rejected—not selected—by \( C \).

Another equivalent notion is that of a choice relation. Indeed, for any choice function \( C \)—and therefore for any rejection function \( R \)—the associated choice relation (Seidenfeld et al., 2010 Section 3) is the binary relation \( \triangleleft \) on \( Q \), defined by:

\[
A_1 \triangleleft A_2 \iff A_1 \subseteq R(A_1 \cup A_2) \text{ for all } A_1 \text{ and } A_2 \text{ in } Q.
\] (1)

The intuition behind \( \triangleleft \) is clear: \( A_1 \triangleleft A_2 \) whenever every option in \( A_1 \) is rejected when presented with the options in \( A_1 \cup A_2 \).

2.1. Useful basic definitions and notation. We call \( \mathbb{N} \) the set of all (positive) integers, \( \mathbb{R}_{>0} \) the set of all (strictly) positive real numbers, and \( \mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup \{ 0 \} \).
Given any subset $A$ of $\mathcal{V}$, we define its linear hull $\text{span}(A)$ as the set of all finite linear combinations of elements of $A$:

$$\text{span}(A) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}, u_k \in A \right\} \subseteq \mathcal{V},$$

its positive hull $\text{pos}(A)$ as the set of all positive finite linear combinations of elements of $A$:

$$\text{pos}(A) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in A \right\} \subseteq \text{span}(A) \subseteq \mathcal{V},$$

and its convex hull $\text{CH}(A)$ as the set of convex combinations of elements of $A$:

$$\text{CH}(A) := \left\{ \sum_{k=1}^{n} \alpha_k u_k : n \in \mathbb{N}, \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1, u_k \in A \right\} \subseteq \text{pos}(A) \subseteq \mathcal{V}.$$

A subset $A$ of $\mathcal{V}$ is called a convex cone if it is closed under positive finite linear combinations, i.e. if $\text{pos}(A) = A$. A convex cone $\mathcal{K}$ is called proper if $\mathcal{K} \cap -\mathcal{K} = \{0\}$.

With any proper convex cone $\mathcal{K} \subseteq \mathcal{V}$, we can associate an ordering $\preceq_{\mathcal{K}}$ on $\mathcal{V}$, defined for all $u$ and $v$ in $\mathcal{V}$ as follows:

$$u \preceq_{\mathcal{K}} v \iff v - u \in \mathcal{K}.$$  

We also write $u \succeq_{\mathcal{K}} v$ for $v \preceq_{\mathcal{K}} u$. The ordering $\preceq_{\mathcal{K}}$ is actually a vector ordering: it is a partial order—reflexive, antisymmetric and transitive—that satisfies the following two characteristic properties:

$$u_1 \preceq_{\mathcal{K}} u_2 \iff u_1 + v \preceq_{\mathcal{K}} u_2 + v;  \quad (2)$$

$$u_1 \preceq_{\mathcal{K}} u_2 \iff \lambda u_1 \preceq_{\mathcal{K}} \lambda u_2;  \quad (3)$$

for all $u_1,$ $u_2$ and $v$ in $\mathcal{V}$, and all $\lambda$ in $\mathbb{R}_{>0}$. Observe, by the way, that as a consequence

$$u \preceq_{\mathcal{K}} v \iff 0 \preceq_{\mathcal{K}} v - u \iff u - v \preceq_{\mathcal{K}} 0$$

for all $u$ and $v$ in $\mathcal{V}$.

Conversely, given any vector ordering $\preceq$, the proper convex cone $\mathcal{K}$ from which it is derived can always be retrieved by $\mathcal{K} = \{ u \in \mathcal{V} : u \succeq 0 \}$. When the abstract options are gambles, $\preceq$ will typically be the point-wise order $\leq$, but it need not necessarily be.

Finally, with any vector ordering $\preceq$, we associate the strict partial ordering $<$ as follows:

$$u < v \iff (u \preceq v \text{ and } u \neq v) \iff v - u \in \mathcal{K} \setminus \{0\} \text{ for all } u \text{ and } v \in \mathcal{V}.$$  

We call $u$ positive if $u > 0$, and collect all positive options in the convex cone $\mathcal{V}_{>0} := \mathcal{K} \setminus \{0\}$.

From now on, we assume that $\mathcal{V}$ is an ordered vector space, with a generic but fixed vector ordering $\preceq_{\mathcal{K}}$. We will refrain from explicitly mentioning the actual proper convex cone $\mathcal{K}$ we are using, and simply write $\mathcal{V}$ to mean the ordered vector space, and use $\preceq$ as a generic notation for the associated vector ordering.

2.2. Rationality axioms. We focus on a special class of choice functions, which we will call coherent.

Definition 2. We call a choice function $C$ on $\mathcal{V}$ coherent if for all $A, A_1$ and $A_2$ in $\mathcal{Q}$, all $u$ and $v$ in $\mathcal{V}$, and all $\lambda$ in $\mathbb{R}_{>0}$:

$C(A) \neq \emptyset$;

C2. if $u < v$ then $\{v\} = C(\{u,v\});$

C3. a. if $C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$ then $C(A) \subseteq A \setminus A_1$;

b. if $C(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$ then $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$;

C4. a. if $A_1 \subseteq C(A_2)$ then $A \setminus A_1 \subseteq C(A_2)$;

b. if $A_1 \subseteq C(A_2)$ then $A_1 + \{u\} \subseteq C(A_2 + \{u\});$

We collect all the coherent choice functions on $\mathcal{V}$ in $\hat{C}(\mathcal{V})$, often denoted as $\hat{C}$ when it is clear from the context what the option space is.
Parts $C_{4a}$ and $C_{4b}$ of Axiom $C_4$ are respectively known as Sen’s condition $a$ and Aizerman's condition. They are more commonly written in terms of the rejection function as, respectively:

$$(A_1 \subseteq R(A_2) \text{ and } A_2 \subseteq A) \Rightarrow A_1 \subseteq R(A), \text{ for all } A, A_1, A_2 \text{ in } \mathcal{Q},$$

and

$$(A_1 \subseteq R(A_2) \text{ and } A \subseteq A_1) \Rightarrow A_1 \setminus A \subseteq R(A_2 \setminus A), \text{ for all } A, A_1, A_2 \text{ in } \mathcal{Q}.$$  

Axiom $C_4$ has multiple equivalent forms, which we will use throughout:

**Proposition 1.** Consider any choice function $C$. Then the following equivalences hold:

1. If $C$ satisfies Axiom $C_{4a}$, then
   $$C \text{ satisfies Axiom } C_{4a} \iff (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in C(A) \Rightarrow \lambda u \in C(\lambda A)).$$
   
2. If $C$ satisfies Axiom $C_{4b}$, then
   $$C \text{ satisfies Axiom } C_{4b} \iff (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in C(A) \iff \lambda u \in C(\lambda A)).$$

Proof. For the equivalences involving Axiom $C_{4a}$ we will establish the following chain of implications:

$$C \text{ satisfies Axiom } C_{4a} \Rightarrow C_{4a1} \Rightarrow C_{4a2} \Rightarrow C_{4a3} \Rightarrow C \text{ satisfies Axiom } C_{4a}$$

To establish the first implication, it suffices to consider $A_1 := C(A_2)$ in the statement of Axiom $C_{4a}$ and to consider any $u$ in $A_1$. That $C_{4a1}$ implies $C_{4a2}$ follows by considering $\frac{1}{\lambda} > 0$ instead of $\lambda$. That then $C_{4a3}$ holds is immediate. Finally, to show that $C_{4a3}$ implies $C_{4a}$ it suffices to note that $C_{4a3}$ implies in particular that $\lambda C(A) \subseteq C(\lambda A)$, and therefore, by considering any $A_1 \subseteq C(A)$, Axiom $C_{4a}$ indeed follows.

For the equivalences involving Axiom $C_{4b}$, we will establish the following chain of implications:

$$C \text{ satisfies Axiom } C_{4b} \Rightarrow C_{4b1} \Rightarrow C_{4b2} \Rightarrow C_{4b3} \Rightarrow C \text{ satisfies Axiom } C_{4b}$$

To establish the first implication, it suffices to consider $A_1 := C(A_2)$ in the statement of Axiom $C_{4b}$ and to consider any $u$ in $A_1$. That $C_{4b1}$ implies $C_{4b2}$ follows by considering $-\nu \in \mathcal{V}$ instead of $\nu$. That then $C_{4b3}$ holds is immediate. Finally, to show that $C_{4b3}$ implies Axiom $C_{4b}$ it suffices to note that in particular $C(A) + \{\nu\} \subseteq C(A + \{\nu\})$, and therefore, by considering any $A_1 \subseteq C(A)$, Axiom $C_{4b}$ indeed follows.

These axioms constitute a subset of the ones introduced by Seidenfeld et al. (2010), duly translated from horse lotteries to our abstract options, which are more general as we will show in Section 3 further on. In this respect, our notion of coherence is less restrictive than theirs. On the other hand, our Axiom $C_4$ is more restrictive than Seidenfeld et al.’s (2010). This is necessary for the link between coherent choice functions and coherent sets of desirable gambles we will establish in Section 4.

One axiom we omit from our coherence definition is the Archimedean one. Typically the preference associated with coherent sets of desirable gambles does not have the Archimedean property (Zaffalon and Miranda 2015, Section 3), so letting go of this axiom is necessary if we want to explore the connection with desirability.

The second axiom that we do not consider as necessary for coherence is what we will call the convexity axiom:

$C_5$. If $A \subseteq A_1 \subseteq CH(A)$ then $C(A) \subseteq C(A_1)$, for all $A$ and $A_1$ in $\mathcal{Q}$. 

As we will show in Section 4, it is incompatible with Walley-Sen maximality [Walley 1991, Troffaes 2007] as a decision rule. Nevertheless, we intend to investigate the connection with desirability for coherent choice functions that do satisfy the convexity axiom.

Two dominance properties are immediate consequences of coherence:

**Proposition 2.** Let $C$ be a coherent choice function on $Q$. Then for all $u_1$ and $u_2$ in $V$ such that $u_1 \preceq u_2$, all $A$ in $Q$ and all $v$ in $A \setminus \{u_1, u_2\}$:

a. if $u_2 \in A$ and $v \not\in C(A \cup \{u_1\})$ then $v \not\in C(A)$;

b. if $u_1 \in A$ and $v \not\in C(A)$ then $v \not\in C(\{u_2\} \cup A \setminus \{u_1\})$.

**Proof.** The result is trivial when $u_1 = u_2$, so let us assume that $u_1 < u_2$.

The first statement is again trivial if $u_1 \in A$. When $u_1 \notin A$, it follows from Axiom $C_2$ that $u_1 \notin C(\{u_1, u_2\})$. By applying Axiom $C_3\alpha$ in the form of Equation (4), we find that $u_1 \not\in C(A \cup \{u_1\})$, and then applying Axiom $C_3\beta$ in the form of Equation (5), together with the assumption that $v \not\in C(A \cup \{u_1\})$, we conclude that $v \not\in C(A \cup \{u_1\} \setminus \{u_1\}) = C(A)$.

For the second statement, it follows from Axiom $C_3\beta$ that $u_1 \notin C(\{u_1, u_2\})$. By applying Axiom $C_3\alpha$ in the form of Equation (4), we find that both $u_1 \notin C(A \cup \{u_2\})$ and $v \not\in C(A \cup \{u_2\})$, so we can apply Axiom $C_3\beta$ in the form of Equation (5) to conclude that $v \not\in C(\{u_2\} \cup A \setminus \{u_1\})$.

We are interested in conservative reasoning with choice functions. We therefore introduce a binary relation $\preceq$ on the set $C$ of all choice functions, having the interpretation of 'not more informative than', or, in other words, 'at least as uninformative as'.

**Definition 3.** Given two choice functions $C_1$ and $C_2$ in $C$, we call $C_1$ not more informative than $C_2$—if ($\forall A \in Q$) $C_1(A) \supseteq C_2(A)$.

This intuitive way of ordering choice functions is also used by Bradley (2015, Section 2) and Van Camp et al. (2017) [Definition 6]. The underlying idea is that a choice function is more informative when it consistently chooses more specifically—or more restrictively—amongst the available options.

Since, by definition, $\preceq$ is a product ordering of set inclusions, the following result is immediate (Davey and Priestley 1990).

**Proposition 3.** The structure $(C; \preceq)$ is a complete lattice:

(i) it is a partially ordered set, or poset, meaning that the binary relation $\preceq$ on $C$ is reflexive, antisymmetric and transitive;

(ii) for any subset $C’$ of $C$, its infimum $\inf C’$ and its supremum $\sup C’$ with respect to the ordering $\preceq$ exist in $C$, and are given by $\inf C’(A) = \bigcup_{C \in C’} C(A)$ and $\sup C’(A) = \bigcap_{C \in C’} C(A)$ for all $A \in Q$.

The idea underlying these notions of infimum and supremum is that $\inf C’$ is the most informative model that is not more informative than any of the models in $C’$, and $\sup C’$ the least informative model that is not less informative than any of the models in $C’$.

We have proved elsewhere (Van Camp et al. 2017) Proposition 3) that coherence is preserved under arbitrary non-empty infima. Because of our interest in the additional Axiom $C_3\alpha$ we prove that it also is preserved under arbitrary non-empty infima.

**Proposition 4.** Given any non-empty collection $C’$ of choice functions that satisfy Axiom $C_3\alpha$, its infimum $\inf C’$ satisfies Axiom $C_3\alpha$ as well.

**Proof.** Consider any $A$ and $A_1$ in $Q$ such that $A \subseteq A_1 \subseteq CH(A)$. Then $C(A) \subseteq C(A_1)$ for all $C$ in $C’$, whence $\inf C’(A) = \bigcup_{C \in C’} C(A) \subseteq \bigcup_{C \in C’} C(A_1) = \inf C’(A_1)$.

3. THE CONNECTION WITH OTHER DEFINITIONS OF CHOICE FUNCTIONS

Before we go on with our exploration of choice functions, let us take some time here to explain why we have chosen to define them in the way we did. Seidenfeld et al. (2010) (see
We collect all the horse lotteries on a vector-valued gamble. The set $\mathcal{L}$ is vector space. We call $\mathcal{L}$ vector space for in the present context.

One reason for our working with the more abstract notion of options—elements of some general vector space—is that they are better suited for dealing with indifference: this involves working with equivalence classes of options, which again constitute a vector space (Van Camp et al., 2017). These equivalence classes can no longer be interpreted easily or directly as gambles, or horse lotteries for that matter. Another reason for using options that are more general than real-valued gambles is that recent work by Zaffalon and Miranda (2015) has shown that a very general theory of binary preference can be constructed using vector-valued gambles, rather than horse lotteries. Such vector-valued gambles again constitute a real vector, or option, space. Here, we show that the conclusions of Zaffalon and Miranda (2015, Section 4) can be extended from binary preferences to choice functions.

We consider an arbitrary possibility space $X$ of mutually exclusive elementary events, one of which is guaranteed to occur. Consider also a countable set $\mathcal{R}$ of prizes, or rewards.

**Definition 4** (Gambles). Any bounded real-valued function on some domain $X$ is called a gamble on $X$. We collect all gambles on $X$ in $\mathcal{L}(X)$, often denoted as $\mathcal{L}$ when it is clear from the context what the domain $X$ is.

When the domain is of the type $X \times \mathcal{R}$, we call elements $f$ of $\mathcal{L}(X \times \mathcal{R})$ vector-valued gambles on $X$. Indeed, for each $x$ in $X$, the partial map $f(x, \cdot)$ is then an element of the vector space $\mathcal{L}(\mathcal{R})$.

The set $\mathcal{L}$, provided with the point-wise addition of gambles, the point-wise multiplication with real scalars, and the point-wise vector ordering $\preceq$, constitutes an ordered vector space. We call $\mathcal{L}_{\geq} := \{ f \in \mathcal{L} : f \geq 0 \}$ the set of all positive (vector-valued) gambles.

Horse lotteries are special vector-valued gambles.

**Definition 5** (Horse lotteries). We call horse lottery $H$ any map from $X \times \mathcal{R}$ to $[0, 1]$ such that for all $x$ in $X$, the partial map $H(x, \cdot)$ is a probability mass function over $\mathcal{R}$. We collect all the horse lotteries on $X$ with reward set $\mathcal{R}$ in $\mathcal{H}(X, \mathcal{R})$, which is also denoted more simply by $\mathcal{H}$ when it is clear from the context what the possibility space $X$ and reward set $\mathcal{R}$ are.

Let us, for the remainder of this section, fix $X$ and $\mathcal{R}$. It is clear that $\mathcal{H} \subseteq \mathcal{L}(X \times \mathcal{R})$. Seidenfeld et al. (2010) consider choice functions whose domain is $\mathcal{Q}(\mathcal{H})$, the set of all finite subsets of $\mathcal{H}$—choice functions on horse lotteries. We will call them choice functions on $\mathcal{H}$. Because of the nature of $\mathcal{H}$, their choice functions are different from ours: they require slightly different rationality axioms. The most significant change is that for Seidenfeld et al. (2010), choice functions need not satisfy Axioms $C_{a}$ and $C_{b}$. In fact, choice functions on $\mathcal{H}$ cannot satisfy these axioms, since $\mathcal{H}$ is no linear space: it is not closed under arbitrary linear combinations, only under convex combinations.

Instead, on their approach a choice function $C^*$ on $\mathcal{H}$ is required to satisfy $C_{a}^*$: $A_{1}^* \triangleq A_{1} + (1 - a) \{ H \} \triangleq A_{2} + (1 - a) \{ H \}$ for all $a$ in $(0, 1]$, all $A_{1}$ and $A_{2}$ in $\mathcal{Q}(\mathcal{H})$ and all $H$ in $\mathcal{H}$.

Note that $H(x, \cdot)$ defines a (sigma-)additive probability measure, and that this countable additivity property is necessary for Lemma below to hold. Actually, Seidenfeld et al. (2010) define choice functions on a larger domain: all possibly infinite but closed sets of horse lotteries (non-closed sets may not have admissible options). This is an extension we see no need for in the present context.
The binary relation \( \prec \) is the choice relation associated with \( C^* \), defined by Equation (1). Furthermore, for a choice function \( C^* \) to be coherent, it needs to additionally satisfy \( \text{Seidenfeld et al.} [2010] \):

\( C^*_1 \): \( C^*(A^*) \neq \emptyset \) for all \( A^* \) in \( Q(\mathcal{H}) \);

\( C^*_2 \): for all \( A^* \) in \( Q(\mathcal{H}) \), all \( H_1 \) and \( H_2 \) in \( \mathcal{H} \) such that \( H_1 \{r, \tau\} \leq H_2 \{r, \tau\} \) and \( H_1 \{r, r\} = H_2 \{r, r\} = 0 \) for all \( r \) in \( \mathcal{R} \} \{r, \tau\} \), and all \( H \) in \( \mathcal{H} \} \{H_1, H_2\} \):

a. if \( H_2 \in \mathcal{A}^* \) and \( H \in R^*((H_1) \cup \mathcal{A}^*) \) then \( H \in R^*(\mathcal{A}^*) \);

b. if \( H_1 \in \mathcal{A}^* \) and \( H \in R^*(\mathcal{A}^*) \) then \( H \in R^*((\{H_2\}) \cup \mathcal{A}^* \{H_1\}) \);

\( C^*_3 \): for all \( A^* \), \( A^*_1 \) and \( A^*_2 \) in \( Q(\mathcal{H}) \):

a. if \( A^*_1 \subseteq R^*(A^*_2) \) and \( A^*_2 \subseteq A^* \) then \( A^*_1 \subseteq R^* (A^*) \);

b. if \( A^*_1 \subseteq R^*(A^*_2) \) and \( A^*_1 \subseteq A^* \) then \( A^*_1 \setminus A^* \subseteq R^*(A^*_2 \setminus A^*) \);

\( C^*_4 \): if \( A^* \subseteq A^*_1 \subseteq \text{CH}(A) \) then \( C^*(A) \subseteq C^*(A^*_1) \), for all \( A^* \) and \( A^*_1 \) in \( Q(\mathcal{H}) \);

\( C^*_5 \): for all \( A^* \), \( A^* \), \( A^* \), \( A^* \), \( A^* \) and \( A^* \) (for \( i \) in \( \mathbb{N} \)) in \( Q(\mathcal{H}) \) such that the sequence \( A^*_i \) converges point-wise to \( A^* \) and the sequence \( A^*_i \) converges point-wise to \( A^* \):

a. if \( i \) in \( \mathbb{N} \) \( A^* \) \( \prec_C \) \( A^*_i \) and \( A^*_i \) \( \prec_C \) \( A^* \) then \( A^* \) \( \prec_C \) \( A^* \);

b. if \( i \) in \( \mathbb{N} \) \( A^* \) \( \prec_C \) \( A^*_i \) and \( A^*_i \) \( \prec_C \) \( A^* \) then \( A^* \) \( \prec_C \) \( A^* \),

where \( \text{Seidenfeld et al.} [2010] \) assume that there is a unique worst reward \( \tau \) and a unique best reward \( \tau \) in \( \mathcal{R} \). This is a somewhat stronger assumption than what we will make: further on in this section, we will only assume that there is a unique worst reward. Axiom \( C^*_5 \) is the counterpart of Proposition 2 for choice functions on horse lotteries, which is a result of our Axioms \( C^*_1, C^*_2, C^*_4, \) \( \text{Seidenfeld et al.} [2010] \) need to impose this property as an axiom, essentially because of the absence in their system of a counterpart for our Axiom \( C^*_2 \).

Axioms \( C^*_a \) and \( C^*_b \) are Archimedean axioms, hard to reconcile with desirability (see for instance \( \text{Zaffalon and Miranda} [2015] \) Section 4), which is why we will not enforce them here.

We now intend to show that under very weak conditions on the rewards set \( \mathcal{R} \), choice functions on horse lotteries that satisfy \( C^*_4 \) are in a one-to-one correspondence with choice functions on a suitably defined option space that satisfy Axioms \( C^*_a \) and \( C^*_b \).

Let us first study the impact of Axiom \( C^*_4 \). We begin by showing that an assessment of \( H \in C(A) \) for some \( A \) in \( Q(\mathcal{H}) \) implies other assessments of this type.

**Proposition 5.** Consider any choice function \( C^* \) on \( Q(\mathcal{H}) \) that satisfies Axiom \( C^*_4 \) any option sets \( A^* \) and \( A'' \) in \( Q(\mathcal{H}) \), and any \( H \) in \( A^* \) and \( H' \) in \( A'' \). If there are \( \lambda \) and \( \lambda' \) in \( \mathbb{R}_{\geq 0} \) such that \( \lambda(A^* - \{H\}) = \lambda'(A'' - \{H'\}) \), then

\[
H \in C^*(A) \Leftrightarrow H' \in C^*(A'').
\]

**Proof.** Fix \( A^* \) and \( A'' \) in \( Q(\mathcal{H}) \), \( H \) in \( A^* \) and \( H' \) in \( A'' \), \( \lambda \) and \( \lambda' \) in \( \mathbb{R}_{\geq 0} \), and assume that \( \lambda(A^* - \{H\}) = \lambda'(A'' - \{H'\}) \). We will show that \( H \in R^*(A^*) \Leftrightarrow H' \in R^*(A'') \). We infer from the assumption that

\[
\frac{\lambda}{\lambda + \lambda'} A^* + \frac{\lambda'}{\lambda + \lambda'} \{H'\} = \frac{\lambda'}{\lambda + \lambda'} A'' + \frac{\lambda}{\lambda + \lambda'} \{H\}.
\]

If we call \( \alpha := \frac{\lambda'}{\lambda + \lambda'} \) to ease the notation along, then \( 1 - \alpha = \frac{\lambda}{\lambda + \lambda'} \) and \( \alpha \in (0,1) \). We now infer from the identity above that \( \alpha A^* + (1 - \alpha) \{H'\} = (1 - \alpha) A'' + \alpha \{H\} \). Consider the following chain of equivalences:

\[
H \in R^*(A^*) \Leftrightarrow \{H\} \prec_C A^* \quad \text{by Equation (1)}
\]

\[
\Leftrightarrow \alpha \{H\} + (1 - \alpha) \{H'\} \prec_C \alpha A^* + (1 - \alpha) \{H'\} \quad \text{using Axiom } C^*_4
\]

\[
\Leftrightarrow \alpha \{H\} + (1 - \alpha) \{H'\} \prec_C (1 - \alpha) A'' + \alpha \{H\}
\]

\[
\Leftrightarrow \{H'\} \prec_C A'' \quad \text{using Axiom } C^*_4
\]

\[
\Leftrightarrow H' \in R^*(A'') \quad \text{by Equation (1).} \]

For any \( r \in \mathcal{R} \), we now introduce \( \mathcal{R}_r := \mathcal{R} \setminus \{r\} \), the set of all rewards without \( r \). For the connection between choice functions on \( \mathcal{H} \) and choice functions on some option space, we need to somehow be able to extend \( \mathcal{H} \) to a linear space. The so-called gamblifier \( \varphi_r \) will play a crucial role in this:

**Definition 6 (Gamblifier).** Consider any \( r \in \mathcal{R} \). The gamblifier \( \varphi_r \) is the linear map

\[
\varphi_r : \mathcal{L}(\mathcal{X} \times \mathcal{R}) \to \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) : f \mapsto \varphi_r f,
\]

where \( \varphi_r f(x, s) := f(x, s) \) for all \( x \in \mathcal{X} \) and \( s \in \mathcal{R}_r \).

In particular, the gamblifier \( \varphi_r \) maps any horse lottery \( H \in \mathcal{H}(\mathcal{X}, \mathcal{R}) \) to an element \( \varphi_r H \) of \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) that satisfies the following two conditions:

\[
\varphi_r H(x, \cdot) \geq 0 \quad \text{and} \quad \sum_{s \in \mathcal{R}_r} \varphi_r H(x, s) \leq 1.
\]

Application of \( \varphi_r \) to sets of the form \( \lambda(A^* - \{H\}) \) essentially leaves the ‘information’ they contain unchanged:

**Lemma 6.** Consider any \( r \in \mathcal{R} \). Then the following two properties hold:

(i) The gamblifier \( \varphi_r \) is one-to-one on \( \mathcal{H} \).

(ii) For any \( A^* \) and \( A^{*'} \) in \( Q(\mathcal{H}) \), any \( H \) in \( A^* \) and \( H' \) in \( A^{*'} \) and any \( \lambda \) and \( \lambda' \) in \( \mathbb{R}_{>0} \):

\[
\lambda(A^* - \{H\}) = \lambda'(A^{*'} - \{H'\}) \Leftrightarrow \varphi_r(\lambda(A^* - \{H\})) = \varphi_r(\lambda'(A^{*'} - \{H'\})).
\]

Proof. We begin with the first statement. Consider any \( H \) and \( H' \) in \( \mathcal{H} \), and assume that \( \varphi_r(H) = \varphi_r(H') \). We infer from Definition 6 that

\[
H(x, r) = 1 - \sum_{s \in \mathcal{R}_r} H(x, s) = 1 - \sum_{s \in \mathcal{R}_r} H'(x, s) = H'(x, r)
\]

for all \( x \in \mathcal{X} \).

Hence indeed \( H = H' \).

The direct implication in the second statement is trivial; let us prove the converse. Assume that \( \varphi_r(\lambda(A^* - \{H\})) = \varphi_r(\lambda'(A^{*'} - \{H'\})) \). We may write, without loss of generality, that \( A^* = \{H, H_1, \ldots, H_n\} \) and \( A^{*'} = \{H', H'_1, \ldots, H'_m\} \) for some \( n \) and \( m \) in \( \mathbb{N} \).

Now, consider any element \( H_l \) in \( A^* \), then \( \varphi_r(\lambda(H_l - H)) \in \varphi_r(\lambda(A^* - \{H\})) \). Consider any \( j \) in \( \{1, \ldots, m\} \) such that \( \varphi_r(\lambda(H_l - H)) = \varphi_r(\lambda'(H'_j - H')) \). It follows from the assumption that there is at least one such \( j \). The proof is complete if we can show that

\[
\lambda(H_l - H) = \lambda'(H'_j - H').
\]

By Definition 6 we already know that

\[
\lambda(H_l(\cdot, r) - H(\cdot, r)) = H'_j(\cdot, s) - H'(\cdot, s)
\]

for all \( s \) in \( \mathcal{R}_r \), and therefore, since \( H, H', H_l \) and \( H'_j \) are horse lotteries, also

\[
\lambda(H_l(\cdot, r) - H(\cdot, r)) = \lambda\left(\sum_{s \in \mathcal{R}_r} H_l(s, r) - \sum_{s \in \mathcal{R}_r} H_l(s, s)\right) = \lambda\left(\sum_{s \in \mathcal{R}_r} H'(s, r) - \sum_{s \in \mathcal{R}_r} H'(s, s)\right)
\]

whence indeed \( \lambda(H_l - H) = \lambda'(H'_j - H') \). \( \square \)

We now lift the gamblifier \( \varphi_r \) to a map \( \tilde{\varphi}_r \) that turns choice functions on gambles into choice functions on horse lotteries:

\[
\tilde{\varphi}_r : \mathcal{C}(\mathcal{L}(\mathcal{X} \times \mathcal{R}_r)) \to \mathcal{C}(\mathcal{H}(\mathcal{X}, \mathcal{R})): C \mapsto \tilde{\varphi}_r C,
\]

(7)
where \( \phi_r(C(A^*)) = \psi_r^{-1}(C(\phi_r, A^*)) \) for every \( A^* \) in \( Q(\mathcal{H}(\mathcal{X}, \mathcal{R})) \). This definition makes sense because we have proved in Lemma 6 that \( \phi_r \) is one-to-one on \( \mathcal{H} \), and therefore invertible on \( \phi_r, \mathcal{H} \). The result of applying \( \phi_r \) to a choice function \( C \) on \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) is a choice function \( \phi_r C \) on \( \mathcal{H}(\mathcal{X}, \mathcal{R}) \). Observe that we can equally well make \( \phi_r \) apply to rejection functions \( R \), and that for every \( A^* \) in \( Q(\mathcal{H}(\mathcal{X}, \mathcal{R})) \):

\[
\phi_r R(A^*) = \psi_r^{-1}(R(\phi_r, A^*)) = \psi_r^{-1}(\phi_r, A^*) = A^* \cup \phi_r C(A^*),
\]

so \( \phi_r R \) is the rejection function associated with the choice function \( \phi_r C \), when \( R \) is the rejection function for \( C \).

One property of the transformation \( \phi_r \) that will be useful in our subsequent proofs is the following:

**Lemma 7.** Consider any \( r \in \mathcal{R} \) and any \( A \in Q(\mathcal{L}(\mathcal{X} \times \mathcal{R}_r)) \), and define \( g \) by \( g(x, s) := \sum f(x, s) \in (f(x, s)) \) for all \( x \in \mathcal{X} \) and \( s \in \mathcal{R}_r \). Consider any \( \lambda \in \mathbb{R} \) such that

\[
\lambda > \max \left\{ \max_{x \in \mathcal{X}} \sum_{s \in \mathcal{R}_r} h(x, s) : h \in A + \{ g \} \right\} > 0.
\]

Then \( \frac{1}{2}(A + \{ g \}) = \phi_r, A^* \) for some \( A^* \) in \( Q(\mathcal{H}(\mathcal{X}, \mathcal{R})) \).

**Proof.** Consider any \( h \) in \( A + \{ g \} \), and let us show that \( \frac{1}{2} h \) satisfies the conditions in Equation 6. The first one is satisfied because \( \lambda > 0 \) and \( h = f + g \) for some \( f \) in \( A \), so \( h = f + g = h + \sum f(x, s) \leq h + f \geq 0 \) and therefore indeed \( \frac{1}{2} h \geq 0 \). For the second condition, recall that \( \lambda > \sum_{s \in \mathcal{R}_r} h(\cdot, s) \) by construction and therefore indeed \( \sum_{x \in \mathcal{X}} \frac{1}{2} h(\cdot, s) \leq 1 \). \( \square \)

**Proposition 8.** Consider any \( r \in \mathcal{R} \). The operator \( \phi_r \) is a bijection between the choice functions on \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) that satisfy Axioms \( \mathcal{C}_a \) and \( \mathcal{C}_b \) and the choice functions on \( \mathcal{H}(\mathcal{X}, \mathcal{R}) \) that satisfy Axiom \( \mathcal{C}_r \).

**Proof.** We first show that \( \phi_r \) is injective. Assume \textit{ex absurdo} that it is not, so there are choice functions \( C \) and \( C' \) on \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) that satisfy Axioms \( \mathcal{C}_a \) and \( \mathcal{C}_b \) such that \( \phi_r C = \phi_r C' \) but nevertheless \( C \neq C' \). The latter means that there are \( A \) in \( Q(\mathcal{L}(\mathcal{X} \times \mathcal{R}_r)) \) and \( f \) in \( A \) such that \( f \in C(A) \) and \( f \notin C'(A) \). Use Lemma 7 to find some \( \lambda \in \mathbb{R}_{>0} \) and \( g \in \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) such that \( \frac{1}{2}(A + \{ g \}) = \phi_r, A^* \) for some \( A^* \) in \( Q(\mathcal{H}(\mathcal{X}, \mathcal{R})) \). If we now apply Axioms \( \mathcal{C}_a \) and \( \mathcal{C}_b \) we find that \( \frac{f + g}{2} \in C \left( \frac{1}{2}(A + \{ g \}) \right) \), or equivalently, \( \psi^{-1} \left( \frac{f + g}{2} \right) \in \phi_r C(A^*) \). Similarly, we find that \( \frac{f + g}{2} \notin C' \left( \frac{1}{2}(A + \{ g \}) \right) \), or equivalently, \( \psi^{-1} \left( \frac{f + g}{2} \right) \notin \phi_r C'(A^*) \). But this contradicts our assumption that \( \phi_r C = \phi_r C' \).

We now show that application of \( \phi_r \) to any choice function \( C \) on \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) that satisfies Axioms \( \mathcal{C}_a \) and \( \mathcal{C}_b \) results in a choice function \( \phi_r, C \) that satisfies Axiom \( \mathcal{C}_r \). Consider any \( A^* \) and \( A^*_2 \) in \( Q(\mathcal{H}(\mathcal{X}, \mathcal{R})) \), any \( H \) in \( \mathcal{H}(\mathcal{X}, \mathcal{R}) \), and any \( \alpha \) in \( (0, 1) \). Iner the following chain of equivalences:

\begin{align*}
A^*_1 & <_r \phi_r, C \ A^*_2^* \\
\iff A^*_1 & \subseteq \phi_r, R(A^*_1 \cup A^*_2^*) & \text{by Equation 1} \\
\iff \phi_r, A^*_1 & \subseteq R(\phi_r, (A^*_1 \cup A^*_2^*)) & \text{by Equation 7} \\
\iff \phi_r, aA^*_1 & \subseteq R(\phi_r, a(A^*_1 \cup A^*_2^*)) & \text{by Axiom \( \mathcal{C}_a \)} \\
\iff \phi_r, (aA^*_1 + (1 - \alpha) \{ H \}) & \subseteq R(\phi_r, (a(A^*_1 \cup A^*_2^*) + (1 - \alpha) \{ H \})) & \text{by Axiom \( \mathcal{C}_b \)} \\
\iff aA^*_1 + (1 - \alpha) \{ H \} & \subseteq \phi_r, R(a(A^*_1 \cup A^*_2^*) + (1 - \alpha) \{ H \}) & \text{by Equation 7} \\
\iff (aA^*_1 + (1 - \alpha) \{ H \}) & \subseteq \phi_r, (aA^*_2^* + (1 - \alpha) \{ H \}) & \text{by Equation 1}, \text{which tells us that indeed } \phi_r, C \text{ satisfies Axiom } \mathcal{C}_r \text{.}
\end{align*}

The proof is complete if we also show that \( \phi_r \) is surjective—that for every choice function \( C^* \) on \( \mathcal{H}(\mathcal{X}, \mathcal{R}) \) that satisfies Axiom \( \mathcal{C}_r \) there is a choice function \( C \) on \( \mathcal{L}(\mathcal{X} \times \mathcal{R}_r) \) that satisfies Axioms \( \mathcal{C}_a \) and \( \mathcal{C}_b \) such that \( \phi_r C = C^* \). So consider any choice
function $C^*$ on $H(\mathcal{X}, \mathcal{R})$ that satisfies Axiom $C_1$. We will show that the special choice function $C$ on $L(\mathcal{X} \times \mathcal{R}^*)$ based on $C^*$, defined as
\[
 f \in C(A) \iff (\exists \lambda \in \mathbb{R}_{>0}, A^* \in Q(H(\mathcal{X}, \mathcal{R})), H \in A^*) 
\]
\[
 (\phi_r(A^* - \{H\}) = \lambda(A - \{f\}) \text{ and } H \in C^*(A^*)) \tag{8}
\]
for all $A$ in $Q(\mathcal{L}(\mathcal{X} \times \mathcal{R}))$ and $f$ in $A$, satisfies Axioms $C_{a1}$ and $C_{ab}$ and $\phi_r, C = C^*$. We first show that $C$ satisfies Axioms $C_{a1}$ and $C_{ab}$. For Axiom $C_{a1}$, we use its equivalent form $C_{a1}$. Consider any $A$ in $Q(\mathcal{L}(\mathcal{X} \times \mathcal{R}))$, any $f$ in $A$, and any $\mu \in \mathbb{R}_{>0}$, and assume that $f \in C(A)$. To show that then $\mu f \in C(\mu A)$, it suffices to consider $\lambda' := \frac{\lambda}{\mu}$ in Equation (8), and note that $\lambda(A - \{f\}) = \lambda'(\mu A - \{\mu f\})$; then the desired statement follows at once from Equation (8). For Axiom $C_{ab}$, we use its equivalent form $C_{ab}$. Consider any $A$ in $Q(\mathcal{L}(\mathcal{X} \times \mathcal{R}))$, any $f$ in $A$, and any $g \in \mathcal{L}(\mathcal{X} \times \mathcal{R})$, and assume that $f \in C(A)$. We show that then $f + g \in C(A + \{g\})$. To this end, it suffices to note that $A - \{f\} = (A + \{g\}) - \{f + g\}$; then the desired statement follows at once from Equation (8). So $C$ as defined in Equation (8) does indeed satisfy Axioms $C_{a1}$ and $C_{ab}$ and is therefore a suitable candidate for showing that $\phi_r, C = C^*$.

We now finish the proof by showing that $\phi_r, C = C^*$. To do so, we consider any $A^*$ in $Q(H(\mathcal{X}, \mathcal{R}))$, and show that $\phi_r, C(A^*) \subseteq C^*(A^*)$ and $C^*(A^*) \subseteq \phi_r, C(A^*)$. To show that $\phi_r, C(A^*) \subseteq C^*(A^*)$, consider any $H$ in $\phi_r, C(A^*)$. By the definition of $\phi_r$ (Equation (7)), then $H \in \phi_r^{-1}(C(\phi_r, A^*))$, and therefore $\phi_r, H \in C(\phi_r, A^*)$. Using Equation (8) with $A = \phi_r A^*$ and $f = \phi_r H$, we find that then
\[
\phi_r(A^* - \{H\}) = \lambda'(\phi_r A^* - \{\phi_r H\}) \text{ and } H' \in C^*(A'^*),
\]
for some $A'$ in $\mathbb{R}_{>0}, A'^* \in Q(H(\mathcal{X}, \mathcal{R}))$ and $H' \in A'^*$. By Lemma and since $\lambda'(\phi_r A^* - \{\phi_r H\}) = \phi_r(\lambda'(A^* - \{H\}))$, infer that then
\[
A'^* - \{H\} = \lambda'(A^* - \{H\}) \text{ and } H' \in C^*(A'^*),
\]
and because $C^*$ satisfies Axiom $C_1$, and using Proposition, this means that indeed $H \in C^*(A^*)$. So we have shown that $H \in C(A^*)$, and since the choice of $H$ was arbitrary in $\phi_r, C(A^*)$, therefore indeed $\phi_r, C(A^*) \subseteq C^*(A^*)$.

To show that $C^*(A^*) \subseteq \phi_r, C(A^*)$, consider any $H$ in $C^*(A^*)$. Let $A := \phi_r A^*$, $f := \phi_r H$ and $\lambda := 1$, then
\[
\phi_r(A^* - \{H\}) = \lambda(A - \{f\}),
\]
whence by Equation (8) $\phi_r, H = f \in C(A) = C(\phi_r, A^*)$. Since $\phi_r$ is one-to-one on $H(\mathcal{X}, \mathcal{R})$ (see Lemma), therefore indeed $H \in \phi_r, C(A^*)$. So we have shown that $\phi_r, C = C^*$, which completes the proof.

Specifying a choice function $C^*$ on $H$ induces a strict preference relation on the reward set, as follows. With any reward $r$ in $\mathcal{R}$ we can associate the constant and degenerate lottery $H_r$ by letting
\[
H_r(x, s) = \begin{cases} 
1 & \text{if } s = r \\
0 & \text{otherwise}
\end{cases}
\]
for all $x$ in $\mathcal{X}$ and $s$ in $\mathcal{R}$.

This is the lottery that associates the certain reward $r$ with all states. Then a reward $r$ is strictly preferred to a reward $s$ when $H_r \in R^*(\{H_r, H_s\})$.

**Definition 7** ($C^*$ has worst reward $r$). Consider any reward $r$ in $\mathcal{R}$, and any choice function $C^*$ on $H(\mathcal{X}, \mathcal{R})$. We say that $C^*$ has worst reward $r$ if $H_r \in R^*(\{H_r, H_s\})$ for all $H$ in $H(\mathcal{X}, \mathcal{R}) \setminus \{H_r\}$.

The worst reward is unique when $C^*$ satisfies Axiom $C_1$, indeed, if there are two different worst rewards $r$ and $s$, by Definition, then $\{H_r, H_s\} = R^*(\{H_r, H_s\})$, contradicting Axiom $C_1$. 
The notion of having worst reward is closely related with what would be the natural translation of Axiom C2 to choice functions C∗ on \( H(X, \mathcal{R}) \): if C∗ satisfies
\[
(\forall H_1, H_2 \in H) \left( \left( H_1 \neq H_2 \text{ and } (\forall s \in \mathcal{R}_r)(H_1(s) \leq H_2(s)) \right) \Rightarrow H_1 \in R^*(\{H_1, H_2\}) \right)
\]
for some \( r \in \mathcal{R}_r \), then we say that C∗ satisfies the dominance relation for worst reward r.

**Proposition 9.** Consider any r in \( \mathcal{R} \) and any choice function C on \( L(X \times \mathcal{R}_r) \) that satisfies Axioms C2a and C2b. Then \( \hat{\phi}_rC \) satisfies the dominance relation for worst reward r (Equation (9)) if and only if \( \hat{\phi}_rC \) has worst reward r.

**Proof.** For the direct implication, consider any H in \( H(X, \mathcal{R}) \setminus \{H_r\} \). Then \( H_r(s) = 0 \leq H(s) \) for all s in \( \mathcal{R}_r \), and also \( H \neq H_r \), whence indeed \( H_r \in \hat{\phi}_rR(\{H, H_r\}) \), because by assumption \( \hat{\phi}_rC \) satisfies Equation (9) for r.

For the converse implication, consider any \( H_1 \) and \( H_2 \) in \( H(X, \mathcal{R}) \) such that \( H_1 \neq H_2 \) and \( H_1(s) \leq H_2(s) \) for all s in \( \mathcal{R}_r \). Then \( \hat{\phi}_rH_1 < \hat{\phi}_rH_2 \), whence \( 0 < \hat{\phi}_r(H_2 - H_1) \). Observe that for the horse lottery \( H' \) in \( H(X, \mathcal{R}) \) defined by
\[
H'(s) = \begin{cases} H_2(s) - H_1(s) & \text{if } s \in \mathcal{R}_r, \\ 1 - \sum_{s \in \mathcal{R}_r}(H_2(s) - H_1(s)) & \text{if } s = r, \end{cases}
\]
we have that \( \hat{\phi}_rH' = \hat{\phi}_r(H_2 - H_1) \). Because \( \hat{\phi}_rC \) is assumed to have worst reward r, we know that in particular \( H_r \in \hat{\phi}_rR(\{H', H_r\}) \), so we infer from Equation (7) that \( 0 = \hat{\phi}_rH_r \in R(\{\hat{\phi}_rH, \hat{\phi}_rH'\}) = R(\{\hat{\phi}_rH_2 - \hat{\phi}_rH_1\}) \). Now use Axioms C2a and C2b to infer that \( \hat{\phi}_rH_1 \in R(\{\hat{\phi}_rH_1, \hat{\phi}_rH_2\}) \), whence indeed \( H_1 \in \hat{\phi}_rR(\{H_1, H_2\}) \), by Equation (7).

Applying the lifting \( \hat{\phi}_r \) furthermore preserves coherence:

**Theorem 10.** Consider any reward r in \( \mathcal{R} \), and any choice function C on \( L(X \times \mathcal{R}_r) \) that satisfies Axioms C2a and C2b. Then the following statements hold:

(i) C satisfies Axiom C1 if and only if \( \hat{\phi}_rC \) satisfies Axiom C1a.

(ii) C satisfies Axiom C2 if and only if \( \hat{\phi}_rC \) has worst reward r.

(iii) C satisfies Axiom C2a if and only if \( \hat{\phi}_rC \) satisfies Axiom C2a.

(iv) C satisfies Axiom C2b if and only if \( \hat{\phi}_rC \) satisfies Axiom C2b.

(v) \( \hat{\phi}_rC \) satisfies Axiom C4.

(vi) C satisfies Axiom C5 if and only if \( \hat{\phi}_rC \) satisfies Axiom C5.

**Proof.** For the direct implication of (ii) assume that C satisfies Axiom C1. Consider any \( A^* \) in \( Q(H(X, R)) \). Then \( \hat{\phi}_rC(A^*) = \hat{\phi}_rC(\hat{\phi}_rA) \neq \varnothing \).

For the converse implication, assume that \( \hat{\phi}_rC \) satisfies Axiom C1. Consider any \( A \) in \( Q(L(X \times \mathcal{R}_r)) \). By Lemma 7 there are \( \lambda \in R_\geq \) and a \( g \) in \( L(X \times \mathcal{R}_r) \) such that \( \frac{1}{\lambda}(A + \{g\}) = \hat{\phi}_rA^* \) for some \( A^* \) in \( Q(H(X, R)) \). Applying Axioms C2a and C2b and the definition of \( \hat{\phi}_r \) (Equation (7)), we infer that indeed
\[
C(A) = \lambda C\left( \frac{1}{\lambda}(A + \{g\}) \right) - \{g\} = \lambda C(\hat{\phi}_rA^*) - \{g\} = \lambda \hat{\phi}_rC(\hat{\phi}_rA^*) - \{g\} \neq \varnothing.
\]

For the direct implication of (iii) assume that C satisfies Axiom C2. Consider any \( H_1 \) and \( H_2 \) in \( H(X, R) \) such that \( H_1 \neq H_2 \) and \( H_1(s) \leq H_2(s) \) for all s in \( \mathcal{R}_r \). Then \( \hat{\phi}_rH_1 < \hat{\phi}_rH_2 \), so Axiom C2 guarantees that \( \hat{\phi}_rH_1 \in R(\{\hat{\phi}_rH_1, \hat{\phi}_rH_2\}) \). Equation (7) now turns this into \( H_1 \in \hat{\phi}_rR(\{H_1, H_2\}) \). Proposition 9 now tells us that \( \hat{\phi}_rC \) has worst reward r.

For the converse implication, assume that \( \hat{\phi}_rC \) has worst reward r. Consider any \( f_1 \) and \( f_2 \) in \( L(X \times \mathcal{R}_r) \) such that \( f_1 < f_2 \). Let
\[
\lambda := \max_{x \in X} \sum_{s \in \mathcal{R}_r} (f_2(x, s) - f_1(x, s)) > 0.
\]
Then clearly \( \frac{1}{\lambda}(f_2 - f_1) = \hat{\phi}_rH \) for some \( H \) in \( H(X, R) \). Also, \( H \neq H_r \) because \( f_1 \neq f_2 \). Using the assumption that \( \hat{\phi}_rC \) has worst reward r, we find that then \( H_r \in \hat{\phi}_rR(\{H_r, H\}) \). As a
consequence, by Equation (7), we find that $0 = \varphi, H, \in R((0, \varphi, H)) = R((0, f_{2} - f_{1}))$. Using Axiom $[\varphi, a]$, we infer that $0 \in R((0, f_{2} - f_{1}))$, and using Axiom $[\varphi, b]$ that indeed $f_{1} \in R((f_{1}, f_{2}))$.

For the direct implication of (iii) assume that $C$ satisfies Axiom $[\varphi, a]$. Consider any $A^{*}, A^{*}_{1}$ and $A^{*}_{2}$ in $Q(H(X, R))$ and assume that $A^{*}_{1} \subseteq \varphi, R(A^{*}_{2})$ and $A^{*}_{2} \subseteq A^{*}$. Then $\varphi, A^{*}_{1} \subseteq R(\varphi, A^{*}_{2})$ by Equation (7), and $\varphi, A^{*}_{1} \subseteq \varphi, A^{*}$. Use version 4 of Axiom $[\varphi, a]$ to infer that then $\varphi, A^{*}_{1} \subseteq R(\varphi, A^{*})$, whence indeed $A^{*}_{1} \subseteq \varphi, R(A^{*})$ by Equation (7).

For the converse implication, assume that $\varphi, C$ satisfies Axiom $[\varphi, a]$. Consider any $A, A_{1}$ and $A_{2}$ in $Q(L(X \times R_{r}))$ and assume that $A_{1} \subseteq R(A_{2})$ and $A_{2} \subseteq A$. Use Lemma 7 to find $\lambda \in R_{o}, a$ and $g$ in $L(X \times R_{r})$ such that $\frac{1}{\lambda}(A_{2} + \{g\}) = \varphi, A^{*}_{2}$ for some $A^{*}_{2} \subseteq A^{*}$. Analogously, we find that $\frac{1}{\lambda}(A_{1} + \{g\}) = \varphi, (A^{*}_{1})$ for some $A^{*}_{1} \subseteq A^{*}_{2}$. So, using Axioms $[\varphi, a]$ and $[\varphi, b]$ we infer from the assumptions that $\frac{1}{\lambda}(A_{1} + \{g\}) \subseteq R(\frac{1}{\lambda}(A_{2} + \{g\}))$, or in other words, $\varphi, A^{*}_{1} \subseteq R(\varphi, A^{*}_{2})$. Equation (7) then yields that $A^{*}_{1} \subseteq \varphi, R(A^{*}_{2})$. As a result, using Axiom $[\varphi, a]$ $A^{*}_{1} \subseteq \varphi, R(A^{*})$, which, again applying Equation (7), results in $\frac{1}{\lambda}(A_{1} + \{g\}) = \varphi, A^{*}_{1} \subseteq R(\varphi, A^{*}) = R(\frac{1}{\lambda}(A + \{g\}))$, and as a consequence, by Axioms $[\varphi, a]$ and $[\varphi, b]$ we find eventually that indeed $A_{1} \subseteq R(A)$.

For the direct implication of (iv) assume that $C$ satisfies Axiom $[\varphi, b]$. Consider any $A_{1}$ and $A_{2}$ in $Q(L(X \times R_{r}))$ and assume that $A_{1} \subseteq R(A_{2})$ and $A \subseteq A_{1}$. Use Lemma 7 to find $\lambda \in R_{o}, a$ and $g$ in $L(X \times R_{r})$ such that $\frac{1}{\lambda}(A_{2} + \{g\}) = \varphi, A^{*}_{2}$ for some $A^{*}_{2} \subseteq A^{*}$. Then $\varphi, A^{*}_{1} \subseteq \varphi, R(A^{*}_{2})$ by Equation (7), and $\varphi, A^{*} \subseteq \varphi, A^{*}_{2}$. Use version 5 of Axiom $[\varphi, a]$ to infer that then $\varphi, (A^{*} \setminus A^{*}_{1}) = (\varphi, A^{*} \setminus \varphi, A^{*}_{2}) \subseteq R((\varphi, A^{*}_{2} \setminus \varphi, A^{*})) = R((\varphi, A^{*}_{2} \setminus \varphi, A^{*}))$, whence indeed $A^{*}_{1} \setminus A^{*} \subseteq \varphi, R(A^{*}_{2} \setminus A^{*})$. Again applying Equation (7) results in

$$\frac{1}{\lambda}(A_{1} \setminus A) + \{g\} = \frac{1}{\lambda}(A_{1} + \{g\}) \setminus \frac{1}{\lambda}(A + \{g\}) = (\varphi, A^{*}_{2}) \setminus (\varphi, A^{*}) = \varphi, (A^{*}_{1} \setminus A^{*})$$

$$\subseteq R(\varphi, (A^{*}_{2} \setminus A^{*})) = R((\varphi, A^{*}_{2}) \setminus (\varphi, A^{*}))$$

$$= R(\frac{1}{\lambda}(A_{2} + \{g\}) \setminus \frac{1}{\lambda}(A + \{g\})) = R(\frac{1}{\lambda}((A_{2} \setminus A) + \{g\})),$$

and as a consequence, by Axioms $[\varphi, a]$ and $[\varphi, b]$ we find eventually that indeed $A_{1} \setminus A \subseteq R(A_{2} \setminus A)$. For (v) since by Proposition 8, $\varphi, a$ is a bijection between the choice functions on $L(X \times R_{r})$ that satisfy Axioms $[\varphi, a]$ and $[\varphi, b]$ and the choice functions on $H(X, R)$ that satisfy Axiom $[\varphi, a]$, therefore indeed $\varphi, C$ satisfies Axiom $[\varphi, a]$.

For the direct implication of (vi) assume that $C$ satisfies Axiom $[\varphi, a]$. Consider any $A^{*}$ and $A^{*}_{1}$ in $Q(H(X, R))$ and assume that $A^{*} \subseteq A^{*}_{1} \subseteq CH(A^{*})$. Then $\varphi, A^{*} \subseteq \varphi, A^{*}_{1} \subseteq CH(\varphi, A^{*})$, whence $C(\varphi, A^{*}) \subseteq C(\varphi, A^{*}_{1})$ by Axiom $[\varphi, a]$. Use Equation (7) to infer that then indeed $\varphi, C(A^{*}) \subseteq \varphi, C(A^{*}_{1})$.

For the converse implication, assume that $\varphi, C$ satisfies Axiom $[\varphi, a]$. Consider any $A$ and $A_{1}$ in $Q(L(X \times R_{r}))$ and assume that $A \subseteq A_{1} \subseteq CH(A)$. Use Lemma 7 to find $\lambda \in R_{o}, a$ and $g$ in $L(X \times R_{r})$ such that $\frac{1}{\lambda}(A_{1} + \{g\}) = \varphi, A^{*}_{1}$ for some $A^{*}_{1} \subseteq CH(X, R))$, and analogously, $\frac{1}{\lambda}(A + \{g\}) = \varphi, A^{*}$ for some $A^{*} \subseteq A^{*}_{1}$. From $A_{1} \subseteq CH(A)$ infer that $\frac{1}{\lambda}(A_{1} + \{g\}) \subseteq CH(\frac{1}{\lambda}(A + \{g\}))$, or in other words, $\varphi, A^{*}_{1} \subseteq CH(\varphi, A^{*})$. Then we claim that $A^{*}_{1} \subseteq CH(A^{*})$. To prove this, consider any $H$ in $A^{*}_{1}$. Then there are $n$ in $N, H_{i}$ in $A,$
We collect all coherent sets of desirable options in the set $\bar{\mathcal{D}}$. As for choice functions, we pay special attention to coherent choice functions. Axiom $\mathcal{C}_4$ (
omenclature{c}{Axiom C})

As an example, given a coherent set of desirable options $\bar{\mathcal{D}}$, we can determine the other sets of desirable options. If for all $\alpha, \beta \in \mathbb{R}_0^+$, we have $\mathcal{C}(A) \supseteq \mathcal{C}(\lambda \varphi + \beta)$ for all $\alpha \geq \beta$, and $\mathcal{C}(A)$ is a convex cone. This convex cone is the most conservative coherent choice function that reflects the binary choices represented by $D$. Since we have shown in earlier work (Van Camp et al., 2017) that maximality leads to the most conservative coherent choice function that reflects the binary choices represented by $D$ (see also Bradley, 2015, Theorem 3), we focus on maximality as the connecting tool between desirability and choice functions. Here, we investigate what remains of this connection when we require in addition that our choice functions should satisfy Axiom $\mathcal{C}_5$.

We recall that a set of desirable options is simply a subset of the vector space $\mathcal{V}$. The underlying idea is that a subject strictly prefers each option in this set to the status quo $0$. For choice functions, we pay special attention to coherent sets of desirable options.

**Definition 8.** A set of desirable options $D$ is called coherent if for all $u$ and $v$ in $\mathcal{V}$, and all $\lambda$ in $\mathbb{R}_0^+$:

- $D_1$: $0 \notin D$;
- $D_2$: $\forall u, v \in D$;
- $D_3$: if $u \in D$ then $\lambda u \in D$;
- $D_4$: if $u, v \in D$ then $u + v \in D$.

We collect all coherent sets of desirable options in the set $\bar{\mathcal{D}}$.

More details can be found in a number of papers and books (Walley, 1991, 2000; Moral, 2005; Miranda and Zaffalon, 2010; Courtois and Moral, 2011; De Cooman and Knepper, 2012; De Cooman and Miranda, 2012; Quaeghebeur, 2014; Quaeghebeur et al., 2015; De Bock and De Cooman, 2015).

Axioms $D_3$ and $D_4$ guarantee that a coherent $D$ is a convex cone. This convex cone induces a strict partial order $\prec_D$ on $\mathcal{V}$, by letting

$$u \prec_D v \iff 0 \prec_D v - u \iff v - u \in \bar{\mathcal{D}},$$

so $D = \{ u \in \mathcal{V} : 0 \prec_D u \}$ (De Cooman and Quaeghebeur, 2012; Quaeghebeur, 2014). $D$ and $\prec_D$ are mathematically equivalent: given one of $D$ or $\prec_D$, we can determine the other unequivocally using the formulas above. When it is clear from the context which set of
desirable options \( D \) we are working with, we often refrain from mentioning the explicit reference to \( D \) in \( \prec_D \) and then we simply write \( \prec \). Coherence for sets of desirable options transfers to binary relations \( \prec \) as follows: \( \prec \) must be a strict partial order—meaning that it is irreflexive and transitive—such that \( \prec \leq \prec \), and must satisfy the two characteristic properties of Equations (2) and (3).

What is the relationship between choice functions and sets of desirable options? Since we have just seen that sets of desirable options represent binary preferences, we see that we can associate a set of desirable options \( D_C \) with every given choice function \( C \) by focusing on its binary choices:

\[ u \prec_{D_C} v \iff v - u \in D_C \iff u \in R(\{u, v\}) \quad \text{for all } u, v \in V. \tag{11} \]

\( D_C \) is a coherent set of desirable options if \( C \) is a coherent choice function (Van Camp et al., 2017, Proposition 12). Conversely (Van Camp et al., 2017, Proposition 13), if we start out with a coherent set of desirable options \( D \) then the set \( \{ C \in \mathcal{C} : D_C = D \} \) of all coherent choice functions whose binary choices are represented by \( D \), is non-empty, and its smallest, or least informative, element \( C_D := \inf\{C \in \mathcal{C} : D_C = D\} \) is given by:

\[ C_D(A) = \{ u \in A : (\forall v \in A) v - u \notin D \} = \{ u \in A : (\forall v \in A) u \not\preceq_C v \} \quad \text{for all } A \in \mathcal{Q}. \tag{12} \]

It selects all options from \( A \) that are undominated, or maximal, under the ordering \( \prec_D \), or in other words, it is the corresponding choice function based on Walley–Sen maximality. This \( C_D \) is easy to characterise:

**Proposition 11.** Given any coherent set of desirable options \( D \), then

\[ 0 \in C_D(\{0\} \cup A) \iff D \cap A = \emptyset \quad \text{for all } A \in \mathcal{Q}. \]

*Proof.* By Equation (12), \( 0 \in C_D(\{0\} \cup A) \iff (\forall v \in \{0\} \cup A) v \notin D \iff (\{0\} \cup A) \cap D = \emptyset \), which is equivalent to \( A \cap D = \emptyset \), because \( 0 \notin D \) for any coherent \( D \).

Although \( C_D \) is coherent when \( D \) is, it does not necessarily satisfy the additional Axiom \( C_5 \) as the following counterexample shows.

**Example 1.** Consider the two-dimensional vector space \( V = \mathbb{R}^2 \). We provide it with the component-wise vector ordering \( \preceq \), and consider the vacuous set of desirable options \( D = \{ u \in V : u > 0 \} = V_{>0} \), which is coherent. By Proposition 11, \( 0 \in C_D(\{0\} \cup A) \iff A \cap V_{>0} = \emptyset \) for all \( A \in \mathcal{Q} \). To show that \( C_D \) does not satisfy Axiom \( C_5 \), consider \( A = \{0, u, v\} \), where \( u := (-1, 2) \) and \( v := (2, -1) \). We find that \( 0 \in C_D(A) \) because \( \{u, v\} \cap V_{>0} = \emptyset \), since \( u \not\preceq 0 \) and \( v \not\preceq 0 \).

However, for the option set \( A_1 = A \cup \{u + v/2\} \subseteq \text{CH}(A) \), we find that \( u + v/2 = (|2|, |2|) > 0 \) and therefore \( 0 \notin C_D(A_1) \), meaning that Axiom \( C_5 \) is not satisfied.

For the specific coherent set of desirable options \( D \) considered in Example 1, the corresponding choice function \( C_D \) fails to satisfy the convexity axiom. However, there are other sets of desirable options \( D \) for which \( C_D \) does satisfy the convexity axiom. They are identified in the next proposition.

**Proposition 12.** Consider any coherent set of desirable options \( D \). Then the corresponding coherent choice function \( C_D \) satisfies Axiom \( C_5 \) if and only if \( D^C \) is a convex cone, or in other words, if and only if \( \text{posi}(D^C) = D^C \), or equivalently, \( \text{posi}(D^C) \cap D = \emptyset \).

*Proof.* Van Camp et al. (2017, Proposition 13) have already shown that \( C_D \) is a coherent choice function.

For necessity, assume that \( \text{posi}(D^C) \neq D^C \), or equivalently, that \( \text{posi}(D^C) \cap D \neq \emptyset \). Then there is some option \( u \in D \) such that \( u \in \text{posi}(D^C) \), meaning that there are \( n \in \mathbb{N} \), \( \lambda_k \) in \( \mathbb{R}_{>0} \) and \( u_k \) in \( D^C \) such that \( u = \sum_{k=1}^n \lambda_k u_k \). Let \( A := \{0, u_1, \ldots, u_n\} \) and \( A_1 := A \cup \{u\} \). Due to the coherence of \( D \) (more precisely Axiom \( D_5 \)), we can rescale \( u \in D \) while keeping the \( u_k \) fixed, in such a way that we achieve that \( \sum_{k=1}^n \lambda_k = 1 \), whence \( A \subseteq A_1 \subseteq \text{CH}(A) \).
We find that \( 0 \in \mathcal{C}_D(A) \) by Proposition 11 because \( A \cap D = \emptyset \), but \( 0 \notin \mathcal{C}_D(A_1) \) because \( u \in D \), so \( A_1 \cap D \neq \emptyset \). This tells us that \( \mathcal{C}_D \) does not satisfy Axiom C, because clearly \( \mathcal{C}_D(A) \notin \mathcal{C}_D(A_1) \).

For sufficiency, assume that \( \mathcal{C}_D \) does not satisfy Axiom C. Therefore, there are \( A \) and \( A_1 \) in \( \mathcal{Q} \) such that \( A \subseteq A_1 \subseteq \text{CH}(A) \) and \( \mathcal{C}_D(A) \notin \mathcal{C}_D(A_1) \), or, in other words, such that \( u \in \mathcal{C}_D(A) \) and \( u \notin \mathcal{C}_D(A_1) \) for some \( u \) in \( A \). Consider such \( A \) and \( A_1 \) in \( \mathcal{Q} \), and \( u \) in \( A \). Due to Axiom C, we find that \( 0 \notin \mathcal{C}_D(A - \{u\}) \) and \( 0 \notin \mathcal{C}_D(A_1 - \{u\}) \), or equivalently, by Proposition 11, that \( A - \{u\} \subseteq D^c \) and \( A_1 - \{u\} \cap D \neq \emptyset \). But \( A_1 - \{u\} \subseteq \text{CH}(A) - \{u\} = \text{CH}(A - \{u\}) \subseteq \text{pos}(A - \{u\}) \subseteq \text{pos}(D^c) \), so \( \text{pos}(D^c) \cap D \neq \emptyset \). \( \square \)

This proposition seems to indicate that there is something special about coherent sets of desirable options whose complement is a convex cone too. We give them a special name that will be motivated and explained in the next section.

Definition 9. A coherent set of desirable options \( D \) is called lexicographic if
\[
\text{pos}(D^c) = D^c, \text{ or, equivalently, if } \text{pos}(D^c) \cap D = \emptyset.
\]
We collect all the lexicographic coherent sets of desirable options in \( \bar{D}_L \).

Another important subclass \( \bar{D} \) of coherent sets of desirable options collects all the maximally informative, or maximal, ones:
\[
\bar{D} := \{ D \in \bar{D} : (\forall D' \in \bar{D}) D \subseteq D' \Rightarrow D = D' \}.
\]

The sets of desirable options in \( \bar{D} \) are the undominated elements of the complete infimum-semilattice \((\bar{D}, \leq)\). Cousu and Moral [2011] have proved the following elegant and useful characterisation of these maximal elements:

Proposition 13. Given any coherent set of desirable options \( D \) and any non-zero option \( u \notin D \), then \( \text{pos}(D \cup \{-u\}) \) is a coherent set of desirable options. As a consequence, a coherent set of desirable options \( D \) is maximal if and only if
\[
(\forall u \in \mathcal{V} \setminus \{0\})(u \in D \text{ or } -u \in D).
\]

De Cooman and Quaeghebeur [2012] have proved that the set of all coherent sets of desirable options is dually atomic, meaning that any coherent set of desirable options \( D \) is the infimum of its non-empty set of dominating maximal coherent sets of desirable options:

Proposition 14. For any coherent set of desirable options \( D \), its set of dominating maximal coherent sets of desirable options \( \bar{D}_D := \{ D \in \bar{D} : D \subseteq \bar{D} \} \) is non-empty, and \( D = \cap \bar{D}_D \).

Any maximal coherent set of desirable options is also a lexicographic one: \( \bar{D} \subseteq \bar{D}_L \). Consider a maximal \( D \) and arbitrary \( n \in \mathbb{N}, u_k \in D^c \) and \( \lambda_k \in \mathbb{R}_{\geq 0} \) for \( k \in \{1, \ldots, n\} \). Then since all \( -u_k \in D \cup \{0\} \) by Proposition 13, we infer that \( -\sum_{k=1}^n \lambda_k u_k \in D \cup \{0\} \), because the coherent \( D \) is in particular a convex cone. If \( \sum_{k=1}^n \lambda_k u_k = 0 \), then \( \sum_{k=1}^n \lambda_k u_k \in D^c \) by Axiom D. If \( \sum_{k=1}^n \lambda_k u_k \neq 0 \), then \( -\sum_{k=1}^n \lambda_k u_k \notin D \), and since coherence (more specifically, a combination of Axioms D1 and D2) implies that a coherent set of gambles cannot include both a gamble and its opposite, we conclude that, here too, \( \sum_{k=1}^n \lambda_k u_k \in D^c \). Therefore, \( D^c \) is indeed a convex cone.

\footnote{As pointed out by a reviewer, this can actually be obtained as a corollary to a result by Hammer [1955] Theorem 2), by taking into account that maximal sets of desirable gambles are semispaces and that lexicographic sets of desirable gambles correspond to hemispaces. To make this paper more self-contained, we give a proof using the coherence axioms of sets of desirable gambles we are employing in this paper.}
5. Lexicographic Choice Functions

In this section, we embark on a more detailed study of lexicographic sets of desirable options, and amongst other things, explain where their name comes from. We will restrict ourselves here to the special case where \( \mathcal{V} \) is the linear space \( \mathcal{L}(\mathcal{X}) \) of all gambles on a finite possibility space \( \mathcal{X} \), provided with the component-wise order \( \leq \) as its vector ordering.

We first show that the lower expectation functional associated with a lexicographic \( D \) is actually a linear prevision [Walley 1991, Troffaes and De Cooman 2014].

**Proposition 15.** For any \( D \) in \( \mathcal{D}_L \), the coherent lower prevision \( P_D \) on \( \mathcal{L}(\mathcal{X}) \) defined by

\[
P_D(f) := \sup\{ \mu \in \mathbb{R} : f - \mu \in D \} \quad \text{for all } f \in \mathcal{L}(\mathcal{X})
\]

is a linear prevision: a real linear functional that is positive—so \( (\forall f \geq 0) P_D(f) \geq 0 \)—and normalised—meaning that \( P_D(1) = 1 \).

**Proof.** Consider any \( f \) in \( \mathcal{L} \) and \( \epsilon \) in \( \mathbb{R}_{>0} \), then we first prove that \( f \in D \) or \( \epsilon - f \in D \). Assume ex absurdo that \( f \notin D \) and \( \epsilon - f \notin D \). Then, because by assumption \( \text{posi}(D^\rho) = D^\rho \) is a convex cone, we also have that \( f + \epsilon - f = \epsilon \notin D \), which contradicts Axiom [D2]. Now, Walley [1991] Theorem 3.8.3 guarantees that for any such \( D \), the corresponding functional \( P_D \) is indeed a linear prevision.

To get some feeling for what these lexicographic models represent, we first look at the special case of binary possibility spaces \( \{a, b\} \), leading to a two-dimensional option space \( \mathcal{V} = \mathcal{L}(\{a, b\}) \) provided with the point-wise order. It turns out that lexicographic sets of desirable options (gambles) are easy to characterise there, so we have a simple expression for \( \mathcal{D}_L \).

**Proposition 16.** All lexicographic coherent sets of desirable gambles on the binary possibility space \( \{a, b\} \) are given by (see also Figure 7):

\[
\mathcal{D}_L := \{ D_\rho, D_\rho^a, D_\rho^b : \rho \in (0, 1) \} \cup \{ D_0, D_1 \} = \{ D_\rho : \rho \in (0, 1) \} \cup \mathcal{D},
\]

where

\[
D_\rho := \{ \lambda(\rho - 1[a]) : \lambda \in \mathbb{R} \} + \mathcal{V}_\rho = \text{span}(\{ \rho - \lambda \}^+)_A + \mathcal{V}_\rho \quad \text{for all } \rho \text{ in } (0, 1)
\]

\[
D_\rho^a := D_\rho \cup \{ \lambda(\rho - 1[a]) : \lambda \in \mathbb{R}_{<0} = D_\rho \cup \text{posi}(\{1(a) - \rho\}) \} \quad \text{for all } \rho \text{ in } (0, 1)
\]

\[
D_\rho^b := D_\rho \cup \{ \lambda(\rho - 1[a]) : \lambda \in \mathbb{R}_{>0} = D_\rho \cup \text{posi}(\{\rho - 1[a]\}) \} \quad \text{for all } \rho \text{ in } (0, 1).
\]

\[
D_0 := \{ f \in \mathcal{V} : f(b) > 0 \} \cup \mathcal{V}_0
\]

\[
D_1 := \{ f \in \mathcal{V} : f(a) > 0 \} \cup \mathcal{V}_0.
\]

**Proof.** We first observe that every set of desirable options in \( \mathcal{D}_L \) is coherent. Indeed, for any \( \rho \) in \( (0, 1) \), \( D_\rho \) is the smallest coherent set of desirable gambles corresponding to the linear prevision \( E_\rho \), with \( \rho := (\rho, 1 - \rho) \), while \( D_\rho^a, D_\rho^b \) are maximal coherent sets of desirable gambles corresponding to the same linear prevision \( E_\rho \). Finally, \( D_0 \) is the maximal (and only) coherent set of desirable gambles corresponding to \( E_\rho \) with \( \rho := (0, 1) \), while \( D_1 \) is the maximal (and only) coherent set of desirable gambles corresponding to \( E_\rho \) with \( \rho := (1, 0) \).

We now prove that we recover all lexicographic coherent sets of desirable gambles in this way. Consider any lexicographic coherent set of desirable gambles \( D^\ast \). Then \( P_{D^\ast} \) is a linear prevision, by Proposition 15 so \( P_{D^\ast} \) is characterised (i) by the mass function \((1, 0)\), (ii) by the mass function \((0, 1)\), or (iii) by the mass function \((\rho^+, 1 - \rho^+)\) for some \( \rho^+ \) in \((0, 1)\). If (i), the only coherent set of desirable gambles that induces the linear prevision with mass function \((1, 0)\) is \( D_1 \in \mathcal{D}_L \). If (ii), the only coherent set of desirable gambles that induces the linear prevision with mass function \((0, 1)\) is \( D_0 \in \mathcal{D}_L \). If (iii), there are...
only three coherent sets of desirable gambles that induce the linear prevision with mass function \((\rho^*, 1 - \rho^*)\): \(D_\rho\), \(D_\rho^a\), and \(D_\rho^b\), and all are elements of \(\mathcal{D}_L\).

In the language of sets of desirable gambles (see for instance [Quaeghebeur, 2014]), this means that in the binary case lexicographic sets of desirable gambles are either maximal or strictly desirable with respect to a linear prevision.

We now turn to the more general finite-dimensional case. Recall that a lexicographic order \(\prec\) with \(\ell \in \mathbb{N}\) layers on a vector space \(\mathcal{V}\) of finite dimension \(n\) is defined by

\[
u \prec u \iff (\exists k \in \{1, \ldots, \ell\})(u_k < v_k \text{ and } (\forall j \in \{1, \ldots, k-1\}) u_j = v_j),
\]

and denote, as usual, its reflexive version \(\preceq\) as \(\forall f, g \in \mathcal{L}(\mathcal{X})\), defined by:

\[
f \prec_p g \iff E_p(f) \prec E_p(g), \text{ for all } f, g \in \mathcal{L}(\mathcal{X}).
\]

We refer to work by [Blume et al., 1991], [Fishburn, 1962] and [Seidenfeld et al., 1990] for more details on generic lexicographic probability systems. The connection between lexicographic probability systems and sets of desirable gambles has also been studied by [Cozman, 2015] and [Benavoli et al., 2017], and the connection with full conditional measures by [Halpern, 2010] and [Hammond, 1994]. Below, we first recall a number of relevant basic properties of lexicographic orders in Propositions 18 and 19. We then provide a characterisation of lexicographic sets of desirable gambles in terms of lexicographic orders in Theorem 23.

Remark that the reflexive version \(\preceq\) of \(\prec\)—defined by \(f \preceq_p g \iff E_p(f) \preceq E_p(g)\) for all \(f, g \in \mathcal{L}(\mathcal{X})\)—is a total order on \(\mathcal{L}(\mathcal{X})\) ([Blume et al., 1991]).

We now turn to the more general finite-dimensional case. Recall that a lexicographic order \(\prec\) with \(\ell \in \mathbb{N}\) layers on a vector space \(\mathcal{V}\) of finite dimension \(n\) is defined by

\[
u \prec u \iff (\exists k \in \{1, \ldots, \ell\})(u_k < v_k \text{ and } (\forall j \in \{1, \ldots, k-1\}) u_j = v_j),
\]

and denote, as usual, its reflexive version \(\preceq\) as \(\forall f, g \in \mathcal{L}(\mathcal{X})\), defined by:

\[
f \prec_p g \iff E_p(f) \preceq E_p(g), \text{ for all } f, g \in \mathcal{L}(\mathcal{X}).
\]

We refer to work by [Blume et al., 1991], [Fishburn, 1962] and [Seidenfeld et al., 1990] for more details on generic lexicographic probability systems. The connection between lexicographic probability systems and sets of desirable gambles has also been studied by [Cozman, 2015] and [Benavoli et al., 2017], and the connection with full conditional measures by [Halpern, 2010] and [Hammond, 1994]. Below, we first recall a number of relevant basic properties of lexicographic orders in Propositions 18 and 19. We then provide a characterisation of lexicographic sets of desirable gambles in terms of lexicographic orders in Theorem 23.

Remark that the reflexive version \(\preceq\) of \(\prec\)—defined by \(f \preceq_p g \iff E_p(f) \preceq E_p(g)\) for all \(f, g \in \mathcal{L}(\mathcal{X})\)—is a total order on \(\mathcal{L}(\mathcal{X})\) ([Blume et al., 1991]).

In what follows, we will restrict our attention to lexicographic probability systems \(p\) that satisfy the following condition:

\[
(\forall x \in \mathcal{X})(\exists k \in \{1, \ldots, \ell\}) p_k(x) > 0.
\]
This condition requires that there should be no possible outcome in $X$ that has zero probability in every layer. It is closely related to the notion of a \textit{Savage-null event} (Savage 1972 Section 2.7):

\textbf{Definition 10.} An event $B \subseteq X$ is called \textit{Savage-null} if $(\forall f, g \in \mathcal{L}(X)) \downarrow_B f \leq_B \downarrow_B g$. The event $\emptyset$ is always Savage-null, and is called the \textit{trivial Savage-null event}.

An important feature of preference relations $\prec_p$ based on lexicographic probability systems is the \textit{incomparability relation} $\parallel_p$, defined by: $f \parallel_p g$ if and only if $f \not\prec_p g$ and $g \not\prec_p f$ for all $f$ and $g$ in $\mathcal{L}(X)$. Since $\leq_p$ is a total order, it follows that

$$f \parallel_p g \iff E_p(f) = E_p(g).$$

Finally, it also follows that

$$f \not\prec_p g \iff g \not\prec_p f \parallel_p f \iff E_p(g) \leq E_p(f).$$

\textbf{Proposition 17.} Consider any lexicographic probability system $p = (p_1, \ldots, p_{\ell})$. Then Condition (14) holds if and only if there are no non-trivial Savage-null events.

\textbf{Proof.} For the direct implication, consider any lexicographic probability system $p$ that satisfies Condition (14), and consider any non-empty event $B \subseteq X$. Consider any $x$ in $B$, then $I_B \subseteq I(x)$, so $E_p(I_B) \geq E_p(I(x))$ for every $k \in \{1, \ldots, \ell\}$. Also, $E_p(0) \leq E_p(I(x))$ by Condition (13), so $0_B \prec_p 1_B$ whence $1_B \not\prec_p 0_B$ and hence, by Definition 10 $B$ is indeed no Savage-null event.

For the converse implication, consider any lexicographic probability system $p$ and assume that Condition (14) does not hold. Then there is some $x^* \in X$ such that $p_k(x^*) = 0$ for all $k$ in $\{1, \ldots, \ell\}$, and therefore $E_p(I_k(x^*)) = 0 = E_p(g_k^* \downarrow_{x^*})$ for all $f$ and $g$ in $\mathcal{L}(X)$ and $k \in \{1, \ldots, \ell\}$, so $E_p(f \downarrow_{x^*}) = E_p(g \downarrow_{x^*})$ for all $f$ and $g$ in $\mathcal{L}(X)$). This implies that $\downarrow_{x^*} f \sim_p \downarrow_{x^*} g$ for all $f$ and $g$ in $\mathcal{L}(X)$, so indeed there is a non-trivial Savage-null event $\{x^*\}$.

\textbf{Proposition 18.} Consider any lexicographic probability system $p$ with $\ell$ layers. Then $\prec_p$ is a strict weak order, meaning that $\prec_p$ is irreflexive, and both $\prec_p$ and $\parallel_p$ are transitive. As a consequence, the relation $\not\prec_p$ is transitive as well.

\textbf{Proof.} This is a consequence of Equations (13), (15) and (16), taking into account that $\prec_p$ and $\leq_p$ are transitive, and that $\preceq_p$ is irreflexive.

We now link the lexicographic orderings $\prec_p$ with the preference relation $\prec_D$ based on desirability, given by Equation (10). We begin with an auxiliary result:

\textbf{Proposition 19.} Consider any lexicographic probability system $p$ with $\ell$ layers, and consider any coherent set of desirable gambles $D$. Then $\prec_p$ and $\prec_D$ are (strict) vector orders compatible with $\prec$; they are irreflexive, transitive and

\begin{enumerate}[i)]
  \item $f \prec_p g \iff f > h \prec_p g + h \iff \lambda f \prec_p \lambda g$;
  \item if there are no non-trivial Savage-null events, then $f \prec_p g$;
  \item $f \prec_D g \iff f > h \prec_D g + h \iff \lambda f \prec_D \lambda g$;
  \item $f < g \iff f \not\prec_D g$,
\end{enumerate}

for all $f$, $g$ and $h$ in $\mathcal{L}(X)$ and $\lambda$ in $\mathbb{R}_{>0}$.

\textbf{Proof.} It is clear from Proposition 18 that $\prec_p$ is irreflexive and transitive. To show that $\prec_D$ is irreflexive, infer from $f - f = 0 \in D$ (Axiom $[\mathbf{D}]$) that indeed $f \not\prec_D f$ for all $f$ in $\mathcal{L}(X)$. To show that $\prec_D$ is transitive, assume that $f \prec_D g$ and $g \prec_D h$. Then $g - f \in D$ and $h - g \in D$, by Equation (10), and hence $h - f = g - f + (h - g) \in D$, by Axiom $[\mathbf{D}]$. Using Equation (10) again, we find that then indeed $f \not\prec_D h$. Let us now prove the remaining statements.

\footnote{Except for the second statement, most of the items in this propositions are well-known Quaeghebeur 2014 Section 1.4.1; we include a simple proof for completeness.}
(i) This follows from the definition of < and the linearity of the expectation operator.
(ii) Assume that there are no non-trivial Savage-null events. Use Proposition 17 to infer that Condition (13) holds. Consider any \( f \in \mathcal{L}(\mathcal{X}) \) such that \( 0 < f \). Then \( 0 \leq f - so \leq \mathbb{E}_\nu(f) \) for every \( k \in \{1, \ldots, \ell\} \) — and \( 0 < f(x^*) \) for some \( x^* \) in \( \mathcal{X} \). Then \( p_k(x^*) > 0 \) for some \( k \in \{1, \ldots, \ell\} \) by Condition (14), so \( 0 \leq \mathbb{E}_\nu(I(\mathbb{I})) \). Use \( f(x^*)I(x^*) \leq f \) to infer that then also \( 0 \leq \mathbb{E}_\nu(f) \), whence indeed \( 0 < \mathcal{D} f \).
(iii) The first equivalence follows immediately from Equation (10), while the second is a consequence of the scaling axiom of coherent sets of desirable options.
(iv) Assume that \( f < g \). Then \( 0 < g - f \), whence \( g - f \in \mathcal{D} \) by Axiom (12). Using Equation (10), we find that then indeed \( f \mathcal{D} g \).

Next we establish a link between lexicographic probability systems and preference relations associated with lexicographic sets of desirable gambles. We refer to papers by Cozman (2015, Section 2.1) and Seidenfeld et al. (1990) for other relevant discussion on the connection between lexicographic probabilities and partial preference relations. Our proof is somewhat reminiscent of the representation of conditional probabilities by Krauss (1968), and will make repeated use of the following separation theorem (Holmes 1975, in the form stated by Walley (1991, Appendix E1):

**Theorem 20 (Separating hyperplane theorem).** Let \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) be two convex subsets of a finite-dimensional linear topological space \( \mathcal{B} \). If \( 0 \in \mathcal{W}_1 \cap \mathcal{W}_2 \) and \( \text{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \emptyset \), then there is a non-zero continuous linear functional \( \Lambda \) on \( \mathcal{B} \) such that

\[
\Lambda(w) \geq 0 \quad \text{for all } w \in \mathcal{W}_1 \quad \text{and} \quad \Lambda(w) < 0 \quad \text{for all } w' \in \mathcal{W}_2.
\]

If \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are finite, \( \mathcal{W}_1 \) non-empty, and \( \sum_{i=1}^{m} \lambda_i w_i = \sum_{k=1}^{n} \mu_k w_k' \neq 0 \) for all \( m \) and \( n \in \mathbb{N} \), all \( \lambda_1, \ldots, \lambda_m \) in \( \mathbb{R}_{\geq 0} \) with \( \lambda_i > 0 \) for at least one \( i \) in \( \{1, \ldots, m\} \), all \( \mu_1, \ldots, \mu_n \) in \( \mathbb{R}_{\geq 0} \), all \( w_1, \ldots, w_m \) in \( \mathcal{W}_1 \), and all \( w_1', \ldots, w_n' \) in \( \mathcal{W}_2 \), then there is a non-zero continuous linear functional \( \Lambda \) on \( \mathcal{B} \) such that

\[
\Lambda(w) > 0 \quad \text{for all } w \in \mathcal{W}_1 \quad \text{and} \quad \Lambda(w) < 0 \quad \text{for all } w' \in \mathcal{W}_2.
\]

Two clarifications here are (i) that we will apply the theorem to linear subsets of \( \mathcal{L}(\mathcal{X}) \), which is a linear topological space (Walley 1991, Appendix D) that is finite-dimensional because \( \mathcal{X} \) is finite, and (ii) that when the linear topological space is finite-dimensional, the assumption \( \text{int}(\mathcal{W}_1) \neq \emptyset \) that Walley (1991, Appendix E1) mentions is not necessary for the separating hyperplane theorem to hold, as shown by Holmes (1975, Theorem 4B).

Our proof will also make use of the following two lemmas.

**Lemma 21.** Consider any coherent set \( D \) of desirable gambles on a finite possibility space \( \mathcal{X} \), and consider any linear subspace \( \Lambda \subseteq \mathcal{L}(\mathcal{X}) \). Then \( \text{int}(\text{cl}(D \cap \Lambda)) \cap D^\mathcal{F} = \emptyset \), where \( \text{int} \) is the topological interior and \( \text{cl} \) the topological closure.

**Proof.** We first prove \( \text{int}(\text{cl}(D)) \cap D^\mathcal{F} = \emptyset \). To show that, we will use the fact that \( D \), and therefore also \( \text{cl}(D) \), is a convex set. Since the interior of a convex set is always included in the relative interior \( \text{ri} \) of that convex set (see Brøndsted 1983, Section 1.3), we find that \( \text{int}(\text{cl}(D)) \subseteq \text{ri}(\text{cl}(D)) \). A well-known result (Brøndsted 1983, Theorem 3.4(d)) states that \( \text{ri}(\text{cl}(C)) = \text{ri}(C) \) for any convex set \( C \) in a finite-dimensional vector space, whence \( \text{int}(\text{cl}(D)) \subseteq \text{ri}(D) \). But \( \text{ri}(D) \) is a subset of \( D \), so \( \text{int}(\text{cl}(D)) \subseteq D \), and hence indeed \( \text{int}(\text{cl}(D)) \cap D^\mathcal{F} = \emptyset \).

Now consider \( D \cap \Lambda \), a subset of \( D \). Since both \( \text{cl} \) and \( \text{int} \) respect set inclusion, we find that \( \text{int}(\text{cl}(D \cap \Lambda)) \subseteq \text{int}(\text{cl}(D)) \subseteq D \), whence indeed \( \text{int}(\text{cl}(D \cap \Lambda)) \cap D^\mathcal{F} = \emptyset \). □

**Lemma 22.** Consider a non-zero real linear functional \( \Lambda_1 \) on the \( n \)-dimensional real vector space \( \mathcal{L}(\mathcal{X}) \), and a sequence of non-zero real linear functionals \( \Lambda_k \) defined on the \( n - k + 1 \)-dimensional real vector space \( \text{ker}\Lambda_{k-1} \) for all \( k \in \{2, \ldots, \ell\} \), where \( \ell \in \{2, \ldots, n\} \). Assume that all \( \Lambda_k \) are positive in the sense that \( \forall f \in \mathcal{L}_{\geq 0} \cap \text{dom} \Lambda_k \), \( \Lambda_k(f) \geq 0 \), for all
We will consider the quotient space \( L \) where we let \( \tilde{\lambda} \) be a non-zero linear functional. Since \( \tilde{\lambda} \) is positive, this implies that \( \tilde{\lambda}_k \) is not zero, and \( \tilde{\lambda}_k \) is \( k \)-dimensional. For all \( f \) in \( \mathcal{L}(X) \) we let \( \Gamma_k(f) \geq 0 \).

\( \Gamma_k(1) > 0 \).

\( \ker \Gamma_k \cap \ker \Lambda_{k-1} = \ker \Lambda_k \).

For all \( f \) in \( \ker \Lambda_{k-1} \), \( \Gamma_k(f) > 0 \) if and only if \( \Lambda_k(f) > 0 \).

**Proof.** Fix any \( k \) in \( \{2, \ldots, \ell \} \). Since the real functional \( \Lambda_k \) on the \( n-k+1 \)-dimensional real vector space \( \ker \Lambda_{k-1} \) is non-zero, there is some \( h_k \) in \( \ker \Lambda_{k-1} \) such that \( \Lambda_k(h_k) > 0 \). We will consider the quotient space \( \mathcal{L}(X)/\ker \Lambda_k \), a \( k \)-dimensional vector space whose elements \( f/\ker \Lambda_k = f + \ker \Lambda_k \) are the affine subspaces through \( f \), parallel to the subspace \( \ker \Lambda_k \), for \( f \in \mathcal{L}(X) \). We first show that it follows from Theorem 20 that there is a non-zero linear functional \( \Gamma_k \) on \( \mathcal{L}(X)/\ker \Lambda_k \) such that

\[
\Gamma_k(u) \leq 0 \text{ for all } u \in \mathcal{W}_k^2 := \{ -l_{(x)} \} \setminus \ker \Lambda_k : x \in X_k \},
\]

\[
\Gamma_k(u) > 0 \text{ for all } u \in \mathcal{W}_k^1 := \{ h_{(x)}/\ker \Lambda_k \} \cup \{ l_{(x)}/\ker \Lambda_k : x \in X_k \},
\]

where we let \( X_k := \{ x \in X : l_{(x)} \neq \ker \Lambda_k \} \subseteq X \). The set \( X_k \) is non-empty: since \( \ker \Lambda_k \) is \( n-k \)-dimensional, at most \( n-k \) of the linearly independent indicators \( l_{(x)} \), \( x \in X \) may lie in \( \ker \Lambda_k \), so \( |X_k| \geq k \). To show that we can apply Theorem 20, we prove that the condition for it is satisfied: \( \sum_{i=1}^{n} \lambda_i w_i^1 - \sum_{j=1}^{m} \mu_j w_j^2 \neq 0 \) for all \( m \) and \( n \) in \( \mathbb{N} \), all \( \lambda_1, \ldots, \lambda_m \) in \( \mathbb{R}_{\geq 0} \) with \( \lambda_j > 0 \) for at least one \( j \) in \( \{1, \ldots, m\} \), all \( \mu_1, \ldots, \mu_n \) in \( \mathbb{R}_{\geq 0} \), all \( w_i \) in \( \mathcal{W}_k^1 \), and all \( w_j \) in \( \mathcal{W}_k^2 \). Since \( \mathcal{W}_k^1 \) and \( \mathcal{W}_k^2 \) are finite, it is not difficult to see that it suffices to consider \( \sum_{i=1}^{n} \lambda_i w_i^1 = \lambda h_k/\ker \Lambda_k + \sum_{x \in X_k} \lambda_x l_{(x)}/\ker \Lambda_k \) and \( \sum_{j=1}^{m} \mu_j w_j^2 = -\sum_{x \in X_k} \mu_x l_{(x)}/\ker \Lambda_k \). Since \( \sum_{x \in X_k} \lambda_x l_{(x)} \) is positive, we have \( \sum_{x \in X_k} \lambda_x l_{(x)} > 0 \) or \( \sum_{x \in X_k} \mu_x l_{(x)} > 0 \), or equivalently, that \( \lambda h_k + \sum_{x \in X_k} \lambda_x l_{(x)} \neq 0, \lambda \neq 0 \), or at least one of \( \lambda_x \) are positive. Let \( \lambda_x^* := \{ x \in X_k : \lambda_x > 0 \} \) and \( g := \sum_{x \in X_k} \lambda_x l_{(x)} \), then we know that \( \lambda h_k + g \in \ker \Lambda_k \).

There are now a number of possibilities. The first is that \( \lambda = 0 \), whence \( \lambda \neq \emptyset \) and therefore \( g \in \ker \Lambda_k \). This implies that \( 0 = \lambda \Lambda_k(g) = \sum_{x \in X_k} \lambda_x l_{(x)} \Lambda_k(l_{(x)}) \) since \( \sum_{x \in X_k} \lambda_x l_{(x)} \neq 0 \) and \( \Lambda_k \) is positive, we find that \( l_{(x)} \in \ker \Lambda_k \) for all \( x \in X_k' \). We can go on in this way until we eventually conclude that \( 0 = \Lambda_k(g) = \sum_{x \in X_k'} \lambda_x l_{(x)} \Lambda_k(l_{(x)}) \). Since \( \sum_{x \in X_k'} \lambda_x l_{(x)} \neq 0 \) and \( \Lambda_k \) is positive, we find that \( l_{(x)} \in \ker \Lambda_k \) for all \( x \in X_k' \). The second possibility is that \( \lambda > 0 \). Now \( \lambda \Lambda_k(h_k) = 0 \), a contradiction. Since \( \Lambda_k \neq \emptyset \), we find that \( \lambda h_k + g \in \ker \Lambda_k \) of all \( x \in X_k \). This implies that \( 0 = \lambda_1(g) = \sum_{x \in X_k} \lambda_x l_{(x)} \Lambda_k(l_{(x)}) \) since \( \sum_{x \in X_k} \lambda_x l_{(x)} \neq 0 \) and \( \Lambda_k \) is positive, we find that \( l_{(x)} \in \ker \Lambda_k \) for all \( x \in X_k' \). This in turn allows us to conclude that \( 0 = \lambda_2(g) = \sum_{x \in X_k} \lambda_x l_{(x)} \Lambda_k(l_{(x)}) \). Since \( \sum_{x \in X_k} \lambda_x l_{(x)} \neq 0 \) and \( \Lambda_k \) is positive, we find that \( l_{(x)} \in \ker \Lambda_k \) for all \( x \in X_k' \). We can go on in this way until we eventually conclude that \( \lambda \Lambda_k(h_k) = 0 \), a contradiction. We conclude that, indeed, there is a non-zero linear functional \( \Gamma_k \) on \( \mathcal{L}(X)/\ker \Lambda_k \) that satisfies Equation 17.

We now define the new real linear functional \( \Gamma_k \) on \( \mathcal{L}(X) \) by letting

\[
\Gamma_k(f) := \Gamma_k(f/\ker \Lambda_k) \text{ for all } f \in \mathcal{L}(X).
\]
Observe that, since \( f = \sum_{x \in \mathcal{X}} f(x) \mathbb{1}_{\{x\}} \), this leads to
\[
\Gamma_k(f) = \sum_{x \in \mathcal{X}} f(x) \bar{\Gamma}_k(\mathbb{1}_{\{x\}} / \ker \Lambda_k) = \sum_{x \in \mathcal{X}} f(x) \bar{\Gamma}_k(\mathbb{1}_{\{x\}} / \ker \Lambda_k),
\]
where the second equality follows from \( \mathbb{1}_{\{x\}} \in \ker \Lambda_k \) and therefore \( \mathbb{1}_{\{x\}} / \ker \Lambda_k = 0 \), for all \( x \in \mathcal{X} \setminus \mathcal{X}_k \). If we also take into account Equation (17), this proves in particular that (ii) and (iii) hold.

For the rest of the proof, consider any \( f \in \ker \Lambda_k \) and \( \lambda = \Lambda_k(f) / \Lambda_k(h_k) \), a well-defined real number because \( \Lambda_k(h_k) > 0 \). Then \( 0 = \Lambda_k(f) - \lambda \Lambda_k(h_k) = \Lambda_k(f - \lambda h_k) \), so \( f - \lambda h_k \in \ker \Lambda_k \). As a result, \( f / \ker \Lambda_k = \lambda h_k / \ker \Lambda_k \) and therefore \( \Gamma_k(f) = \Gamma_k(f / \ker \Lambda_k) = \bar{\Gamma}_k(\lambda h_k / \ker \Lambda_k) \). Substituting back for \( \lambda \), we get the equality:
\[
\Gamma_k(f) \Lambda_k(h_k) = \bar{\Gamma}_k(h_k / \ker \Lambda_k) \Lambda_k(f).
\]

Since both \( \Lambda_k(h_k) > 0 \) and \( \bar{\Gamma}_k(h_k / \ker \Lambda_k) > 0 \) [by Equation (17)], we see that \( \Gamma_k(f) \) and \( \Lambda_k(f) \) are either both zero, both (strictly) positive, or both (strictly) negative. This proves (iii) and (iv).

\[\square\]

**Theorem 23.** Given a lexicographic probability system \( p = (p_1, \ldots, p_r) \) that has no non-trivial Savage-null events, the set of desirable gambles \( D_p := \{ f \in \mathcal{L}(\mathcal{X}) : 0 <_p f \} \) corresponding with the preference relation \( <_p \), is an element of \( \mathcal{D}_L \)—a coherent and lexicographic set of desirable gambles. Conversely, given a lexicographic set of desirable gambles \( D \) in \( \mathcal{D}_L \), its corresponding preference relation \( <_D \) is a preference relation based on some lexicographic probability system \( p = (p_1, \ldots, p_r) \) that has no non-trivial Savage-null events.

**Proof.** We begin with the first statement. We first show that \( D_p \) is coherent. For Axiom (ii), infer from \( 0 \leq p(0) \) by the irreversibility of \( <_p \) [see Proposition (18)] that indeed \( 0 \notin D_p \). For Axiom (iii), consider any \( f \in \mathcal{L}_{>0} \). Use Proposition (19) to infer that \( 0 <_p f \), whence indeed \( f \in D_p \). For Axiom (iv), consider any \( f \in D_p \) and \( \lambda \in \mathbb{R}_{>0} \). Then \( 0 <_p f \), and hence \( 0 <_p \lambda f \) using Proposition (19). Then indeed \( \lambda f \in D_p \). For Axiom (v), consider any \( f \) and \( g \) in \( D_p \), whence \( 0 <_p f \) and \( 0 <_p g \). From \( 0 <_p g \) infer that \( f <_p f + g \) by Proposition (19) and using \( 0 <_p f \), that \( 0 <_p f + g \) by the transitivity of \( <_p \) [see Proposition (18)]. Then indeed \( f + g \in D_p \).

So it only remains to show that \( \text{pos}(D_p^c) = D_p^c \). Consider any \( f \) and \( g \) in \( D_p^c \) and any \( \lambda_1 \) and \( \lambda_2 \) in \( \mathbb{R}_{>0} \), then we must prove that \( \lambda_1 f + \lambda_2 g \in D_p^c \). Since by assumption \( 0 \notin f \) and \( 0 \notin g \), Equation (10) guarantees that
\[
E_p(f) \leq L \ E_p(0) = 0 \quad \text{and} \quad E_p(g) \leq L \ E_p(0).
\]

By the linearity of the expectation operator,
\[
E_p(\lambda_1 f + \lambda_2 g) \leq L \ E_p(0) = 0,
\]
whence \( 0 \notin \lambda_1 f + \lambda_2 g \). Then indeed \( \lambda_1 f + \lambda_2 g \in D_p^c \).

For the second statement, we consider any \( D \) in \( \mathcal{D}_L \), and we construct a lexicographic probability system \( p \) with no non-trivial Savage-null events and such that \( <_p \) equals \( <_D \). Define the real functional \( \Lambda_1 \) on \( \mathcal{L}(\mathcal{X}) \) by letting \( \Lambda_1(f) := \sup \{ \alpha \in \mathbb{R} : f - \alpha \in D \} \) for all \( f \in \mathcal{L}(\mathcal{X}) \). Proposition (15) guarantees that \( \Lambda_1 \) is a linear functional. Its kernel \( \ker \Lambda_1 \) is an \( n - 1 \)-dimensional linear space \(^5\) where \( n \) is the finite dimension of the real vector space \( \mathcal{L}(\mathcal{X}) \)—the cardinality of \( \mathcal{X} \). Since both \( D^c \) and \( \ker \Lambda_1 \) are convex cones, so is their intersection \( D^c \cap \ker \Lambda_1 \), and it contains 0 because \( 0 \in D^c \) and \( 0 \in \ker \Lambda_1 \). Using similar arguments, we see that \( D \cap \ker \Lambda_1 \) is either a convex cone or empty. When \( D \cap \ker \Lambda_1 = \emptyset \),

\(^5\)To see that \( \ker \Lambda_1 \) is a linear space, note that by Proposition (15) \( \Lambda_1 = E_p \) is a linear previsionso \( \Lambda_1 \) is a linear map from the \( n \)-dimensional linear space \( \mathcal{L}(\mathcal{X}) \) to \( \mathbb{R} \). Since \( \Lambda_1 \) is a linear map, its kernel is closed under addition and scalar multiplication, so it is a linear space, and by the rank-nullity theorem, its dimension is \( \dim \mathcal{L}(\mathcal{X}) - \dim \mathbb{R} = n - 1 \).
let \( \ell := 1 \), and stop. When \( D \cap \ker A_1 \neq \emptyset \), it follows from Theorem 20 that there is some non-zero (continuous) linear functional \( A_2 \) on \( \ker A_1 \) such that
\[
A_2(f) \leq 0 \text{ for all } f \in D^c \cap \ker A_1 \text{ and } A_2(f) \geq 0 \text{ for all } f \in D \cap \ker A_1.
\]

(Apply Theorem 20 with \( B = \ker A_1 \), \( W_1 = D^c \cap \ker A_1 \) and \( W_2 = \text{cl}(D^c \cap \ker A_1) \) (the topological closure of \( D^c \cap \ker A_1 \) in \( \ker A_1 \)); then \( \text{int}(W_1) \cap W_2 = \emptyset \) by Lemma 21, and \( 0 \in W_1 \cap W_2 \) ker \( A_2 \) is a \( n - 2 \)-dimensional linear space. Also, \( D \cap \ker A_2 \) is either empty or a non-empty convex cone. If it is empty, let \( \ell = 2 \); otherwise, we repeat the same procedure again: it follows from Theorem 20 that there is some non-zero (continuous) linear functional \( A_3 \) on \( \ker A_2 \) such that
\[
A_3(f) \leq 0 \text{ for all } f \in D^c \cap \ker A_2 \text{ and } A_3(f) \geq 0 \text{ for all } f \in D \cap \ker A_2.
\]

(Apply Theorem 20 with \( B = \ker A_2 \), \( W_1 = D^c \cap \ker A_2 \) and \( W_1 = \text{cl}(D^c \cap \ker A_2) \) (the topological closure of \( D^c \cap \ker A_2 \) in \( \ker A_2 \)); then \( \text{int}(W_1) \cap W_2 = \emptyset \) by Lemma 21, and \( 0 \in W_1 \cap W_2 \) ker \( A_3 \) is a \( n - 3 \)-dimensional linear space. Also, \( D \cap \ker A_3 \) is either empty or a non-empty convex cone. If it is empty, let \( \ell = 3 \); if not, continue in the same vein. This leads to successive linear functionals \( A_k \) defined on the \( n - k + 1 \)-dimensional linear spaces \( \ker A_{k-1} \) such that
\[
A_k(f) \leq 0 \text{ for all } f \in D^c \cap \ker A_{k-1} \text{ and } A_k(f) \geq 0 \text{ for all } f \in D \cap \ker A_{k-1}.
\]

This sequence stops as soon as \( D \cap \ker A_k = \emptyset \), and we then let \( \ell = k \). Because the finite dimensions of the successive ker \( A_k \) decrease with 1 at each step, we are guaranteed to stop after at most \( n \) repetitions: should \( D \cap \ker A_k = \emptyset \) for all \( k \in \{1, \ldots, n-1\} \) then ker \( A_n \) will be the 0-dimensional linear space \( \{0\} \), and then necessarily \( D \cap \ker A_n = \emptyset \). For the last functional \( A_{\ell} \), we have moreover that
\[
A_{\ell}(f) > 0 \text{ for all } f \in D \cap \ker A_{\ell-1}.
\]

To see this, recall that by construction \( A_{\ell}(f) \geq 0 \) for all \( f \) in \( D \cap \ker A_{\ell-1} \), and that \( D \cap \ker A_{\ell} = \emptyset \).

In this fashion we obtain \( \ell \) linear functionals \( A_1, \ldots, A_{\ell} \), each defined on the kernel of the previous functional—except for the domain \( L(\mathcal{X}) \) of \( A_1 \). We now show that we can turn the \( A_2, \ldots, A_{\ell} \) into expectation operators: positive and normalised linear functionals on the linear space \( L(\mathcal{X}) \). Indeed, consider their respective extensions \( \Gamma_2, \ldots, \Gamma_{\ell} \) to \( L(\mathcal{X}) \) from Lemma 22 and let \( \Gamma_1 := A_1 \). They satisfy \( \Gamma_k(1) > 0 \) for all \( k \in \{1, \ldots, \ell\} \); see Proposition 15 and Lemma 24. Now consider the real linear functionals on \( L(\mathcal{X}) \) defined by \( E_k := \Gamma_1 \), and \( E_k(f) := \Gamma_k(f) / \Gamma_k(1) \) for all \( k \in \{2, \ldots, \ell\} \) and \( f \) in \( L(\mathcal{X}) \). It is obvious from Proposition 15 and Lemma 24 that these linear functionals are normalised and positive, and therefore expectation operators on \( L(\mathcal{X}) \). Indeed each \( E_k \) is the expectation operator associated with the mass function \( p_k(x) := E_k(f(x)) \) for all \( x \) in \( \mathcal{X} \). In this way, \( p := (p_1, \ldots, p_\ell) \) defines a lexicographic probability system.

We now prove that \( p \) has no non-trivial Savage-null events, using Proposition 17. Assume \textit{ex absurd"o} that there is some \( x^* \) in \( \mathcal{X} \) such that \( p_k(x^*) = E_k(f(x^*)) = 0 \) for all \( k \in \{1, \ldots, \ell\} \). Then \( E_1(x^*) = E_2(x^*) = \ldots = E_\ell(x^*) = 0 \) for all \( k \in \{2, \ldots, \ell\} \). Invoke Lemma 24 to find that \( E_1(x^*) \in \ker A_1 \cap \ker A_2 = \ker A_2 \). Repeated application of this same lemma eventually leads us to conclude that \( \{x^*\} \in \ker A_{\ell-1} \) and \( \{x^*\} \notin \ker A_{\ell} \). Since also \( x^* \in D \) and hence \( x^* \in D \cap \ker A_{\ell-1} \) [Axiom D2], Equation 19 implies that \( A_{\ell}(f(x^*)) > 0 \), a contradiction.

It now only remains to prove that \( \prec_D \) is the lexicographic ordering with respect to \textit{this} lexicographic probability system, or in other words that
\[
f \in D \iff 0 < L(E_1(f), \ldots, E_\ell(f)) \text{ for all } f \in L(\mathcal{X}).
\]

For necessity, assume that \( f \in D \). Then \( E_1(f) \geq 0 \) by the definition of \( A_1 \). If \( E_1(f) > 0 \), then we are done. So assume that \( E_1(f) = 0 \). Then \( f \in \ker A_1 \) and \( A_2(f) \geq 0 \) by Equation 18. Again, if \( A_2(f) > 0 \), we can invoke Lemma 24 to find that \( \Gamma_2(f) > 0 \) and
We see that there are two different lexicographic probability systems that map to the same coherent set of desirable gambles. It makes sense to talk about a coherent set of desirable gambles when there is a coherent lexicographic probability system. It makes sense to talk about a coherent set of desirable gambles when there is a coherent lexicographic probability system. This means that we can infer the following chain of equivalences:

\[
\text{Example 1).}
\]

For the first statement, infer the following chain of equivalences:

\[
\text{If } E_\ell(f) = 0 \text{ for all } \ell \text{ in } \{1, \ldots, k-1\}, \text{ or in other words, the smallest number for which } E_k(f) > 0, \text{ then } k \leq \ell \text{ by construction—see Equation (19)—, whence indeed } 0 <_\ell E_1(f), \ldots, E_{\ell}(f).
\]

For sufficiency, assume that \(0 <_\ell E_1(f), \ldots, E_{\ell}(f)\), meaning that there is some \(k\) in \(\{1, \ldots, \ell\}\) for which \(E_j(f) = 0 = \Gamma_j(f)\) for all \(j\) in \(\{1, \ldots, k-1\}\) and \(E_k(f) > 0\), whence also \(\Gamma_k(f) > 0\). So \(f \in \ker \Gamma_j\) for all \(j\) in \(\{1, \ldots, k-1\}\) and therefore repeated application of Lemma 22(iii) tells us that \(f \in \ker \Lambda_j\) for all \(j\) in \(\{1, \ldots, k-1\}\). Since \(\Gamma_k(f) > 0\), we infer from Lemma 22(iv) that also \(\Lambda_k(f) > 0\), whence indeed \(f \in D\) by Equation (18).

We conclude that the sets of desirable options in \(\bar{D}_L\) are exactly the ones that are representable by a lexicographic probability system that has no non-trivial Savage-null events. This is, of course, the reason why we have called the coherent sets of desirable options in \(\bar{D}_L := \{D \in D : \pos(D^c) = D^c\}\) lexicographic.

This does not mean, however, that the correspondence between the two families is bijective. Indeed, consider the binary possibility space \(\mathcal{X} = \{a, b\}\) and the lexicographic systems \(p = (p_1, p_2)\) and \(p' = (p_1', p_2')\), associated with the respective mass functions \(p_1 = (1/2, 1/2)\), \(p_2 = (0, 1)\) and \(p_2' = (1/4, 3/4)\) on \(\{a, b\}\). Then \(p\) and \(p'\) have no non-trivial Savage-null events. However, \(D_p = D_{p'} = \{f : f(a) + f(b) > 0 \text{ or } f(b) = -f(a) > 0\}\), so we see that there are two different lexicographic probability systems that map to the same lexicographic set of desirable gambles. Note, however, that \(\prec_p\) and \(\prec_{p'}\) are both equal to \(\prec_D\) — and therefore \(\prec_p = \prec_{p'}\) — as is also guaranteed by the following corollary.

**Corollary 24.** Consider any coherent lexicographic set of desirable gambles \(D\) in \(\bar{D}_L\) and any lexicographic probability system \(p\) that has no non-trivial Savage-null events. Then \(\prec_D = \prec_p\) if and only if \(D = D_p\). As a consequence, given any coherent lexicographic set of desirable gambles \(D\) in \(\bar{D}_L\), then \(\prec_D = \prec_p\) for all lexicographic probability systems \(p\) that have no non-trivial Savage-null events such that \(D_p = D\).

**Proof.** For the first statement, infer the following chain of equivalences:

\[
\begin{align*}
\prec_D &= \prec_p \iff \forall f \in \mathcal{L} \left( 0 <_D f \iff 0 <_p f \right) \\
&\iff \forall f \in \mathcal{L} \left( f \in D \iff f \in D_p \right) \\
&\iff D = D_p
\end{align*}
\]

The second statement now follows immediately.

This corollary is important, since it guarantees that \(\prec_p\) and \(\prec_{p'}\) differ if and only if \(D_p\) and \(D_{p'}\) differ: it rules out that two different preference relations \(\prec_p\) and \(\prec_{p'}\) (based on two different lexicographic probability systems that have no non-trivial Savage-null events) map to the same coherent lexicographic set of desirable gambles \(D_p = D_{p'}\). For more information about this relation—and also taking updating into account—can be found in work by [Benavoli et al. 2017], which builds on the important lexicographic separation theorem by [Martínez-Legaz 1983].

**Remark 1.** As pointed out by a reviewer, the second part of Theorem 23 can also be obtained as a consequence of an earlier result by [Martínez-Legaz and Vicente-Pérez 2012, Corollary 3.5], considering that lexicographic sets of desirable gambles are hemispheres and the representation of lexicographic probability systems as stochastic matrices. This same result was also used by [Benavoli et al. 2017] in their study of the connection between sets of desirable gambles and sets of lexicographic probability systems. It makes sense to talk about a coherent set of desirable gambles when there is a coherent lexicographic probability system.
use of a lexicographic separation result [Martínez-Legaz [1983] that is more directly suited for our purpose than the separation results (see Theorem 20) we borrowed from Walley [1991]. However, we feel that there is value in our giving a more direct proof, since it is more directly tailored to, and ‘translated’ in, the language of sets of desirable gambles. ♦

Lexicographic probability systems can now also be related to specific types of choice functions, through Proposition 12: given a coherent set of desirable options D, the most conservative coherent choice function C_D whose binary choices are represented by D_C = D satisfies the convexity axiom C^* if and only if D is a lexicographic set of desirable options. We will call \( \mathcal{D}_l := \{ D_D : D \in \mathcal{D}_l \} \) the set of lexicographic choice functions.

Looking first at the most conservative coherent choice function that corresponds to D and then checking whether it is ‘convex’, leads rather restrictively to lexicographic choice functions, and is only possible for lexicographic D: convexity and choice based on Walley–Sen maximality are only compatible for lexicographic binary choice. But suppose we turn things around somewhat, first restrict our attention to all ‘convex’ coherent choice functions from the outset, and then look at the most conservative such choice function that makes the same binary choices as present in some given D:

\[
\inf \{ C \in \mathcal{L} : C \text{ satisfies Axiom } C_5 \text{ and } D_C = D \}.
\]

We infer from Proposition 4 that this infimum is still ‘convex’ and coherent. It will, of course, no longer be lexicographic, unless D is. The following proposition tells us it still is an infimum of lexicographic choice functions.

**Proposition 25.** Consider an arbitrary coherent set of desirable options D. The most conservative coherent choice function C that satisfies Axiom C_5 and D_C = D is the infimum of all lexicographic choice functions C_D' with D' in \( \mathcal{D}_l \) such that D \( \subseteq \) D'.

\[
\inf \{ C \in \mathcal{L} : C \text{ satisfies Axiom } C_5 \text{ and } D_C = D \} = \inf \{ C_D' : D' \in \mathcal{D}_l \text{ and } D \subseteq D' \}.
\]

**Proof.** Denote the choice function on the left-hand side by C_{left}, and the one on the right-hand side by C_{right}. Both are coherent, and so by Axiom C_5, completely characterised by the option sets from which 0 is chosen. Consider any A in C_0, then we have to show that 0 \in C_{left}(\{0\} \cup A) \Leftrightarrow 0 \in C_{right}(\{0\} \cup A).

For the direct implication, we assume that 0 \in C_{left}(\{0\} \cup A), meaning that there is some C^* in C that satisfies Axiom C_5: D_{C^*} = D and 0 \in C^*(\{0\} \cup A). We have to prove that there is some D' in \( \mathcal{D}_l \) such that D \( \subseteq \) D' and D' \cap A = \emptyset [by Proposition 11], and we will do so by constructing a suitable lexicographic probability system, by a repeated application of an appropriate version of the separating hyperplane theorem (Theorem 20, as in the proof of Theorem 23).

To prepare for this, we prove that pos(C^*) \cap D = \emptyset. Indeed, assume ex absurdo that pos(C^*) \cap D \neq \emptyset, so there is some f in D such that f \in pos(C^*) \cup A). Then there is some \lambda in R_+ such that g := \lambda f \in CH((0) \cup A). Let A' := A \cup \{g\}, so \{0\} \cup A' \subseteq CH((0) \cup A), whence 0 \in C^*(\{0\} \cup A') by Axiom C_5, if we recall that 0 \in C^*(\{0\} \cup A). But f \in D implies that g \in D, and since D_{C^*} = D, also that g \in D_{C'}, or equivalently, 0 \in R^*(\{0,g\}), by Proposition 11 Version 4 of Axiom C_5 then guarantees that 0 \in R^*(\{0\} \cup A'), a contradiction.

It follows from this observation that we can apply Theorem 20 to show that there is some non-zero linear functional \( \Lambda_1 \) on \( \mathcal{L} \) such that \( \Lambda_1(f) \leq 0 \) for all f in pos(\{0\} \cup A) and \( \Lambda_1(f) \geq 0 \) for all f in D. [Apply Theorem 20 with B = L(X), W_2 = pos(C^*(\{0\} \cup A) and W_1 = D \cup \{0\}, then inf(W_1 \cap W_2 = \emptyset since inf(W_1) \subseteq D_1, and 0 \in W_1 \cap W_2.] Its kernel ker \( \Lambda_1 \) is an n - 1-dimensional linear space, where n is the dimension of L(X)—the cardinality of X. Since both D and ker \( \Lambda_1 \) are convex cones, their intersection ker \( \Lambda_1 \cap D \) is either empty or a convex cone. When ker \( \Lambda_1 \cap D = \emptyset \), we let \( \ell := 1 \), and stop.
When \( \ker \Lambda_1 \cap D \neq \emptyset \), it follows from the same version of the separating hyperplane theorem that there is some non-zero linear functional \( \Lambda_2 \) on \( \ker \Lambda_1 \) such that

\[
\Lambda_2(f) \leq 0 \text{ for all } f \text{ in } \ker \Lambda_1 \cap \text{posi}(\{0\} \cup A) \text{ and } \Lambda_2(f) \geq 0 \text{ for all } f \text{ in } \ker \Lambda_1 \cap D.
\]

[Apply Theorem 20 with \( B = \ker \Lambda_1, W_2 = \text{posi}(\{0\} \cup A) \cap \ker \Lambda_1 \) \( \cup \) \( \{0\} \), then int\( (W_1) \cap W_2 = \emptyset \) since \( W_2 \subseteq \text{posi}(\{0\} \cup A) \) and \( \text{int}(W_1) \subseteq D \), and \( 0 \in W_1 \cap W_2 \).

\( \ker \Lambda_2 \) is a \( n - 2 \)-dimensional linear space. As before, \( D \cap \ker \Lambda_2 \) is either empty or a non-empty convex cone. If it is empty, let \( \ell = 2 \); otherwise, repeat the same procedure over and over again, leading to successive non-zero linear functionals \( \Lambda_k \) on \( \ker \Lambda_{k-1} \) such that

\[
\Lambda_k(f) \leq 0 \text{ for all } f \text{ in } \ker \Lambda_{k-1} \cap \text{posi}(\{0\} \cup A) \text{ and } \Lambda_k(f) \geq 0 \text{ for all } f \text{ in } \ker \Lambda_{k-1} \cap D,
\]

(21)

until eventually we get to the first \( k \) such that \( D \cap \ker \Lambda_k = \emptyset \), and then let \( \ell := k \) and stop. We are guaranteed to stop after at most \( n \) repetitions, since \( \ker \Lambda_n \) is the \( 0 \)-dimensional linear space \( \{0\} \), for which \( D \cap \ker \Lambda_n = \emptyset \). For the last functional \( \Lambda_\ell \), we have that

\[
\Lambda_\ell(f) > 0 \text{ for all } f \text{ in } D \cap \ker \Lambda_{\ell-1}.
\]

(22)

To see this, recall that by construction \( \Lambda_\ell(f) \geq 0 \) for all \( f \) in \( D \cap \ker \Lambda_{\ell-1} \), and that \( D \cap \ker \Lambda_\ell = \emptyset \).

In this fashion we obtain \( \ell \) linear functionals \( \Lambda_1, \ldots, \Lambda_\ell \), each defined on the kernel of the previous functional—except for the domain \( \mathcal{L}(X) \) of \( \Lambda_1 \). We now show that we can turn the \( \Lambda_1, \ldots, \Lambda_\ell \) into expectation operators: positive and normalised linear functionals on the linear space \( \mathcal{L}(X) \). Indeed, consider their respective extensions \( \Gamma_1, \ldots, \Gamma_\ell \) to \( \mathcal{L}(X) \) from Lemma 22 and let \( \Gamma_1 := \Lambda_1 \). They satisfy \( \Gamma_k(1) > 0 \) for all \( k \) in \( \{1, \ldots, \ell\} \); see Proposition 15 and Lemma 22(ii). Now consider the real linear functionals on \( \mathcal{L}(X) \) defined by \( \mathbb{E}_k(f) := \Gamma_k(f)/\Gamma_k(1) \) for all \( k \) in \( \{1, \ldots, \ell\} \) and \( f \) in \( \mathcal{L}(X) \). It is obvious from Lemma 24(iii) that these linear functionals are normalised and positive, and therefore expectation operators on \( \mathcal{L}(X) \). Indeed each \( \mathbb{E}_k \) is the expectation operator associated with the mass function \( p_k \) defined by \( p_k(x) := \mathbb{E}_k(\mathbb{I}_X(x)) \) for all \( x \) in \( X \). In this way, \( p := (p_1, \ldots, p_\ell) \) defines a lexicographic probability system.

We now prove that \( p \) has no non-trivial Savage-null events, using Proposition 17. Assume \textit{ex absurdo} that there is some \( x^* \in \mathcal{X} \) such that \( p_k(x^*) = \mathbb{E}_k(\mathbb{I}_{\{x^*\}}) = 0 \) for all \( k \) in \( \{1, \ldots, \ell\} \). Then \( \mathbb{I}_{\{x^*\}} \in \ker \Gamma_1 = \ker \Lambda_1 \) and \( \mathbb{I}_{\{x^*\}} \in \Gamma_k \) for all \( k \) in \( \{2, \ldots, \ell\} \). Invoke Lemma 24(iii) to find that \( \mathbb{I}_{\{x^*\}} \in \ker \Lambda_1 \cap \ker \Gamma_2 = \ker \Lambda_2 \). Repeated application of this same lemma eventually leads us to conclude that \( \mathbb{I}_{\{x^*\}} \in \ker \Lambda_{\ell-1} \cap \ker \Gamma_\ell = \ker \Lambda_\ell \). Since also \( \mathbb{I}_{\{x^*\}} \in D \) and hence \( \mathbb{I}_{\{x^*\}} \in D \cap \ker \Lambda_{\ell-1} \) [Axiom D2], Equation (22) implies that \( \Lambda_\ell(\mathbb{I}_{\{x^*\}}) > 0 \), a contradiction.

If we now let \( D^* := \{ f \in \mathcal{L}(X) : 0 \leq \mathbb{E}_i(f), \ldots, \mathbb{E}_\ell(f) \} \), then \( D^* \in \mathcal{D}_1 \) by Theorem 23. If we can show that \( D \subseteq D^* \) and \( A \cap D^* = \emptyset \), we are done. So first, consider any \( f \) in \( D \). Then \( \Lambda_1(f) \geq 0 \) by Equation 20. If \( \Lambda_1(f) > 0 \) then also \( \mathbb{E}_1(f) > 0 \) by Lemma 22(iii) and therefore \( f \in D^* \). If \( \Lambda_1(f) = 0 \) then \( \Lambda_2(f) \geq 0 \) by Equation 21. If \( \Lambda_2(f) > 0 \) then also \( \mathbb{E}_2(f) > 0 \) by Lemma 24(ii) and therefore \( f \in D^* \). We can go on in this way until we get to the first \( k \) for which \( \Lambda_k(f) < 0 \), and therefore also \( \mathbb{E}_k(f) > 0 \) by Lemma 24(ii), whence therefore \( f \in D^* \). We are guaranteed to find such a \( k \) because we infer from Equation (22) that \( \Lambda_k(f) > 0 \). This shows that indeed \( D \subseteq D^* \).

Secondly, consider any \( f \) in \( A \). Then \( \Lambda_1(f) \leq 0 \) by Equation 20. If \( \Lambda_1(f) < 0 \) then also \( \mathbb{E}_1(f) < 0 \) by Lemma 24(ii) and therefore \( f \notin D^* \). If \( \Lambda_1(f) = 0 \) then \( \Lambda_2(f) \leq 0 \) by Equation 21. If \( \Lambda_2(f) < 0 \) then also \( \mathbb{E}_2(f) < 0 \) by Lemma 24(ii) and therefore \( f \notin D^* \). If we go on in this way, only two things can happen: either there is a first \( k \) for which \( \Lambda_k(f) < 0 \), and therefore also \( \mathbb{E}_k(f) < 0 \) by Lemma 24(ii), whence therefore \( f \notin D^* \). Or we find that \( \Lambda_k(f) \leq 0 \), and therefore also \( \mathbb{E}_k(f) \leq 0 \) by Lemma 24(ii) for all \( k \) in \( \{1, \ldots, \ell\} \), whence again \( f \notin D^* \). This shows that indeed \( A \cap D^* = \emptyset \).
For the converse implication, assume that \(0 \in C_{\text{right}}(\{0\} \cup A)\). We must prove that there is some \(\tilde{C}\) in \(\bar{C}\) that satisfies Axiom \(C_5\), \(D_{\tilde{C}} = D\) and \(0 \in \tilde{C}(\{0\} \cup A)\). We claim that \(\tilde{C} := C_{\text{right}}\) does the job. Because we know by assumption that \(0 \in C_{\text{right}}(\{0\} \cup A)\), and from Propositions 12 and 14 that \(C_{\text{right}}\) is coherent and satisfies Axiom \(C_5\), it only remains to prove that \(D_{C_{\text{right}}} \equiv D\). To this end, consider any \(f\) in \(\mathcal{L}(X)\) and recall the following equivalences:

\[
\begin{align*}
f \in D_{C_{\text{right}}} &\iff 0 \in R_{\text{right}}(\{0, f\}) \quad \text{[Equation (11)]} \\
&\iff (\forall D' \in \bar{D}_L)(D \subseteq D' \Rightarrow 0 \in R_{D'}(\{0, f\})) \quad \text{[definition of inf]} \\
&\iff (\forall D' \in \bar{D}_L)(D \subseteq D' \Rightarrow f \in D') \quad \text{[Proposition 11]} \\
&\iff f \in D, \quad \text{[Proposition 14 and \(D \subseteq \bar{D}_L\)]}
\end{align*}
\]

which completes the proof. \(\square\)

As a consequence of this result, we also have that, for any coherent set of desirable options \(D\),

\[
\inf\{C \in \bar{C} : C \text{satisfies Axiom } C_5 \text{ and } D \subseteq D_C\} = \inf\{C_{D'} : D' \in \bar{D}_L \text{ and } D \subseteq D'\}.
\]

### 6. Discussion and Future Research

One of the advantages of lexicographic probability systems is that they are more informative than single probability measures, and that they allow us to deal with some of the issues that arise when conditioning on events of probability zero. This is also the underlying idea behind some imprecise probability models, such as sets of desirable gambles. In this paper, we have investigated the connection between the two models, by means of the more general theory of coherent choice functions. We have shown that lexicographic probability systems correspond to the convexity axiom that was considered by Seidenfeld et al. when considering choice functions on horse lotteries. The study of this axiom has led to the consideration of what we have called lexicographic sets of desirable gambles.

In addition, we have also discussed the connection between our notion of coherent choice functions on abstract vectors, and the earlier notion for horse lotteries, developed mostly by Seidenfeld et al. (2010). We have proved that by defining choice functions on arbitrary vector spaces—something which also proves useful when studying the implications of an indifference assessment (Van Camp et al., 2017)—we can include choice functions on horse lotteries as a particular case. This allows us in particular to formulate our results for that framework. Note, nevertheless, that there are some differences between Seidenfeld et al.'s (2010) approach and ours, due to the rationality axioms considered and also to the fact that they deal with possibly infinite (but closed) sets of options, whereas our model assumes that choices are always made between finite sets of alternatives. It would be interesting to investigate the extent to which our results can be generalised to infinite option sets.

One of the advantages of Seidenfeld et al.'s (2010) approach is that it leads to a representation theorem, in the sense that any coherent choice function can be obtained as the infimum of an arbitrary family of more informative convex coherent choice functions that essentially correspond to probability mass functions. Based on the results and conclusions derived here, it seems natural to wonder if a similar result can be established in our framework. Unfortunately, the answer to this question is negative: it turns out that in addition to convexity we need another axiom, which we have called weak Archimedeanity. With this extra axiom, at least for binary possibility spaces, it turns out a similar representation result can be proved: every such choice function is an infimum of its lexicographic dominating choice functions, showing the importance of a study of lexicographic choice functions also from another angle of perspective. The observation
that we need an Archimedean axiom is in agreement with [Seidenfeld et al., 2010] need of their Archimedean axiom, which is—unlike our weak Archimedeanity—difficult to join with desirability. We intend to report on these results elsewhere.

On the other hand, we would also like to combine our results with the discussion by [Van Camp et al., 2017], and investigate indifference and conditioning for the special case of lexicographic choice functions. In particular, this should allow us to link our work with [Blume et al., 1991] discussion of conditioning lexicographic probabilities. In this context, [Benavoli et al., 2017] have recently made a similar connection between sets of desirable gambles and sets of lexicographic probabilities. Given our characterisation of the special subclass of lexicographic sets of desirable options (or gambles for that matter)—those sets of desirable gambles that are representable by a single lexicographic probability—we believe that exploring this might provide additional insight into this relation. Finally, it would be of obvious interest to extend our results in Section 5 to lexicographic probability systems defined on infinite spaces.

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