BIVARIATE P-BOXES AND MAXITIVE FUNCTIONS

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Abstract. We give necessary and sufficient conditions for a maxitive function to be the upper probability of a bivariate p-box, in terms of its associated possibility distribution and its focal sets. This allows us to derive conditions in terms of the lower and upper distribution functions of the bivariate p-box. In particular, we prove that only bivariate p-boxes with a non-informative lower or upper distribution function may induce a maxitive function. In addition, we also investigate the extension of Sklar's theorem to this context.

Keywords: Uni- and bivariate p-boxes, maxitive functions, focal sets, comonotonicity, Sklar's theorem.

1. Introduction

In a context of imprecise information the use of probability theory can prove cumbersome, since the available data may not justify the elicitation of a unique probability model. This may be due to the scarcity of the observations, the existence of conflicts between the sources of knowledge, or the presence of ambiguity in the transmission of data. This predicament has given rise to a number of alternative proposals, such as Choquet capacities [3], belief functions [20] or possibility measures [6], that can all be considered as part of what has been called *imprecise probability theory* [24, 25].

One of these models, that was analyzed in detail by Ferson et al. [7] after some earlier work by Williamson and Downs [26], are probability boxes (p-boxes for short). These model the available information by means of a set of cumulative distribution functions that is determined by a lower and an upper bound. This approach has the advantage of being computationally simpler than other imprecise probability models, and as a consequence easier to apply in practical problems [8, 9, 10, 19].

Probability boxes can be given different interpretations, such as confidence bands [2, 17] or the result of interval measurements [27]; our main interpretation here will be that of envelopes of a set of probability distributions [7, 9]. Nevertheless, this will not affect the technical developments that follow.

We focus here on *bivariate* p-boxes, that were recently introduced in [15, 18] as a generalization of p-boxes to the multivariate case. They arise in cases where we have imprecise knowledge about a bivariate distribution function, either because we cannot determine precisely its marginals, because we have uncertain information about the dependence between the underlying variables, or both.

In order to be able to take advantage of all the machinery that has already been developed within imprecise probability theory, it is important to clarify the connection between p-boxes and other imprecise probability models. This was done in the univariate case by Troffaes and Destercke [22], who showed that any univariate p-box is a particular case of a plausibility function. This was complemented in

[23], where it was determined under which conditions a p-box can be regarded as a particular case of a maxitive or a possibility measure. The interest of maxitive models is that they are determined by the restrictions to singletons, being thus more attractive from the computational point of view. They are linked in particular to fuzzy models.

Here we study to which extent the aforementioned results can be established for the bivariate case. This continues our work in [18], where we investigated when the lower/upper probability models associated with a bivariate p-box satisfy the properties of avoiding sure loss and coherence. We shall focus on finite spaces, and shall determine which maxitive functions can be obtained as the upper probability of the bivariate p-box they induce. As we shall show in Section 3, this only holds when one of the bounds of the bivariate p-box is non-informative: either the lower distribution function is constant on 0 or the upper distribution function is constant on 1. In addition to the necessary and sufficient conditions we shall establish in terms of the possibility distribution, we shall determine the focal sets of the maxitive measures that can be attained by means of bivariate p-boxes. Our results are then applied in Section 4 by characterizing maxitive bivariate p-boxes in terms of their lower and upper distribution functions. Section 5 studies the role of Sklar's Theorem in maxitive bivariate p-boxes and shows how to build a maxitive bivariate p-box with fixed marginals. Our paper concludes with some additional discussion in Section 6. In order to ease the reading of the main bulk of the paper, we have relegated to the appendix the proofs of our results.

Two important technical assumptions we shall make throughout are (i) that the possibility distribution is strictly positive, which implies the logical independence between the marginals; and (ii) that the order in the finite spaces $\mathcal{X}, \mathcal{Y}$ is fixed, which also restricts the possibility to represent a maxitive measure by means of a p-box, unlike the developments made in [23]. We shall discuss these assumptions in more detail later on.

2. Preliminary notions

Consider two finite ordered spaces $\mathcal{X} = \{x_1, \dots, x_n\}, \mathcal{Y} = \{y_1, \dots, y_m\}$, where $x_i < x_{i+1}$ and $y_j < y_{j+1}$ for every $i = 1, \dots, n-1$ and every $j = 1, \dots, m-1$, and let $\mathcal{X} \times \mathcal{Y}$ denote their product. Following the notation introduced in [18][1], we shall consider the cumulative sets

$$A_{x_i} = \{x \in \mathcal{X} \mid x \leq x_i\}, \quad A_{y_j} = \{y \in \mathcal{Y} \mid y \leq y_j\}$$

for $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$, as well as the *cumulative rectangles*

$$A_{x_i,y_j} = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x \leq x_i, y \leq y_j\}$$

for $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$.

2.1. **Probabilities and cumulative distribution functions.**

We shall consider probability measures P defined on either $\mathcal{X}, \mathcal{Y}$ or the product space $\mathcal{X} \times \mathcal{Y}$. Since these are finite spaces, any such P is determined by its restriction to singletons (its probability mass function). It is also determined by its cumulative distribution function (cdf for short).

[1]Although there may be some confusion when the spaces $\mathcal{X}, \mathcal{Y}$ are not disjoint, this will not be the case for this paper, so have opted for sticking to this notation for simplicity; otherwise, we may employ the notation $A^{\mathcal{X}}_{x_i}, A^{\mathcal{Y}}_{y_j}$.

Definition 1. *A* (univariate) cdf *on $\mathcal{X}$ is a increasing function $F : \mathcal{X} \to [0,1]$ satisfying $F(x_n) = 1$.*

Any probability P on $\mathcal{X}$ defines a cdf by $F(x_i) = P(A_{x_i})$ for any $x_i \in \mathcal{X}$, and conversely any cdf F on $\mathcal{X}$ defines a probability P_F by $P_F(\{x_1\}) = F(x_1)$ and $P_F(\{x_i\}) = F(x_i) - F(x_{i-1})$, for any $i = 2, \ldots, n$.

In the bivariate case, cdfs are defined as follows.

Definition 2. *A function $F : \mathcal{X} \times \mathcal{Y} \to [0,1]$ is a* bivariate cdf *when it is componentwise increasing, $F(x_n, y_m) = 1$ and it satisfies the* rectangle inequality*:*

$$\Delta_F^{i,j} = F(x_i, y_j) + F(x_{i-1}, y_{j-1}) - F(x_{i-1}, y_j) - F(x_i, y_{j-1}) \geq 0 \tag{1}$$

for any $i = 2, \ldots, n$, $j = 2, \ldots, m$.

As in the univariate case, there is a one-to-one correspondence between probability measures and distribution functions: any probability P on $\mathcal{X} \times \mathcal{Y}$ defines a bivariate cdf by $F(x_i, y_j) = P(A_{x_i, y_j})$ for any $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$. Conversely, any cdf F on $\mathcal{X} \times \mathcal{Y}$ defines a probability P_F by:

$$P_F(\{(x_i, y_j)\}) = \Delta_F^{i,j}, \quad \forall i, j \geq 2, \tag{2}$$

and with $P_F(\{(x_1, y_1)\}) = F(x_1, y_1), P_F(\{(x_1, y_j)\}) = F(x_1, y_j) - F(x_1, y_{j-1})$ and $P_F(\{(x_i, y_1)\}) = F(x_i, y_1) - F(x_{i-1}, y_1)$ for $i = 2, \ldots, n$, $j = 2, \ldots, m$.

A bivariate cdf F defines two marginal cdfs F_X and F_Y by $F_X(x) = F(x, y_m)$ and $F_Y(y) = F(x_n, y)$ for any $x \in \mathcal{X}, y \in \mathcal{Y}$. On the other hand, a bivariate cdf can always be obtained as a function of its marginals, by means of a function called *copula*:

Definition 3. [16] *A* copula *is a function $C : [0,1] \times [0,1] \to [0,1]$ satisfying:*

(a) *$C(0, u) = C(u, 0) = 0$ and $C(u, 1) = C(1, u) = u$ for any $u \in [0,1]$.*
(b) *$C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0$ for any $x_1, y_1, x_2, y_2 \in [0,1]$ such that $x_1 \leq x_2, y_1 \leq y_2$.*

Two of the most important copulas are the lower and upper *Fréchet-Hoeffding bounds*, given by $C_W(x, y) = \max\{x + y - 1, 0\}$ and $C_M(x, y) = \min\{x, y\}$ for any $(x, y) \in [0,1]^2$. Any copula C is bounded by the lower and upper Fréchet-Hoeffding bounds: $C_W \leq C \leq C_M$. Another important example is the *product* copula, given by $C_P(x, y) = x \cdot y$.

Sklar's Theorem shows the connection between copulas and bivariate p-boxes.

Theorem 1. [21]*Given a bivariate cdf $F_{X,Y}$ with marginals F_X and F_Y, there exists a copula C such that*

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)) \tag{3}$$

for any (x, y). Conversely, given two marginal cdfs F_X and F_Y and a copula C, the function $F_{X,Y}$ given by Equation (3) is a bivariate cdf.

When the copula C in the equation above is the product or the lower or upper Fréchet-Hoeffding bounds, we say that the bivariate distribution function models *independence, countermonotonicity* and *comonotonicity*, respectively.

Remark 1. *Comonotonicity (respectively, countermonotonicity) refers to situations where there is an increasing (resp. decreasing) relationship between both components. It can be equivalently expressed in the following manner (see for instance*

[16]*): F is comonotone if and only if the support* $\text{Supp}(P_F)$ *of its associated probability measure* P_F*, given by*

$$\text{Supp}(P_F) = \{(x,y) \in \mathcal{X} \times \mathcal{Y} \mid P(\{(x,y)\}) > 0\},$$

is an increasing subset of $\mathcal{X} \times \mathcal{Y}$[2]*. Thus, if* F *is a comonotone bivariate cdf on* $\mathcal{X} \times \mathcal{Y}$*, there exists* $S = \{(u_1,v_1),\dots,(u_k,v_k)\} \subset \mathcal{X} \times \mathcal{Y}$ *increasing such that* $P_F(\{(u_i,v_i\}) > 0$ *for any* $i = 1,\dots,k$ *and* $P_F(\{(u_1,v_1)\})+\ldots+P_F(\{(u_k,v_k)\}) = 1$. ♦

2.2. **Lower and upper probabilities.** As we mentioned in the introduction, imprecise probabilities [1] is a generic term that is used to refer to most of the mathematical models that serve as an alternative to probability theory in cases of imprecise or ambiguous information. In this paper we shall work with *lower* and *upper* probabilities and some particular cases of interest, such as plausibility or maxitive functions[3].

Definition 4. [24, Sect. 2.7] *Consider a space* Ω*. An* upper probability $\overline{P}$ *with domain* $K \subseteq \mathcal{P}(\Omega)$ *is a map* $\overline{P} : \mathcal{K} \to [0,1]$*. Its conjugate* lower probability $\underline{P} : \mathcal{K}^c \to [0,1]$ *is defined on* $\mathcal{K}^c = \{A^c \mid A \in \mathcal{K}\}$ *by*

$$\underline{P}(A) = 1 - \overline{P}(A^c) \quad \forall A \in \mathcal{K}^c. \tag{4}$$

Lower and upper probabilities may be given a behavioral interpretation: $\underline{P}(A)$ (respectively, $\overline{P}(A)$) may be understood as the supremum betting rate on A (respectively, infimum betting rate against A) (see [11, 24] for more details). Alternatively, they can also be regarded from an *epistemic* point of view, as lower and upper bounds for some unknown probability $P(A)$.

Following this second interpretation, any upper or lower probability determines a set of probabilities, or *credal set*, by $\mathcal{M}(\overline{P}) = \{P \text{ prob.} \mid P(A) \leq \overline{P}(A)\ \forall A \in \mathcal{K}\}$ and $\mathcal{M}(\underline{P}) = \{P \text{ prob.} \mid P(A) \geq \underline{P}(A)\ \forall A \in \mathcal{K}^c\}$. Taking into account Equation (4), $\mathcal{M}(\underline{P}) = \mathcal{M}(\overline{P})$. Because of this, $\underline{P}, \overline{P}$ carry the same probabilistic information, and as a consequence it suffices to work with one of them. In this paper we will focus on upper probabilities.

Definition 5. *An upper probability* $\overline{P}$ *defined on* $\mathcal{K}$ avoids sure loss *when* $\mathcal{M}(\overline{P}) \neq \emptyset$*, and it is* coherent *when* $\overline{P}(A) = \sup\{P(A) \mid P \in \mathcal{M}(\overline{P})\}$ *for any* $A \in \mathcal{K}$*. A lower probability* $\underline{P}$ *avoids sure loss (respectively, it is coherent) when its conjugate* $\overline{P}$ *does.*

Given a coherent upper probability with domain $\mathcal{K}$, we can always extend it to a larger domain such as $\mathcal{P}(\Omega)$ by means of a procedure called *natural extension*:

$$\overline{E}(A) = \sup\{P(A) \mid P \in \mathcal{M}(\overline{P})\}, \quad \forall A \subseteq \Omega. \tag{5}$$

$\overline{E}$ is the greatest coherent upper probability on $\mathcal{P}(\Omega)$ that agrees with $\overline{P}$ on $\mathcal{K}$.

A particular family of coherent lower and upper probabilities is that of belief and plausibility functions, that play a key role in Shafer's Evidence Theory [20].

[2]Given two ordered spaces $\mathcal{X}$ and $\mathcal{Y}$, a subset $S \subseteq \mathcal{X} \times \mathcal{Y}$ is called *increasing* when the elements of S can be listed as $(u_1,v_1),\dots,(u_k,v_k)$ such that $u_i \leq u_{i+1}$ and $v_i \leq v_{i+1}$ for any $i = 1,\dots,k-1$.

[3]Even though Definition 4 does not impose any condition on the domain or on the properties satisfied by the lower/upper probabilities, the ones we shall consider in this paper later on will be defined on $\mathcal{P}(\Omega)$ and will satisfy the usual monotonicity constraints imposed on non-additive measures, due to being a particular case of coherent lower/upper previsions.

Definition 6. *A* probability mass function *is a function* $m : \mathcal{P}(\Omega) \to [0,1]$ *such that* $m(\emptyset) = 0$ *and* $\sum_{A\subseteq\Omega} m(A) = 1$. *It determines a* belief *Bel and a* plausibility *Pl function by:*

$$\mathrm{Bel}(A) = \sum_{B\subseteq A} m(B) \text{ and } \mathrm{Pl}(A) = \sum_{B\cap A\neq\emptyset} m(B).$$

Bel and Pl are coherent lower and upper probabilities, respectively, and they satisfy $\mathrm{Bel}(A) \leq \mathrm{Pl}(A)$ for every set A. The subsets E of Ω such that $m(E) > 0$ are called the *focal sets* of m, and can be used to determine the associated belief and plausibility functions. When these focal sets are nested, that is, when we can establish an order $E_1 \subseteq \ldots \subseteq E_k$, the associated belief and plausibility function become minitive and maxitive functions.

Definition 7. *A* maxitive function $\Pi : \mathcal{P}(\Omega) \to [0,1]$ *is a function satisfying:*

$$\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}, \quad \forall A, B \subseteq \Omega.$$

Its conjugate function N *obtained by* (4) *is called a* minitive function*:*

$$N(A \cap B) = \min\{N(A), N(B)\}, \quad \forall A, B \subseteq \Omega.$$

Since we are dealing with finite spaces, minitive and maxitive functions are also *necessity* and *possibility* measures [6, 28]. In particular, a maxitive measure is determined by its restriction to singletons, called its *possibility distribution*. The possibility distribution also helps to determine the focal sets: if we consider the family of sets $E_\alpha := \{\omega : \Pi(\{\omega\}) \geq \alpha\}$ for $\alpha \in [0,1]$, the finiteness of Ω allows us to conclude that $\{E_\alpha : \alpha \in [0,1]\} = \{E_{\alpha_1}, \ldots, E_{\alpha_n}\}$ for some $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$. Then the focal sets of Π are

$$\Omega = E_{\alpha_1} \supset E_{\alpha_2} \supset \cdots \supset E_{\alpha_n}, \tag{6}$$

and their mass functions are $m(E_{\alpha_1}) = \alpha_1, m(E_{\alpha_i}) = \alpha_i - \alpha_{i-1}$ for $i = 2, ..., n$.

2.3. **Probability boxes.** As mentioned before, lower and upper probabilities can be regarded as lower and upper bounds for a imprecisely known probability measure. A similar approach for distribution functions produces the model of probability boxes.

Definition 8 ([7, 18])**.** *A* (univariate) probability box *(or p-box)* $(\underline{F}_X, \overline{F}_X)$ *defined on* $\mathcal{X}$ *is a pair of ordered cdfs* $\underline{F}_X \leq \overline{F}_X$, *where* $\underline{F}_X, \overline{F}_X : \mathcal{X} \to [0,1]$. *A* bivariate *p*-box $(\underline{F}, \overline{F})$ *defined on* $\mathcal{X}\times\mathcal{Y}$ *is a pair of ordered componentwise increasing functions* $\underline{F}, \overline{F} : \mathcal{X} \times \mathcal{Y} \to [0,1]$ *such that* $\underline{F} \leq \overline{F}$ *and* $\underline{F}(x_n, y_m) = \overline{F}(x_n, y_m) = 1$.

In particular, we shall say that $\underline{F}_X$ (resp., $\underline{F}_Y, \underline{F}$) is *vacuous* when $\underline{F}_X = I_{x_n}$ (resp., $\underline{F}_Y = I_{y_m}, \underline{F} = I_{(x_n,y_m)}$). Note that the lower and upper bounds of a bivariate p-box are not required to satisfy the rectangle inequality given in Equation (1). The reason is that, as shown in [18, Example 1], the lower and upper envelopes of a set of bivariate cdfs may not satisfy such an inequality in general.

As we said before, there is a one-to-one correspondence between probability measures and cdfs, both in the univariate and in the bivariate case. It is also possible to make a connection in the imprecise case, although the correspondence will no longer be one-to-one. To see this, consider an upper probability $\overline{P}$ on $\mathcal{P}(\mathcal{X})$. It defines a univariate p-box $(\underline{F}_X, \overline{F}_X)$ by:

$$\overline{F}_X(x) = \overline{P}(A_x) \text{ and } \underline{F}_X(x) = 1 - \overline{P}(A_x^c)$$

for any $x \in \mathcal{X}$. Conversely, we also have the following correspondence.

Theorem 2 ([22, 24])**.** *Let $(\underline{F}_X, \overline{F}_X)$ be a univariate p-box defined on $\mathcal{X}$. It defines a coherent upper probability on $\mathcal{K}_1 = \{A_x \mid x \in \mathcal{X}\} \cup \{A_x^c \mid x \in \mathcal{X}\}$ by:*

$$\overline{P}(A_x) = \overline{F}_X(x) \text{ and } \overline{P}(A_x^c) = 1 - \underline{F}_X(x) \tag{7}$$

for any $x \in \mathcal{X}$. Its associated coherent lower probability is given by:

$$\underline{P}(A_x) = \underline{F}_X(x) \text{ and } \underline{P}(A_x^c) = 1 - \overline{F}_X(x).$$

Thus, any coherent upper probability on $\mathcal{P}(\mathcal{X})$ whose restriction to $\mathcal{K}_1$ satisfies Equation (7) will be compatible with the p-box $(\underline{F}_X, \overline{F}_X)$.

Let us now turn to the bivariate case. Consider $\mathcal{K}_2 = \{A_{x,y} \mid (x,y) \in \mathcal{X} \times \mathcal{Y}\} \cup \{A_{x,y}^c \mid (x,y) \in \mathcal{X} \times \mathcal{Y}\}$. Any coherent upper probability on $\mathcal{K} \supseteq \mathcal{K}_2$ defines a bivariate p-box $(\underline{F}, \overline{F})$ by:

$$\overline{F}(x,y) = \overline{P}(A_{x,y}) \text{ and } \underline{F}(x,y) = 1 - \overline{P}(A_{x,y}^c) \tag{8}$$

for any $(x,y) \in \mathcal{X} \times \mathcal{Y}$.

Conversely [18, Definition 8], given a bivariate p-box $(\underline{F}, \overline{F})$ on $\mathcal{X} \times \mathcal{Y}$, its associated upper probability $\overline{P}_{(\underline{F},\overline{F})}$ on $\mathcal{K}_2$ is given by:

$$\overline{P}_{(\underline{F},\overline{F})}(A_{x,y}) = \overline{F}(x,y) \text{ and } \overline{P}_{(\underline{F},\overline{F})}(A_{x,y}^c) = 1 - \underline{F}(x,y) \tag{9}$$

for any $(x,y) \in \mathcal{X} \times \mathcal{Y}$, and its lower probability is given by:

$$\underline{P}_{(\underline{F},\overline{F})}(A_{x,y}) = \underline{F}(x,y) \text{ and } \underline{P}_{(\underline{F},\overline{F})}(A_{x,y}^c) = 1 - \overline{F}(x,y).$$

In general, $\overline{P}$ may not be coherent ([18, Example 2]). We will call a bivariate p-box *coherent* when its associated upper probability given by Equation (9) is. In what follows, we shall assume that the upper probability derived from a (uni or bivariate) p-box by means of Equations (7) or (9) is coherent and that it is defined on the power set, using the notion of natural extension in Equation (5).

Given a bivariate p-box $(\underline{F}, \overline{F})$ defined on $\mathcal{X} \times \mathcal{Y}$, its *marginal* univariate p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are given by:

$$\underline{F}_X(x_i) = \underline{F}(x_i, y_m) \text{ and } \overline{F}_X(x_i) = \overline{F}(x_i, y_m) \ \forall i.$$
$$\underline{F}_Y(y_j) = \underline{F}(x_n, y_j) \text{ and } \overline{F}_Y(y_j) = \overline{F}(x_n, y_j) \ \forall j.$$

Similarly, a coherent upper probability $\overline{P}$ on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ defines marginal upper probabilities $\overline{P}_X, \overline{P}_Y$ on $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$, respectively, by:

$$\overline{P}_X(A) = \overline{P}(A \times \mathcal{Y}) \text{ and } \overline{P}_Y(B) = \overline{P}(\mathcal{X} \times B),$$

for any $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.

In this paper, we study when the upper probability induced by a bivariate p-box is maxitive. This problem was solved in the univariate case in [23]:

Theorem 3 ([23, Corollary 13])**.** *Let $(\underline{F}_X, \overline{F}_X)$ be a univariate p-box defined on $\mathcal{X}$, and denote by $\overline{P}$ its associated upper probability given by Equation (7) and extended to $\mathcal{P}(\mathcal{X})$ using the natural extension. Then, $\overline{P}$ is maxitive if and only if one of the following conditions holds:*

(a) $\underline{F}_X$ is 0-1 valued.
(b) $\overline{F}_X$ is 0-1 valued.

3. Maxitive functions induced by bivariate p-boxes

In this section we consider a maxitive function defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ and we investigate the conditions it must satisfy in order to be the upper probability of the bivariate p-box it determines. One difference with the results established in [23, Section 5] for the univariate case is that we shall assume that the order in the spaces $\mathcal{X}, \mathcal{Y}$ is fixed, and cannot be adapted to the values of the maxitive measure.

By (8), a maxitive function Π defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, induces a p-box $(\underline{F}, \overline{F})$ by

$$\overline{F}(x,y) = \Pi(A_{x,y}), \quad \underline{F}(x,y) = N(A_{x,y}) = 1 - \Pi(A_{x,y}^c) \ \ \forall (x,y) \in \mathcal{X} \times \mathcal{Y}. \tag{10}$$

From Equation (9), $(\underline{F}, \overline{F})$ defines an upper probability $\overline{P}_{(\underline{F},\overline{F})}$. Therefore, our problem is to investigate under which conditions Π coincides with the natural extension of $\overline{P}_{(\underline{F},\overline{F})}$ to $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, or equivalently, when the diagram on Figure 1 commutes.

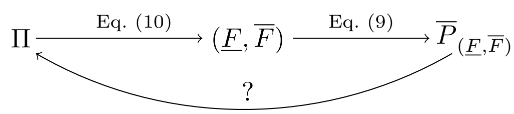

Figure 1. Description of the problem.

We begin by showing that indeed not every maxitive function on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ can be obtained as the upper probability of a bivariate p-box.

Example 1. *Consider $\mathcal{X} = \{x_1, x_2\}, \mathcal{Y} = \{y_1, y_2\}$ and let Π be the maxitive measure determined by the following possibility distribution:*

$$\pi(x_1, y_1) = 1 = \pi(x_2, y_2), \quad \pi(x_1, y_2) = 0.5 = \pi(x_2, y_1).$$

Assume that $(\underline{F}, \overline{F})$ is a bivariate p-box such that $\overline{P}_{(\underline{F},\overline{F})} = \Pi$. In particular, we should have $\overline{P}_{(\underline{F},\overline{F})}(\{x_1, y_1\}) = 1 = \overline{P}_{(\underline{F},\overline{F})}(\{x_2, y_2\})$, meaning that there must be P_1, P_2 in $\mathcal{M}(\overline{P}_{(\underline{F},\overline{F})})$ such that $P_1(\{x_1, y_1\}) = 1 = P_2(\{x_2, y_2\})$. But this implies that the p-box $(\underline{F}, \overline{F})$ must be vacuous, because

$$\underline{F}(x_1, y_1) \leq \underline{F}(x_2, y_1) = \underline{F}(x_1, y_2) \leq F_{P_2}(x_1, y_2) = F_{P_2}(x_2, y_1) = 0,$$

and $\overline{F}(x_1, y_1) \geq F_{P_1}(x_1, y_1) = 1$. As a consequence, taking the cdf F such that $F(x_i, y_1) = 0$ and $F(x_i, y_2) = 1$ for $i = 1, 2$, F belongs to $(\underline{F}, \overline{F})$. Hence:

$$\overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_2)\}) \geq P_F(\{(x_1, y_2)\}) = 1 > \Pi(\{(x_1, y_2)\}),$$

a contradiction. Therefore $\overline{P}_{(\underline{F},\overline{F})}$ does not coincide with Π.

Note that in this case the upper probability of the vacuous p-box $(\underline{F}, \overline{F})$ that is determined by Equation (10) is maxitive: we have $\overline{P}_{(\underline{F},\overline{F})}(A) = 1$ for every non-empty A. Thus, the maxitivity of $\overline{P}_{(\underline{F},\overline{F})}$ does not imply its equality with Π. ♦

In this example, the upper probability of the p-box determined by Π is maxitive but it does not coincide with Π. However, in general $\overline{P}_{(\underline{F},\overline{F})}$ may not even be maxitive:

Example 2. *Consider $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ and the maxitive function Π whose focal sets are $E_1 = \{(x_2, y_1)\}$, $E_2 = \mathcal{X} \times \mathcal{Y}$, each one with a mass of 0.5.*

Its associated bivariate p-box $(\underline{F}, \overline{F})$ is given by

	(x_1, y_1)	(x_1, y_2)	(x_2, y_1)	(x_2, y_2)
$\underline{F}$	0	0	0.5	1
$\overline{F}$	0.5	0.5	1	1

From this it follows that $\overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1)\}) = 0.5 = \overline{P}_{(\underline{F},\overline{F})}(\{(x_2, y_2)\})$. The first equality follows from Equation (9). To see the second, note that

$$\overline{P}_{(\underline{F},\overline{F})}(\{(x_2, y_2)\}) = 1 - \underline{P}_{(\underline{F},\overline{F})}(\{(x_2, y_2)\}^c) \leq 1 - \underline{P}_{(\underline{F},\overline{F})}(A_{(x_2,y_1)}) = 0.5,$$

and that the distribution function $\underline{F} \in (\underline{F}, \overline{F})$ satisfies $0.5 = P_{\underline{F}}(\{(x_2, y_2)\}) \leq \overline{P}_{(\underline{F},\overline{F})}(\{(x_2, y_2)\})$ (that $\underline{F}$ is indeed a distribution function follows from Lemma 1 below).

Now, if we consider the distribution function $F_1 \in (\underline{F}, \overline{F})$ given by $F_1(x_1, y_1) = F_1(x_1, y_2) = F_1(x_2, y_1) = 0.5, F_1(x_2, y_2) = 1$, we have that

$$1 = P_{F_1}(\{(x_1, y_1), (x_2, y_2)\}) \leq \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1), (x_2, y_2)\}).$$

Thus, $\overline{P}_{(\underline{F},\overline{F})}$ is not maxitive because:

$$0.5 = \max\{\overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1)\}), \overline{P}_{(\underline{F},\overline{F})}(\{(x_2, y_2)\})\} \neq 1 = \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1), (x_2, y_2)\}). \blacklozenge$$

Remark 2. *The upper probability $\overline{P}_{(\underline{F},\overline{F})}$ is the natural extension of the restriction of Π to $\mathcal{K}_2 = \{A_{x,y} \mid (x, y) \in \mathcal{X} \times \mathcal{Y}\} \cup \{A^c_{x,y} \mid (x, y) \in \mathcal{X} \times \mathcal{Y}\}$ and as a consequence we always have $\overline{P}_{(\underline{F},\overline{F})} \geq \Pi$. However, the equality need not hold in general, as Example 2 shows. On the other hand, the correspondence $\Pi \to \overline{P}_{(\underline{F},\overline{F})}$ considered in Figure 1 is in general many-to-one, in the sense that we may have $\Pi_1 \neq \Pi_2$ giving rise to the same bivariate p-box $(\underline{F}, \overline{F})$ (and as a consequence also to the same $\overline{P}_{(\underline{F},\overline{F})}$): note for instance that $\underline{F}$ is vacuous and $\overline{F} = 1$ as soon as $\pi(x_1, y_1) = 1 = \pi(x_n, y_m)$, irrespective of the other values of the possibility distribution.* $\blacklozenge$

In what follows, we characterize those maxitive functions Π satisfying $\Pi = \overline{P}_{(\underline{F},\overline{F})}$, where $(\underline{F}, \overline{F})$ is the bivariate p-box determined by Equation (10). We shall establish necessary and sufficient conditions for this equality, in terms of the focal elements of Π and its possibility distribution π. We shall assume throughout that the possibility of any pair (x_i, y_j) is strictly positive:

$$\pi(x_i, y_j) > 0 \ \forall i, j. \tag{11}$$

This assumption means that all elements of $\mathcal{X} \times \mathcal{Y}$ are deemed possible, and implies the logical independence between the variables X, Y. In particular, it implies that $\overline{F}(x_1, y_1) = \pi(x_1, y_1) > 0$ and as a consequence $\overline{F}(x_i, y_j) > 0$ for any i, j. This assumption will have some implications on the results we shall establish, as we shall see in Section 3.3.

We begin by establishing some necessary conditions. One helpful property will be the following:

Lemma 1. *Let* Π *be a maxitive function and let* $(\underline{F}, \overline{F})$ *be the bivariate p-box determined by Equation* (10). *Then* $\underline{F}(x_i, y_j) = \min\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\}$ *for any* i, j *and as a consequence* $\underline{F}$ *is a bivariate distribution function.*

This implies that $\underline{F}$ is a bivariate distribution function that is the comonotonic combination of its marginals, and therefore we can make use of the properties of comonotone cdfs explained in Remark 1.

This lemma allows us to establish the following result.

Proposition 1. *Let* Π *be a maxitive function and denote by* $(\underline{F}, \overline{F})$ *the bivariate p-box it induces. If* $\Pi = \overline{P}_{(\underline{F}, \overline{F})}$, *then one of the following conditions must hold:*

(a) $(x_1, y_1) \in E$ *for any focal set* E.
(b) $(x_n, y_m) \in E$ *for any focal set* E.

This result implies that one of the bounds of the p-box associated with the bivariate maxitive function must be non-informative.

Corollary 1. *Let* Π *be a maxitive function and denote by* $(\underline{F}, \overline{F})$ *the bivariate p-box it induces. If* $\Pi = \overline{P}_{(\underline{F}, \overline{F})}$, *then either* $\underline{F}$ *is vacuous or* $\overline{F}$ *is constantly 1.*

Of course, condition (a) from Proposition 1 corresponds to the case of $\overline{F}$ constantly 1 and condition (b) corresponds to the case of a vacuous $\underline{F}$. This property motivates the following definition, which allows us to split our study in two simpler cases.

Definition 9. *A bivariate p-box* $(\underline{F}, \overline{F})$ *is called of* type 1 *when* $\underline{F}$ *is vacuous, and of* type 2 *when* $\underline{F}$ *is non-vacuous and* $\overline{F}$ *is constant on* 1.

Thus, a maxitive function Π for which the diagram in Figure 1 commutes is either of type 1 or of type 2. In what follows, we shall investigate in detail each of these two cases, establishing necessary and sufficient conditions for the equality $\Pi = \overline{P}_{(\underline{F}, \overline{F})}$ in terms of the possibility distribution of Π and in terms of its focal sets.

3.1. **Type-1 bivariate p-boxes.** We begin by considering maxitive function whose associated bivariate p-box is of type 1, meaning that $\pi(x_n, y_m) = 1$ and therefore $\underline{F}$ is vacuous. The following proposition establishes some helpful properties of these p-boxes.

Proposition 2. *Let* $(\underline{F}, \overline{F})$ *be a type 1 bivariate p-box. Then:*

(a) $\overline{P}_{(\underline{F}, \overline{F})}(x_i, y_j) = \overline{F}(x_i, y_j)$ *for any* i, j.

If $\overline{P}_{(\underline{F}, \overline{F})}$ *coincides with a maxitive function* Π, *then:*

(b) *Either* $\overline{F}_X = 1$ *or* $\overline{F}_Y = 1$.
(c) $\overline{F}(x_i, y_j) = \max\{\overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\}$ *for any* i, j.
(d) $\pi(x, y) = \overline{F}(x, y)$ *for any* (x, y) *and* π *is component-wise increasing.*

Thus, if Π is a maxitive function such that the diagram on Figure 1 commutes and its associated p-box $(\underline{F}, \overline{F})$ is of type 1, then the upper distribution function $\overline{F}$ coincides with the possibility distribution π associated with Π. From Equation (6), this means that if $\overline{F}$ takes the values $\gamma_1 < \ldots < \gamma_k = 1$, then Π has k focal sets, given by:

$$E_i = \{(x, y) : \overline{F}(x, y) \geq \gamma_i\}, \tag{12}$$

and their masses are $m(E_i) = \gamma_i - \gamma_{i-1}$ for any $i = 1, \dots, k$ (we assume $\gamma_0 = 0$). Obviously, $E_i \supseteq E_{i+1}$ for any $i = 1, \dots, k-1$ and $E_0 = \mathcal{X} \times \mathcal{Y}$.

We can give a geometrical interpretation to, and the explicit form of these focal sets taking into account the following result.

Corollary 2. *Let* Π *be a maxitive function, and* $(\underline{F}, \overline{F})$ *the bivariate p-box it induces by Equation* (10)*. Assume that* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$ *and that* $(\underline{F}, \overline{F})$ *is of type 1. Then, for any* (x_i, y_j)*, either*

$$\overline{F}(x_1, y_j) = \ldots = \overline{F}(x_{i-1}, y_j) = \overline{F}(x_i, y_j) \text{ or}$$
$$\overline{F}(x_i, y_1) = \ldots = \overline{F}(x_i, y_{j-1}) = \overline{F}(x_i, y_j).$$

As a consequence, if we use the notation $\overline{F}(x_{n+1}, y_j) = 1$ *and* $\overline{F}(x_i, y_{m+1}) = 1$ *for any* i, j*, then the set*

$$S = \{(x_i, y_j) \mid \overline{F}(x_i, y_j) < \min\{\overline{F}(x_{i+1}, y_j), \overline{F}(x_i, y_{j+1})\}\}. \tag{13}$$

is increasing.

Thus, the set S of Equation (13) can be expressed as

$$S = \{(x_{i_1}, y_{j_1}), \dots, (x_{i_k}, y_{j_k})\}, \tag{14}$$

where $x_{i_1} \leq \ldots \leq x_{i_k}$ and $y_{j_1} \leq \ldots \leq y_{j_k}$. Using this notation we can establish the following result that gives the form of the focal sets.

Proposition 3. *Let* Π *be a maxitive function, and* $(\underline{F}, \overline{F})$ *the bivariate p-box it induces by Equation* (10)*. Assume that* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$ *and that* $(\underline{F}, \overline{F})$ *is of type 1. Consider the set* S *defined in Equation* (13) *and expressed as in Equation* (14)*. Then, the focal sets of* Π *are:*

$$E_0 = \mathcal{X} \times \mathcal{Y}, \; E_l = A^c_{x_{i_l}, y_{j_l}}, \quad \forall l = 1, \dots, k,$$

with $m(E_l) = \overline{F}(x_{i_{l+1}}, y_{j_{l+1}}) - \overline{F}(x_{i_l}, y_{j_l})$*, where* $\overline{F}(x_{i_{k+1}}, y_{j_{k+1}}) = 1$*, and* $m(\mathcal{X} \times \mathcal{Y}) = \overline{F}(x_1, y_1)$*.*

In Proposition 2, we have established a number of necessary conditions for a maxitive function Π to be the upper probability of a bivariate p-box of type 1. Next we show that these conditions are also sufficient.

Proposition 4. *Consider a maxitive function* Π *whose possibility distribution* π *is component-wise increasing, satisfies* $\pi(x, y) = \max\{\pi(x, y_1), \pi(x_1, y)\}$ *and* $\pi(x_n, y_m) = 1$*. Then* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$*, where* $(\underline{F}, \overline{F})$ *is given by Equation* (10)*.*

When π is component wise increasing, Equation (10) implies that:

$$\overline{F}(x, y) = \Pi(A_{x,y}) = \pi(x, y).$$

Therefore, $1 = \overline{F}(x_n, y_m) = \max\{\pi(x_n, y_1), \pi(x_1, y_m)\}$, whence either $\pi(x_n, y_1) = 1$ or $\pi(x_1, y_m) = 1$. In the first case, this means that $\overline{F}_Y(y_1) = \pi(x_n, y_1) = 1$, and therefore $\overline{F}_Y$ is constantly 1. Similarly, in the second case $\overline{F}_X$ is constantly 1.

The following example illustrates the previous results:

Example 3. *Consider* $\mathcal{X} = \{x_1, x_2, x_3\}$ *and* $\mathcal{Y} = \{y_1, y_2, y_3\}$ *and the maxitive function* Π *with possibility distribution:*

	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)
π	0.1	0.5	0.7	0.8	0.8	0.8	1	1	1

This maxitive function induces the following bivariate p-box:

	(x_1,y_1)	(x_1,y_2)	(x_1,y_3)	(x_2,y_1)	(x_2,y_2)	(x_2,y_3)	(x_3,y_1)	(x_3,y_2)	(x_3,y_3)
$\underline{F}$	0	0	0	0	0	0	0	0	1
$\overline{F}$	0.1	0.5	0.7	0.8	0.8	0.8	1	1	1

Since π *satisfies the conditions in Proposition 4, we deduce that* $\overline{P}_{(\underline{F},\overline{F})} = \Pi$.

Using the notation of Equation (13)*, the set* S *is given by:*

$$S = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_3)\}.$$

Therefore, according to Proposition 3, the focal sets of $\overline{P}_{(\underline{F},\overline{F})}$ *are:*

$$\begin{aligned} E_0 &= \mathcal{X}\times\mathcal{Y}, \quad m(E_0) = 0.1. \\ E_1 &= A^c_{x_1,y_1}, \quad m(E_1) = 0.4. \\ E_2 &= A^c_{x_1,y_2}, \quad m(E_2) = 0.2. \\ E_3 &= A^c_{x_1,y_3}, \quad m(E_3) = 0.1. \\ E_4 &= A^c_{x_2,y_3}, \quad m(E_4) = 0.2 \end{aligned}$$

They are depicted in Figure 2.⧫

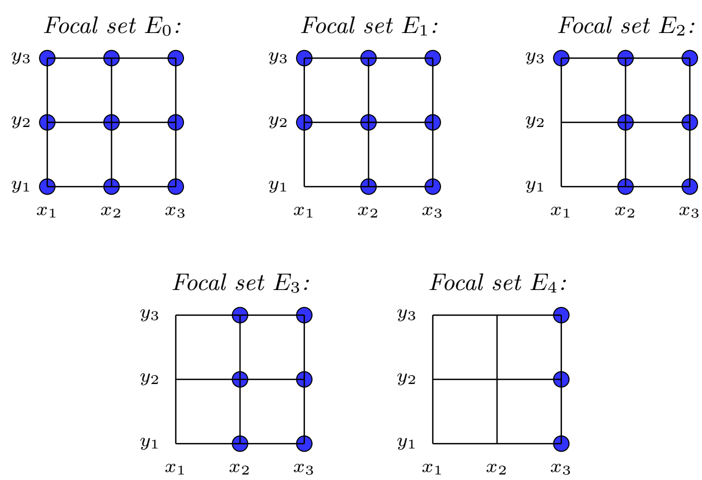

FIGURE 2. Description of the focal sets of the maxitive function of Example 3.

3.2. **Type-2 bivariate p-boxes.** We focus next on bivariate p-boxes of type 2, which are those $(\underline{F},\overline{F})$ such that $\overline{F}$ is constant on 1 and $\underline{F}$ is non-vacuous[4]. From Proposition 1 and Corollary 1, they correspond to maxitive functions such that $\pi(x_1,y_1) = 1$.

We begin by determining the focal sets of the maxitive functions inducing a bivariate p-box of type 2. In order to do this, recall that, by Lemma 1, the lower distribution function $\underline{F}$ is actually a comonotone bivariate cdf, and therefore the support of $\underline{F}$ is an increasing subset of $\mathcal{X}\times\mathcal{Y}$. From Remark 1, there is a set

[4]Although in this paper we have opted for making type 1 and type 2 exclusive categories, not much changes if we define the latter as those satisfying $\overline{F} = 1$: the only difference is that we should then include the case of vacuous $\underline{F}$ and $\overline{F} = 1$, that corresponds simply to $m(\mathcal{X}\times\mathcal{Y}) = 1$.

$S = \{(u_1, v_1), \ldots, (u_k, v_k)\} \subset \mathcal{X} \times \mathcal{Y}$ such that $u_i \leq u_{i+1}, v_i \leq v_{i+1}$ for any $i = 1, \ldots, k-1$, $P_{\underline{F}}(\{(u_i, v_i)\}) > 0$ and $\sum_{i=1}^k P_{\underline{F}}(\{(u_i, v_i\}) = 1$. This set allows us to determine the focal sets.

Proposition 5. *Let* Π *be a maxitive function, and* $(\underline{F}, \overline{F})$ *the bivariate p-box it induces by Equation* (10). *Assume that* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$ *and that* $(\underline{F}, \overline{F})$ *is of type 2. Then given the support* S *of* $\underline{F}$ *as denoted above, the focal sets of* $\overline{P}_{(\underline{F},\overline{F})}$ *are*

$$E_i = A_{u_i,v_i}, \quad m(E_i) = \underline{F}(u_i, v_i) - \underline{F}(u_{i-1}, v_{i-1})$$

for any $l = 1, \ldots, k$*, where* $\underline{F}(u_0, v_0) := 0$.

Next, we characterize the conditions a maxitive function inducing a type 2 bivariate p-box must satisfy in order to make the diagram of Figure 1 commute.

Theorem 4. *Let* Π *be a maxitive function on* $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$*, and denote by* π *its associated possibility distribution. Then,* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$ *and* $(\underline{F}, \overline{F})$ *is of type 2 if and only if* π *is component-wise decreasing,* $\pi(x, y) = \min\{\pi(x_1, y), \pi(x, y_1)\}$ *for any* (x, y) *and* $\pi(x_1, y_1) = 1$.

These two results are illustrated in the following example.

Example 4. *Consider the maxitive function* Π *with the following possibility distribution:*

	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)
π	1	0.8	0.4	0.7	0.7	0.4	0.1	0.1	0.1

The bivariate p-box $(\underline{F}, \overline{F})$ *it induces is given by:*

	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)
$\underline{F}$	0.2	0.3	0.3	0.2	0.6	0.9	0.2	0.6	1
$\overline{F}$	1	1	1	1	1	1	1	1	1

Since π *satisfies the conditions of Theorem 4, we deduce that* $\overline{P}_{(\underline{F},\overline{F})} = \Pi$. *On the other hand, the increasing support set* S *of* $P_{\underline{F}}$ *is:*

$$S = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_3)\}.$$

Applying Proposition 5, the focal sets of Π *are:*

$$E_1 = A_{x_1,y_1}, \quad m(E_1) = \underline{F}(x_1, y_1) = 0.2.$$
$$E_2 = A_{x_1,y_2}, \quad m(E_2) = \underline{F}(x_1, y_2) - \underline{F}(x_1, y_1) = 0.1.$$
$$E_3 = A_{x_2,y_2}, \quad m(E_3) = \underline{F}(x_2, y_2) - \underline{F}(x_1, y_2) = 0.3.$$
$$E_4 = A_{x_2,y_3}, \quad m(E_4) = \underline{F}(x_2, y_3) - \underline{F}(x_2, y_2) = 0.3.$$
$$E_5 = A_{x_3,y_3}, \quad m(E_5) = \underline{F}(x_3, y_3) - \underline{F}(x_2, y_3) = 0.1.$$

We have graphically depicted these focal sets in Figure 3. ♦

3.3. **Discussion.** The following theorem summarizes our findings in the previous subsections.

Theorem 5. *Let* Π *be a maxitive function and consider the bivariate p-box* $(\underline{F}, \overline{F})$ *it induces by Equation* (10). *Then* $\Pi = \overline{P}_{(\underline{F},\overline{F})}$ *if and only if one of the following conditions holds:*

(a) π *is component-wise increasing,* $\pi(x, y) = \max\{\pi(x, y_1), \pi(x_1, y)\}$ *for any* (x, y) *and* $\pi(x_n, y_m) = 1$.
(b) π *is component-wise decreasing,* $\pi(x, y) = \min\{\pi(x_1, y), \pi(x, y_1)\}$ *for any* (x, y) *and* $\pi(x_1, y_1) = 1$.

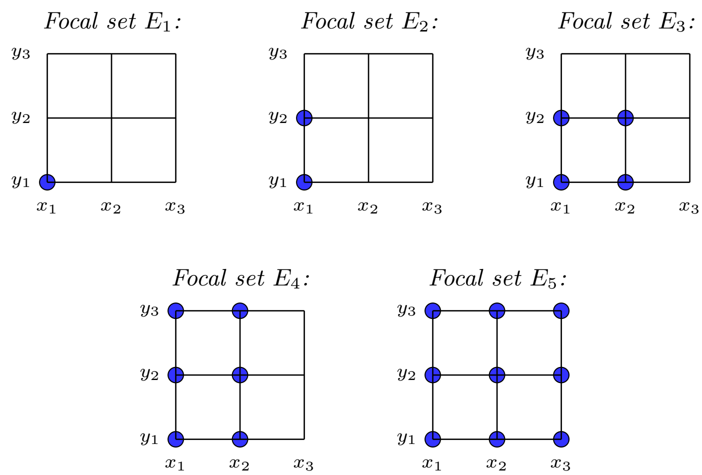

FIGURE 3. Description of the focal sets of the maxitive function Π of Example 4.

This shows that the maxitive functions that can be obtained as upper probabilities of bivariate p-boxes have quite a particular structure, because they are determined by $n+m-1$ values only: $\pi(x,y_1),\pi(x_1,y)$, for $(x,y)\in\mathcal{X}\times\mathcal{Y}$. Although this implies that maxitive functions making the diagram of Figure 1 commute are computationally easier to handle than general maxitive functions on $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$, it also implies that these models have limited expressivity.

Moreover, we have also established the shape of the focal sets of these maxitive functions:

- In the case of bivariate p-boxes of type 2, they are of the form $E_i = A_{u_i,v_i}$ for $i=1,\ldots,k$, where $\{(u_1,v_1),\ldots,(u_k,v_k)\}$ is the increasing support set of the cdf $\underline{F}$.
- When the bivariate p-box is of type 1, they are of the form $E_0 = \mathcal{X}\times\mathcal{Y}$, $E_l = A^c_{x_{i_l},y_{j_l}}, \forall l=1,\ldots,k$, where $\{(x_{i_1},y_{j_1}),\ldots,(x_{i_k},y_{j_k})\} = \{(x_i,y_j) \mid \overline{F}(x_i,y_j) < \min\{\overline{F}(x_{i+1},y_j),\overline{F}(x_i,y_{j+1})\}\}$.

At the beginning of this section we have mentioned one restriction that we are imposing on the possibility distribution: that the possibility of any pair (x_i,y_j) in $\mathcal{X}\times\mathcal{Y}$ is strictly positive. It is important to remark that this restriction is instrumental in our results, in the sense that there exist maxitive functions making the diagram of Figure 1 commute whose possibility distribution does not satisfy this condition. Their associated bivariate p-boxes need not satisfy the conditions we have summarized in Theorem 5, as next example shows.

Example 5. *Consider $\mathcal{X}=\{x_1,x_2\}$ and $\mathcal{Y}=\{y_1,y_2\}$ and the possibility distribution given by:*

	(x_1,y_1)	(x_1,y_2)	(x_2,y_1)	(x_2,y_2)
π	0	1	0.5	0.5

Its associated bivariate p-box $(\underline{F}, \overline{F})$ *is given by:*

	(x_1, y_1)	(x_1, y_2)	(x_2, y_1)	(x_2, y_2)
$\underline{F}$	0	0.5	0	1
$\overline{F}$	0	1	0.5	1

It is easy to check that $\overline{P}_{(\underline{F},\overline{F})}$ *is a maxitive function whose associated focal sets are* $\{(x_1, y_2)\}$ *and* $\{(x_1, y_2), (x_2, y_1), (x_2, y_2)\}$*, each with mass* 0.5*, whence* $\overline{P}_{(\underline{F},\overline{F})} = \Pi$*. However,* π *is neither component-wise increasing nor decreasing, and* $(\underline{F}, \overline{F})$ *does not satisfy the necessary condition of Proposition 1, since* $\overline{F}$ *is not constant on 1 and* $\underline{F}$ *is not vacuous.*

The key for this result is that $\pi(x_1, y_1) = 0$*, and therefore our results from this section cannot be applied.* ♦

4. Bivariate p-boxes inducing a maxitive function

In our previous section we determined which conditions a maxitive function must satisfy in order to make the diagram of Figure 1 commute, or equivalently, the conditions under which a maxitive function coincides with the upper probability of the bivariate p-box it induces.

In this section we start with a bivariate p-box and investigate when the upper probability it determines by Equation (9) is maxitive. For this aim we only need to take into account that if $\overline{P}_{(\underline{F},\overline{F})}$ is maxitive, then it is uniquely determined by the possibility distribution π that is its restriction to events, so that $\pi(x, y) = \overline{P}_{(\underline{F},\overline{F})}(\{(x, y)\})$. Then $(\underline{F}, \overline{F})$ is induced by the maxitive function Π associated with π:

$$\overline{F}(x, y) = \overline{P}_{(\underline{F},\overline{F})}(A_{x,y}) = \Pi(A_{x,y}) \text{ and } \underline{F}(x, y) = 1 - \overline{P}_{(\underline{F},\overline{F})}(A^c_{x,y}) = 1 - \Pi(A^c_{x,y}),$$

meaning that $(\underline{F}, \overline{F})$ is determined by Π by means of Equation (10). This, together with Equation (11), motivates the following assumption:

$$\overline{P}_{(\underline{F},\overline{F})}(\{(x_i, y_j)\}) > 0 \ \forall i, j. \tag{15}$$

Equation (15) means in particular that there is at least one precise model P that is compatible with our p-box $(\underline{F}, \overline{F})$, in the sense that $\underline{F} \leq F_P \leq \overline{F}$, and that gives positive probability to the pair (x_i, y_j). In other words, any singleton is a focal element of at least one of the compatible precise models.

Thus, Proposition 1 and Corollary 1 imply that a bivariate p-box whose upper probability is maxitive is either of type 1 (meaning that $\underline{F}$ is vacuous) or type 2 (meaning that $\underline{F}$ is non-vacuous and $\overline{F} = 1$).

This motivates us to introduce the following notation.

Definition 10. *A bivariate p-box* $(\underline{F}, \overline{F})$ *is called* maxitive *when its upper probability* $\overline{P}_{(\underline{F},\overline{F})}$ *is. If a maxitive bivariate p-box is of type 1 (respectively, type 2), it will be called* type 1 *(respectively,* type 2*)* maxitive *p-box.*

In the case of type 1 bivariate p-boxes, Propositions 2 and 4 allow us to establish the following:

Theorem 6. *Let* $(\underline{F}, \overline{F})$ *be a type 1 p-box. It is maxitive if and only if for any* i, j $\overline{F}(x_i, y_j) = \max\{\overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\}$ *and either* $\overline{F}_X = 1$ *or* $\overline{F}_Y = 1$*.*

Moreover, we deduce from Proposition 3 that the focal sets of $\overline{P}_{(\underline{F},\overline{F})}$, when $(\underline{F},\overline{F})$ is a type 1 maxitive p-box, are an increasing sequence of nested complementaries of cumulative rectangles.

Similarly, we can characterize type 2 maxitive p-boxes.

Theorem 7. *Let $(\underline{F},\overline{F})$ be a type 2 p-box. It is maxitive if and only for every $(x,y)\in\mathcal{X}\times\mathcal{Y}$ it holds that $\underline{F}(x,y)=\min\{\underline{F}_X(x),\underline{F}_Y(y)\}$.*

Moreover, in the case of type 2 maxitive bivariate p-boxes the focal sets are a nested family of rectangles, by Proposition 5.

5. Maxitive bivariate p-boxes as combination of their marginals

As we mentioned in Section 2.1, Sklar's Theorem tells us that any bivariate cdf can be expressed as a copula of its marginal distribution functions. When we have imprecision about either the marginals or the copula that links them, we can obtain an imprecise version of the theorem [15, Theorem 2]. Our aim in this section is to investigate this result in the particular case of maxitive bivariate p-boxes.

5.1. Imprecise Sklar's Theorem.

Let us begin by recalling the extension of Sklar's theorem to the imprecise case.

Theorem 8 (Imprecise Sklar's Theorem)**.** [15, Theorem 2] *Consider two marginal p-boxes $(\underline{F}_X,\overline{F}_X)$ and $(\underline{F}_Y,\overline{F}_Y)$. If $\mathcal{C}$ denotes a non-empty set of copulas, we can define a bivariate p-box $(\underline{F},\overline{F})$ by:*

$$\underline{F}(x,y)=\inf_{C\in\mathcal{C}} C(\underline{F}_X(x),\underline{F}_Y(y)),\quad \overline{F}(x,y)=\sup_{C\in\mathcal{C}} C(\overline{F}_X(x),\overline{F}_Y(y)). \tag{16}$$

Furthermore, $\overline{P}_{(\underline{F},\overline{F})}$ is a coherent upper probability.

However, the converse implication of Theorem 1 does not hold in general in the bivariate case: not every bivariate p-box $(\underline{F},\overline{F})$ associated with a coherent $\overline{P}_{(\underline{F},\overline{F})}$ can be expressed in terms of its marginal p-boxes by means of a set of copulas [15, Example 1]. In this section, we shall investigate whether the converse holds in the particular case where $\overline{P}_{(\underline{F},\overline{F})}$ is not only coherent but maxitive. Our first result is a simple consequence of Theorem 7.

Corollary 3. *Let $(\underline{F},\overline{F})$ be a maxitive bivariate p-box of type-2. Then, the converse implication on the Imprecise Sklar's Theorem holds using the upper Fréchet-Hoeffding copula C_M:*

$$\underline{F}(x,y)=\min\{\underline{F}_X(x),\underline{F}_Y(y)\} \text{ and } \overline{F}(x,y)=\min\{\overline{F}_X(x),\overline{F}_Y(y)\} \tag{17}$$

for any $x\in\mathcal{X},y\in\mathcal{Y}$.

Although we may think that a similar result holds for type-1 bivariate p-boxes, our next example shows that this is not the case.

Example 6. *Consider the bivariate p-box $(\underline{F},\overline{F})$ defined on $\mathcal{X}\times\mathcal{Y}=\{x_1,x_2\}\times\{y_1,y_2\}$ by:*

	(x_1,y_1)	(x_1,y_2)	(x_2,y_1)	(x_2,y_2)
$\underline{F}$	0	0	0	1
$\overline{F}$	0.1	0.7	1	1

From Theorem 6, $(\underline{F},\overline{F})$ is a type 1 maxitive bivariate p-box. Assume there is a set of copulas allowing to express $(\underline{F},\overline{F})$ as in Equation (16)*. Then we should have*

$\overline{F}(x_1, y_1) = \sup_{C\in\mathcal{C}} C(\overline{F}_X(x_1), \overline{F}_Y(y_1)) = \sup_{C\in\mathcal{C}} C(0.7, 1) = 0.7$, *contradicting the assessment* $\overline{F}(x_1, y_1) = 0.1$. ♦

In fact, it is possible to characterize those maxitive bivariate p-boxes for which the converse of Sklar's theorem holds.

Proposition 6. *Let* $(\underline{F}, \overline{F})$ *be a maxitive bivariate p-box of type-1. Then,* $(\underline{F}, \overline{F})$ *can be expressed as in Equation* (16) *if and only if the following holds:*

- *If* $\overline{F}_X = 1$*, then* $\overline{F}(x_1, y_j) = \ldots = \overline{F}(x_n, y_j) = \overline{F}_Y(y_j)$ *for any* j.
- *If* $\overline{F}_Y = 1$*, then* $\overline{F}(x_i, y_1) = \ldots = \overline{F}(x_i, y_m) = \overline{F}_X(x_i)$ *for any* i.

5.2. **Independence and comonotonicity for maxitive bivariate p-boxes.** In imprecise probability theory there are several extensions of the notion of independence [4, 5, 12]. The cases of extreme dependence modeled by comonotonicity and countermonotonicity were studied in [13] for lower probabilities and bivariate p-boxes. In this section we investigate independence and comonotonicity for maxitive bivariate p-boxes.

Among the several extensions of independence to the imprecise case, we shall consider here the notion of strong independence, that gives rise to the so-called *strong product*.

Definition 11. *Given two lower probabilities* $\underline{P}_X$ *and* $\underline{P}_Y$ *defined on* $\mathcal{P}(\mathcal{X})$ *and* $\mathcal{P}(\mathcal{Y})$*, respectively, their strong product,* $\underline{P}_X \boxtimes \underline{P}_Y$*, is defined by:*

$$\underline{P}_X \boxtimes \underline{P}_Y = \inf\{P_X \times P_Y \mid P_X \in \mathcal{M}(\underline{P}_Y), P_X \in \mathcal{M}(\underline{P}_Y)\}.$$

It can be connected to the imprecise version of Sklar's Theorem as follows.

Proposition 7. [15, Prop. 6] *Let* $(\underline{F}_X, \overline{F}_X)$*,* $(\underline{F}_Y, \overline{F}_Y)$ *be two marginal p-boxes, and denote by* $\overline{P}_X, \overline{P}_Y$ *their associated coherent upper probabilities. Let* $(\underline{F}, \overline{F})$ *be the coherent bivariate p-box defined from the marginals by:*

$$\underline{F}(x, y) = \underline{F}_X(x) \cdot \underline{F}_Y(y) \text{ and } \overline{F}(x, y) = \overline{F}_X(x) \cdot \overline{F}_Y(y), \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

Then, $\underline{P}_{(\underline{F},\overline{F})} = \underline{P}_X \boxtimes \underline{P}_Y$.

This implies that a maxitive bivariate p-box with vacuous $\underline{F}$ is the strong product of the marginals.

Proposition 8. *Let* $(\underline{F}, \overline{F})$ *be a maxitive bivariate p-box of type-1. If* $(\underline{F}, \overline{F})$ *satisfies Equation* (16)*, then* $\overline{P}_{(\underline{F},\overline{F})}$ *is the strong product of the marginals.*

We shift our focus now towards comonotonicity. According to [13], an upper probability in $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone when all the probabilities in its associated credal set are comonotone. This holds if and only if for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ either $\overline{P}(\{(u, v) : u \leq x, v > y\}) = 0$ or $\overline{P}(\{(u, v) : u > x, v \leq y\}) = 0$. Similarly, $\overline{P}$ is countermonotone if and only if for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ either $\overline{P}(\{(u, v) : u \leq x, v < y\}) = 0$ or $\overline{P}(\{(u, v) : u < x, v \leq y\}) = 0$.

Moreover (see [13, Thm. 5]), when $\overline{P}$ is comonotone, its associated bivariate p-box $(\underline{F}, \overline{F})$ is coupled by the upper Fréchet-Hoeffding bound, but the converse does not hold in general.

Taking into account Corollary 3, we may think that the maxitive function associated induced by a bivariate p-box with $\overline{F} = 1$ is comonotone. However, this is not the case when both $\mathcal{X}, \mathcal{Y}$ have at least two elements. The reason is that we are assuming in Equation (15) that the upper probability $\overline{P}_{(\underline{F},\overline{F})}(\{x_i, y_j\}) > 0$ for every (x_i, y_j). This means that the union of the supports of the probabilities in $\mathcal{M}(\overline{P}_{(\underline{F},\overline{F})})$ equals $\mathcal{X} \times \mathcal{Y}$, which is not an increasing set. Thus, $\overline{P}_{(\underline{F},\overline{F})}$ is not comonotone.

5.3. **Building a maxitive bivariate p-box with given marginals.** We conclude this section investigating how can we define a maxitive bivariate p-box with given marginals. Taking Proposition 1 into account, we consider two cases: type 1 and type 2 p-boxes.

First of all, if $(\underline{F}, \overline{F})$ is a maxitive type 1 bivariate p-box, it follows from Theorem 6 that is necessary that either $\overline{F}_X = 1$ or $\overline{F}_Y = 1$. Assume for instance that $\overline{F}_X = 1$; the second case is analogous.

Proposition 9. *Let $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ be two marginal p-boxes such that $\underline{F}_X$ and $\underline{F}_Y$ are vacuous and $\overline{F}_X$ is constantly 1, and let $(\underline{F}, \overline{F})$ be given by Equation* (17)*. Then, $(\underline{F}, \overline{F})$ is a maxitive bivariate p-box.*

Perhaps surprisingly, there may be more than one maxitive bivariate p-box with given marginals. They are characterized in our next result.

Proposition 10. *Let $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ be two marginal p-boxes such that $\underline{F}_X$ and $\underline{F}_Y$ are vacuous and $\overline{F}_X = 1$. Let $j^* \in \{1, \dots, m-1\}$ be the element satisfying $\overline{F}_Y(y_1) = \overline{F}_Y(y_2) = \ldots = \overline{F}_Y(y_{j^*}) < \overline{F}_Y(y_{j^*+1})$. Then, any maxitive bivariate p-box $(\underline{F}, \overline{F})$ with marginals $(\underline{F}_X, \overline{F}_X), (\underline{F}_Y, \overline{F}_Y)$ is given by:*

$$\overline{F}(x, y) = \begin{cases} \overline{F}_Y(y) & \text{if } y \in \{y_{j^*+1}, \dots, y_m\}, \\ \overline{F}_Y(y_1) & \text{if } x = x_n, y \in \{y_1, \dots, y_{j^*}\}, \\ \max\{\alpha_i, \beta_j\} & \text{if } x = x_i < x_n \text{ and } y = y_j \le y_{j^*}, \end{cases} \tag{18}$$

for some $0 < \alpha_1 \le \ldots \le \alpha_{n-1} \le \overline{F}_Y(y_1)$, $0 < \beta_1 \le \ldots \le \beta_{j^*} \le \overline{F}_Y(y_1)$ and $\alpha_1 = \beta_1$.

A consequence of this result is that any maxitive bivariate p-box with fixed marginals is bounded by the bivariate p-box $(\underline{F}^{C_M}, \overline{F}^{C_M})$ generated by Equation (17), in the sense that $\underline{F} = \underline{F}^{C_M}$ but $\overline{F} \le \overline{F}^{C_M}$.

Example 7. *Consider $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$ and the marginal p-boxes given by:*

	x_1	x_2	x_3
$\underline{F}_X$	0	0	1
$\overline{F}_X$	1	1	1

	y_1	y_2	y_3	y_4
$\underline{F}_Y$	0	0	0	1
$\overline{F}_Y$	0.5	0.5	0.7	1

Then, any maxitive bivariate p-box $(\underline{F}, \overline{F})$ with these marginals satisfies that $\underline{F}$ is vacuous and that $\overline{F}$ is given by:

y_4	1	1	1
y_3	0.7	0.7	0.7
y_2	β_2	$\min\{\alpha_2, \beta_2\}$	0.5
y_1	α_1	α_2	0.5
$\overline{F}(x_i, y_j)$	x_1	x_2	x_3

where $0 < \alpha_1 \leq \alpha_2, \beta_2 \leq 0.5$. The upper bound (most conservative) $\overline{F}$ of these bivariate p-boxes is:

y_4	1	1	1
y_3	0.7	0.7	0.7
y_2	0.5	0.5	0.5
y_1	0.5	0.5	0.5
$\overline{F}^{C_M}(x_i, y_j)$	x_1	x_2	x_3

and it is obtained by applying C_M to $\overline{F}_X, \overline{F}_Y$. ♦

On the other hand, the case of type-2 bivariate p-boxes is really simple. We know that $\underline{F}$ is a bivariate cdf and Corollary 3 ensures that $(\underline{F}, \overline{F})$ can be expressed as the upper Fréchet-Hoeffding bound of the marginals. This means that given two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ such that both $\overline{F}_X, \overline{F}_Y$ are constantly 1, we can build the maxitive bivariate p-box by applying Equation (17) to the marginals.

6. Conclusions

Our results in this paper show that the connection between p-boxes and maxitive measures established in [23] does not extend straightforwardly to the bivariate case: it does not suffice that either the lower or the upper distribution function is 0-1-valued, but they need moreover satisfy some additional properties, as we have established in Theorem 5. In addition, not every maxitive measure can be established as the upper probability of a bivariate p-box, as shown in Example 1.

There are several open lines of research that arise from our work. On the one hand, we should extend our results to arbitrary spaces, not necessarily finite. In that case, we should distinguish between maxitive and possibility measures, and we envisage that some additional continuity properties should be imposed if a bivariate p-box is to induce a possibility measure, similarly to what has been established in [23] for the univariate case. Related to this, our assumption of positive upper probability on the singletons, that has been instrumental in many of our results, is more problematic when one of the marginal spaces is uncountable, as cannot hold for instance in the precise case.

Even if we focus on the finite case, we should also study the connection between bivariate p-boxes and maxitive functions when we remove the assumption in Equation (11). As we have already shown in Example 5, if we lift this restriction we may have other bivariate p-boxes whose upper probability is a maxitive measure. We should then determine what is the connection between these two models in the general case.

Another interesting open problem would be the use of maxitive bivariate p-boxes as outer approximations of bivariate p-boxes, taking into account the particular structure of the focal sets that we have established in this paper. Since maxitive bivariate p-boxes are more efficient from a computational point of view, it would be useful to study to which extent this outer approximation produces a loss of information.

Note also that, although in this paper we have focused on events only, imprecise probability models can be expressed in terms of lower and upper expectations of gambles [24], and both models are not equivalent in general. In this sense, it should be studied the properties of (maxitive) bivariate p-boxes as lower and upper *previsions*, in the vein of the work carried out in [22].

Acknowledgements

We acknowledge the financial support by project TIN2014-59543-P. We would also like to thank the anonymous reviewers for their helpful comments, and specially one reviewer for the proof of Proposition 1, which helped simplifying some results from an earlier version of this paper. This paper presents an extension, with proofs and additional results, of previous work [14] presented at the IPMU'2016 conference.

Appendix A. Proofs and additional results

Proof of Lemma 1. For every $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$, it holds that

$$\begin{aligned}\underline{F}(x_i, y_j) = N(A_{x_i,y_j}) &= N(A_{x_i,y_m} \cap A_{x_n,y_j}) \\ &= \min\{N(A_{x_i,y_m}), N(A_{x_n,y_j})\} = \min\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\},\end{aligned}$$

where the first and last equalities follow from Equation (10) and the fourth one because N is a minitive function. Therefore, $\underline{F}$ is built by applying the upper Fréchet-Hoeffding bound to marginal cdfs, and therefore from Sklar's Theorem, $\underline{F}$ is bivariate cdf. ☐

Lemma 2. *Let F be a bivariate cdf defined on $\mathcal{X} \times \mathcal{Y}$. Take $\varepsilon \in (0, 1]$ and define $F^* : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ by $F^*(x, y) = \max\{\varepsilon, F(x, y)\}$ for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Then, F^* is also a bivariate cdf.*

As a consequence, given $Z = \{(x, y) \mid F(x, y) \neq 0\}$ and $0 < \epsilon \leq \min\{F(x, y) \mid (x, y) \in Z\} > 0$, the function F^* defined by:

$$F^* = \begin{cases} F(x, y) & \text{if } (x, y) \in Z, \\ \varepsilon & \text{if } (x, y) \notin Z, \end{cases}$$

is a bivariate cdf.

Proof. On the one hand, it is obvious that F^* is component-wise increasing and that $F(x_n, y_m) = 1$. Let us see that it also fulfils the rectangle inequality (1). For this aim, take $i \in \{2, \ldots, n\}, j \in \{2, \ldots, m\}$ and let us show that $\Delta^{i,j}_{F^*} \geq 0$. Consider the following cases:

(a) If $F^*(x_{i-1}, y_j) = \epsilon$, then by monotonicity we also have $F^*(x_{i-1}, y_{j-1}) = \epsilon$. We obtain then that

$$\Delta^{i,j}_{F^*} = F^*(x_i, y_j) + \varepsilon - \varepsilon - F^*(x_i, y_{j-1}) \geq 0,$$

taking into account that F^* is component-wise increasing.

Similarly, $\Delta^{i,j}_{F^*} \geq 0$ when $F^*(x_i, y_{j-1}) = \epsilon$.

(b) If $F^*(x_{i-1}, y_j) = F(x_{i-1}, y_j)$ and $F^*(x_i, y_{j-1}) = F(x_i, y_{j-1})$, then by monotonicity also $F^*(x_i, y_j) = F(x_i, y_j)$, whence

$$\begin{aligned}\Delta^{i,j}_{F^*} &= F(x_i, y_j) + F^*(x_{i-1}, y_{j-1}) - F(x_i, y_{j-1}) - F(x_{i-1}, y_j) \\ &= F(x_i, y_j) + \max\{\varepsilon, F(x_{i-1}, y_{j-1})\} - F(x_i, y_{j-1}) - F(x_{i-1}, y_j) \geq \Delta^{i,j}_F \geq 0.\end{aligned}$$

Thus, F^* is a distribution function.

The second part is an immediate consequence of the first. ☐

Proof of Proposition 1. From Equation (11), $\pi(x_i, y_j) > 0$ for every i, j; since the focal sets of Π are nested we deduce that $\mathcal{X} \times \mathcal{Y}$ must be a focal set. Thus, $m(\mathcal{X} \times \mathcal{Y}) > 0$, whence

$$\max\{\underline{F}(x_{n-1}, y_m), \underline{F}(x_n, y_{m-1})\} \leq 1 - m(\mathcal{X} \times \mathcal{Y}) < 1.$$

Assume ex-absurdo the existence of focal sets E_1, E_2 such that $(x_1, y_1) \notin E_1$, $(x_n, y_m) \notin E_2$ (it may occur that $E_1 = E_2$). Since the focal sets are nested, one of E_1 or E_2, say E^*, satisfies $(x_1, y_1), (x_n, y_m) \notin E^*$. Consider the set $\mathcal{F} = \{E \text{ focal} \mid \{(x_1, y_1), (x_n, y_m)\} \cap E \neq \emptyset\}$. Since the focal sets are nested, any $E \in \mathcal{F}$ must include either (x_1, y_1) or (x_n, y_m). We split the proof in these two cases.

Case 1: $(x_1, y_1) \in E$ for any $E \in \mathcal{F}$. Then

$$\overline{F}(x_1, y_1) = \Pi(\{(x_1, y_1)\}) = \Pi(\{(x_1, y_1), (x_n, y_m)\}) = \sum_{E \in \mathcal{F}} m(E) \leq 1 - m(E^*) < 1.$$

Now, let us define $F : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ by $F(x, y) = \max\{\underline{F}(x, y), \overline{F}(x_1, y_1)\}$ for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Since $\underline{F}$ is a distribution function by Lemma 1 and $\overline{F}(x_1, y_1) > 0$, we deduce from Lemma 2 that F is a distribution function. Moreover, by construction $F \in (\underline{F}, \overline{F})$ and $F(x_n, y_{m-1}), F(x_{m-1}, y_m) < 1$. Let us prove that $\overline{P}_{(\underline{F}, \overline{F})}(\{(x_1, y_1), (x_n, y_m)\}) > \Pi(\{(x_1, y_1), (x_n, y_m)\})$, that will contradict the hypotheses. For this aim, consider the following cases:

(a) $F(x_n, y_{m-1}) = \underline{F}(x_n, y_{m-1})$ and $F(x_{n-1}, y_m) = \underline{F}(x_{n-1}, y_m)$. Since $\underline{F}$ is a comonotone cdf by Lemma 1,

$$\underline{F}(x_{n-1}, y_{m-1}) = \min\{\underline{F}(x_n, y_{m-1}), \underline{F}(x_{n-1}, y_m)\} < 1,$$

so

$$\begin{aligned} P_{\underline{F}}(\{(x_n, y_m)\}) &= 1 + \underline{F}(x_{n-1}, y_{m-1}) - \underline{F}(x_{n-1}, y_m) - \underline{F}(x_n, y_{m-1}) \\ &= 1 - \max\{\underline{F}(x_{n-1}, y_m), \underline{F}(x_n, y_{m-1})\} > 0. \end{aligned}$$

This implies that:

$$\begin{aligned} P_F(\{(x_n, y_m)\}) &= 1 + F(x_{n-1}, y_{m-1}) - \underline{F}(x_{n-1}, y_m) - \underline{F}(x_n, y_{m-1}) \\ &\geq 1 + \underline{F}(x_{n-1}, y_{m-1}) - \underline{F}(x_{n-1}, y_m) - \underline{F}(x_n, y_{m-1}) > 0. \end{aligned}$$

(b) $F(x_{n-1}, y_m) = \overline{F}(x_1, y_1)$ or $F(x_n, y_{m-1}) = \overline{F}(x_1, y_1)$. Assume, without loss of generality, that the first equality holds. This implies that $F(x_{n-1}, y_{m-1}) = \overline{F}(x_1, y_1)$, because:

$$\overline{F}(x_1, y_1) = F(x_1, y_1) \leq F(x_{n-1}, y_{m-1}) \leq F(x_{n-1}, y_m) = \overline{F}(x_1, y_1).$$

Therefore:

$$\begin{aligned} P_F(\{(x_n, y_m)\}) &= 1 + F(x_{n-1}, y_{m-1}) - F(x_{n-1}, y_m) - F(x_n, y_{m-1}) \\ &= 1 - F(x_n, y_{m-1}) \geq 1 - \underline{F}(x_n, y_{m-1}) > 0. \end{aligned}$$

In both cases we have seen that $P_F(\{(x_n, y_m)\}) > 0$ and $P_F(\{(x_1, y_1)\}) = \overline{F}(x_1, y_1)$, whence:

$$\begin{aligned} \overline{P}_{(\underline{F}, \overline{F})}(\{(x_1, y_1), (x_n, y_m)\}) &\geq P_F(\{(x_1, y_1), (x_n, y_m)\}) \\ &= P_F(\{(x_1, y_1)\}) + P_F(\{(x_n, y_m)\}) > P_F(\{(x_1, y_1)\}) \\ &= \overline{F}(x_1, y_1) = \Pi(\{(x_1, y_1)\}) = \Pi(\{(x_1, y_1), (x_n, y_m)\}), \end{aligned}$$

a contradiction with the equality $\Pi = \overline{P}_{(\underline{F}, \overline{F})}$.

Case 2: $(x_n, y_m) \in E$ for any $E \in \mathcal{F}$. Then

$$\Pi(\{(x_n, y_m)\}) = \Pi(\{(x_1, y_1), (x_n, y_m)\}) = \sum_{E\in\mathcal{F}} m(E) < 1.$$

Since $\overline{P}_{(\underline{F},\overline{F})} = \Pi$, this implies that:

$$\overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_m)\}) = \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1), (x_n, y_m)\}). \tag{19}$$

Take P in the credal set of $\overline{P}_{(\underline{F},\overline{F})}$ such that $P(\{(x_n, y_m)\}) = \Pi(\{(x_n, y_m)\})$. From Equation (19) we deduce that $P(\{(x_1, y_1)\}) = F_P(x_1, y_1) = 0$. There are two cases:

(a) $F_P(x_{n-1}, y_{m-1}) > 0$. Take $0 < \varepsilon < \min\{\overline{F}(x_1, y_1), F_P(x_{n-1}, y_{m-1})\}$, and define $F^* = \max\{\varepsilon, F_P\}$. From Lemma 2, F^* is a cdf. Furthermore, it trivially belongs to $(\underline{F}, \overline{F})$. However:

$$\begin{aligned} P_{F^*}(\{(x_n, y_m)\}) &= 1 + F^*(x_{n-1}, y_{m-1}) - F^*(x_n, y_{m-1}) - F^*(x_{n-1}, y_m) \\ &= 1 + F_P(x_{n-1}, y_{m-1}) - F_P(x_n, y_{m-1}) - F_P(x_{n-1}, y_m) \\ &= P(\{(x_n, y_m)\}) = \overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_m)\}) \end{aligned}$$

and $P_{F^*}(\{(x_1, y_1)\}) = \varepsilon > 0$. Therefore, we conclude that:

$$\begin{aligned} \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_1), (x_n, y_m)\}) &\geq P_{F^*}(\{(x_1, y_1)\}) + P_{F^*}(\{(x_n, y_m)\}) \\ &> \overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_m)\}), \end{aligned}$$

a contradiction with Equation (19).

(b) If $F_P(x_{n-1}, y_{m-1}) = 0$, then also

$$0 = \underline{F}(x_{n-1}, y_{m-1}) = \min\{\underline{F}(x_{n-1}, y_m), \underline{F}(x_n, y_{m-1})\}.$$

Assume for instance that $\underline{F}(x_{n-1}, y_m) = 0$; the other case is similar. Then it must be $\underline{F}(x_n, y_{m-1}) > 0$, or we would obtain $\overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_m)\}) \geq P_{\underline{F}}(\{(x_n, y_m)\}) = 1$, a contradiction. Thus, we also have $F_P(x_{n-1}, y_m) > 0$. Take

$$0 < \varepsilon < \min\{\overline{F}(x_1, y_1), \underline{F}(x_n, y_{m-1})\}$$

and define $F^* = \max\{\varepsilon, F_P\}$, which from Lemma 2 is a cdf. Furthermore, by construction it trivially belongs to $(\underline{F}, \overline{F})$. However:

$$\begin{aligned} P_{F^*}(\{(x_n, y_m)\}) &= 1 + F^*(x_{n-1}, y_{m-1}) - F^*(x_{n-1}, y_m) - F^*(x_n, y_{m-1}) \\ &= 1 + \varepsilon - F^*(x_{n-1}, y_m) - F_P(x_n, y_{m-1}) \\ &> P(\{(x_n, y_m)\}) = \overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_m)\}), \end{aligned}$$

because $\epsilon - F^*(x_{n-1}, y_m) = \min\{0, \epsilon - F_P(x_{n-1}, y_m)\} > -F_P(x_{n-1}, y_m) = F_P(x_{n-1}, y_{m-1}) - F_P(x_{n-1}, y_m)$. This is a contradiction with Equation (19).

Since in any case we arrive a contradiction, we conclude either (x_1, y_1) or (x_n, y_m) must belong to all the focal sets. □

Proof of Corollary 1. From Equation (10) and Proposition 1, if (x_n, y_m) belongs to all the focal sets, it follows that $\Pi(\{(x_n, y_m)\}) = 1$, whence for every $(x, y) \neq (x_n, y_m)$ it holds that:

$$\underline{F}(x, y) = 1 - \Pi(A^c_{x,y}) \leq 1 - \Pi(\{(x_n, y_m)\}) = 0,$$

so $\underline{F}$ is vacuous. On the other hand, if (x_1, y_1) belongs to all the focal sets, then $\Pi(\{(x_1, y_1)\}) = 1$, whence $\overline{F}(x_1, y_1) = \Pi(\{(x_1, y_1)\}) = 1$ and by monotonicity we get $\overline{F}(x, y) = 1$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, so $\overline{F}$ is constantly 1. □

Proof of Proposition 2. (a) The case $(x_i, y_j) = (x_n, y_m)$ holds trivially by considering $P_{\underline{F}} \leq \overline{P}_{(\underline{F},\overline{F})}$. Consider $(x_i, y_j) \neq (x_n, y_m)$, and let P be the probability measure determined by the mass function

$$P(\{(x_i, y_j)\}) = \overline{F}(x_i, y_j), \quad P(\{(x_n, y_m)\}) = 1 - \overline{F}(x_i, y_j).$$

Its associated distribution function is

$$F_P(x_r, y_s) = \begin{cases} 0 & \text{if } r < i \text{ or } s < j \\ \overline{F}(x_i, y_j) & \text{if } r \geq i, s \geq j, (r, s) \neq (n, m) \\ 1 & \text{if } (r, s) = (n, m), \end{cases}$$

whence $F_P \in (\underline{F}, \overline{F})$. Thus, $\overline{P}_{(\underline{F},\overline{F})}(\{(x_i, y_j)\}) \geq P(\{(x_i, y_j)\}) = \overline{F}(x_i, y_j)$. Since by monotonicity $\overline{P}_{(\underline{F},\overline{F})}(\{(x_i, y_j)\}) \leq \overline{P}_{(\underline{F},\overline{F})}(A_{(x_i,y_j)}) = \overline{F}(x_i, y_j)$, we deduce the equality.

(b) Assume ex-absurdo that neither $\overline{F}_X$ nor $\overline{F}_Y$ are constant on 1. Since $\overline{F}(x_1, y_1) > 0$, this means that $\overline{F}(x_n, y_1) \in (0, 1)$ and $\overline{F}(x_1, y_m) \in (0, 1)$. Without loss of generality, assume that $\overline{F}(x_1, y_m) \geq \overline{F}(x_n, y_1)$.

Let F be given by:

$$F(x, y) = \begin{cases} \overline{F}(x_1, y_m) & \text{if } y = y_m \text{ and } x_n > x. \\ \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} & \text{if } x = x_n \text{ and } y_m > y. \\ 1 & \text{if } x = x_n \text{ and } y = y_m. \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove that this function is a bivariate cdf bounded by $\underline{F}$ and $\overline{F}$. Since obviously $\underline{F} \leq F$ because $\underline{F}$ is vacuous, it suffices to show that $F \leq \overline{F}$:

– If $y = y_m$ and $x_n > x$, $\overline{F}(x, y) \geq \overline{F}(x_1, y_m) = F(x, y)$.
– If $x = x_n$ and $y_m > y$,

$$\overline{F}(x, y) \geq \overline{F}(x_n, y_1) \geq \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} = F(x, y).$$

– If $x = x_n$ and $y = y_m$, $\overline{F}(x_n, y_m) = 1 = F(x_n, y_m)$.
– Otherwise, $\overline{F}(x, y) \geq 0 = F(x, y)$.

Next we are going to establish that F is a bivariate cdf. For this, we compute its associated probability mass function $P_F(\{(x_i, y_j)\})$, that by (2) coincides with $\Delta_F^{i,j}$. We obtain the following:

$$\Delta_F^{i,j} = \begin{cases} \overline{F}(x_1, y_m) & \text{if } (x_i, y_j) = (x_1, y_m). \\ \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} & \text{if } (x_i, y_j) = (x_n, y_1). \\ 1 - \overline{F}(x_1, y_m) - \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} & \text{if } (x_i, y_j) = (x_n, y_m). \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} P_F(\{(x, y)\}) = 1$. To see that $P_F(\{(x_n, y_m)\})$ is non-negative, note that

$$\begin{aligned} P_F(\{(x_n, y_m)\}) &= 1 - \overline{F}(x_1, y_m) - \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} \\ &\geq 1 - \overline{F}(x_1, y_m) - 1 + \overline{F}(x_1, y_m) = 0. \end{aligned}$$

Then, F is a bivariate cdf bounded by $\underline{F}, \overline{F}$. This means that P_F belongs to the credal set of $\overline{P}_{(\underline{F},\overline{F})}$. Moreover, since both $\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)$ belong to $(0, 1)$,

$$\begin{aligned} P_F(\{(x_n, y_1), (x_1, y_m)\}) &= P_F(\{(x_n, y_1)\}) + P_F(\{(x_1, y_m)\}) \\ &= \min\{\overline{F}(x_n, y_1), 1 - \overline{F}(x_1, y_m)\} + \overline{F}(x_1, y_m) \\ &> \overline{F}(x_1, y_m), \end{aligned}$$

and then:

$$\begin{aligned} &\overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_1), (x_1, y_m)\}) \geq P_F(\{(x_n, y_1), (x_1, y_m)\}) > \overline{F}(x_1, y_m) \\ &= \max\{\overline{F}(x_1, y_m), \overline{F}(x_n, y_1)\} = \max\{\overline{P}_{(\underline{F},\overline{F})}(\{(x_n, y_1)\}), \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_m)\})\}, \end{aligned}$$

where the last equality follows from the first statement. This is a contradiction with the maxitivity of $\overline{P}_{(\underline{F},\overline{F})}$.

(c) From the second statement, either $\overline{F}_X$ or $\overline{F}_Y$ must be constant on 1. Consider the case where $\overline{F}_X = 1$; the proof when $\overline{F}_Y = 1$ is analogous. If $i = 1$ or $j = 1$, the result is trivial. Assume ex-absurdo that there exist $i \in \{2, \ldots, n\}, j \in \{2, \ldots, m\}$ such that $\overline{F}(x_i, y_j) > \max\{\overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\}$. We can assume without loss of generality that $\overline{F}(x_i, y_1) \geq \overline{F}(x_1, y_j)$. Let us define the function F by:

$$F(x, y) := \begin{cases} 0 & \text{if } x < x_i \text{ and } y < y_j. \\ \min\{\overline{F}(x_i, y_j) - \overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\} & \text{if } x < x_i \text{ and } y \geq y_j. \\ \overline{F}(x_i, y_1) & \text{if } x \geq x_i \text{ and } y < y_j. \\ 1 & \text{if } (x, y) = (x_n, y_m). \\ \overline{F}(x_i, y_j) & \text{otherwise.} \end{cases}$$

Obviously, F is bounded by $\underline{F}$ and $\overline{F}$. In order to check that it is also a bivariate cdf, we compute its probability mass function P_F:

$$\begin{aligned} P_F(\{(x_1, y_j)\}) &= \min\{\overline{F}(x_i, y_j) - \overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\} \\ P_F(\{(x_i, y_1)\}) &= \overline{F}(x_i, y_1) \\ P_F(\{(x_i, y_j)\}) &= \overline{F}(x_i, y_j) - \overline{F}(x_i, y_1) - \min\{\overline{F}(x_i, y_j) - \overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\} \\ P_F(\{(x_n, y_m)\}) &= 1 - \overline{F}(x_i, y_j) \end{aligned}$$

and $P_F(\{(x, y)\}) = 0$ otherwise. This function is non-negative and moreover $\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} P_F(\{(x, y)\}) = 1$. We conclude that $P_F \in \mathcal{M}(\overline{P}_{(\underline{F},\overline{F})})$ and, since both $\overline{F}(x_i, y_j) - \overline{F}(x_i, y_1)$ and $\overline{F}(x_1, y_j)$ are strictly positive,

$$\begin{aligned} P_F(\{(x_1, y_j), (x_i, y_1)\}) &= P_F(\{(x_1, y_j)\}) + P_F(\{(x_i, y_1)\}) \\ &= \min\{\overline{F}(x_i, y_j) - \overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\} + \overline{F}(x_i, y_1) \\ &> \overline{F}(x_i, y_1) = \max\{\overline{P}_{(\underline{F},\overline{F})}(\{(x_i, y_1)\}), \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_j)\})\}. \end{aligned}$$

This means that:

$$\begin{aligned} \overline{P}_{(\underline{F},\overline{F})}((\{(x_1, y_j), (x_i, y_1)\})) &\geq P_F(\{(x_1, y_j), (x_i, y_1)\}) \\ &> \max\{\overline{P}_{(\underline{F},\overline{F})}(\{(x_i, y_1)\}), \overline{P}_{(\underline{F},\overline{F})}(\{(x_1, y_j)\})\}, \end{aligned}$$

and then $\overline{P}_{(\underline{F},\overline{F})}$ would not be maxitive, a contradiction.

(d) This follows from (a) taking into account that $\overline{F}$ is component-wise increasing and $\overline{P}_{(\underline{F},\overline{F})}(\{(x,y)\}) = \Pi(\{(x,y)\}) = \pi(x,y)$. □

Proof of Corollary 2. To see the first statement, note that, from Proposition 2(c), for any i,j it holds that either:

(a) $\overline{F}(x_i,y_j) = \overline{F}(x_1,y_j)$, that by componentwise monotonicity implies that
$$\overline{F}(x_1,y_j) = \ldots = \overline{F}(x_{i-1},y_j) = \overline{F}(x_i,y_j); \text{ or}$$
(b) $\overline{F}(x_i,y_j) = \overline{F}(x_i,y_1)$, that by componentwise monotonicity implies that
$$\overline{F}(x_i,y_1) = \ldots = \overline{F}(x_i,y_{j-1}) = \overline{F}(x_i,y_j).$$

To see that S is increasing, assume ex-absurdo the existence of $(x_i,y_j),(x_k,y_l) \in S$ such that neither $x_i \leq x_k, y_j \leq y_l$ nor $x_i \geq x_k, y_j \geq y_l$ holds. Consider for instance that $x_i < x_k$, $y_j > y_l$. Then, $\overline{F}(x_i,y_j) < \overline{F}(x_{i+1},y_j) \leq \overline{F}(x_k,y_j)$ and also $\overline{F}(x_k,y_l) < \overline{F}(x_k,y_{l+1}) \leq \overline{F}(x_k,y_j)$. However, the first statement implies that it should be either $\overline{F}(x_k,y_j) = \overline{F}(x_i,y_j)$ or $\overline{F}(x_k,y_j) = \overline{F}(x_k,y_l)$, a contradiction. □

Proof of Proposition 3. Taking into account Equation (12), we are going to prove that for any $\alpha_i \in [0,1]$ there is some $l \in \{0,1,\ldots,k\}$ such that
$$\{(x,y) : \overline{F}(x,y) \geq \alpha_i\} = E_l.$$
To see that this is indeed the case, fix $\alpha_i \in [0,1]$ and consider $A := \{(x,y) : \overline{F}(x,y) < \alpha_i\}$. If this set is empty, then $\{(x,y) : \overline{F}(x,y) \geq \alpha_i\} = \mathcal{X} \times \mathcal{Y} = E_0$. If it is non-empty, then we are going to prove that there is some (x^*,y^*) such that $\overline{F}(x,y) < \alpha_i$ if and only if $x \leq x^*$ and $y \leq y^*$.

Define
$$x_A := \max\{x : (x,y) \in A \text{ for some } y\}$$
$$y_A := \max\{y : (x,y) \in A \text{ for some } x\}.$$
If $(x_A,y_A) \notin A$, this means that $\overline{F}(x_A,y_A) \geq \alpha_i$. There are $x' < x_A, y' < y_A$ such that $(x',y_A) \in A$ and $(x_A,y') \in A$. Proposition 2(c) implies then that either $\overline{F}(x_A,y_A) = \overline{F}(x_A,y_1) = \overline{F}(x_A,y') < \alpha_i$ or $\overline{F}(x_A,y_A) = \overline{F}(x_1,y_A) = \overline{F}(x',y_A) < \alpha_i$. This is a contradiction.

Now, if $(x_A,y_A) \in A$, this means that $\overline{F}(x_A,y_A) < \alpha_i$, and by monotonicity any $(x,y) \in A$ satisfies $\overline{F}(x,y) \leq \overline{F}(x_A,y_A) < \alpha_i$. Therefore $A = A_{x_A,y_A}$, and we deduce that $\{(x,y) : \overline{F}(x,y) \geq \alpha_i\} = A^c = A^c_{x_A,y_A}$. It remains only to see that $(x_A,y_A) \in S$. If this was not the case then, with $x_i := x_A \in \mathcal{X}$ and $y_j := y_A \in \mathcal{Y}$, it will hold that $\overline{F}(x_i,y_j) = \overline{F}(x_i,y_{j+1})$ or $\overline{F}(x_i,y_j) = \overline{F}(x_{i+1},y_j)$, meaning that either (x_i,y_{j+1}) or (x_{i+1},y_j) belong to A, a contradiction with the definition of $(x_A,y_A) = (x_i,y_j)$.

Conversely, for any $(x_{i_l},y_{j_l}) \in S$ it holds that $A^c_{x_{i_l},y_{j_l}} = \{(x,y) : \overline{F}(x,y) > \overline{F}(x_{i_l},y_{j_l})\}$ is a focal element of $\overline{P}_{(\underline{F},\overline{F})}$. Thus, the focal sets of $\overline{P}_{(\underline{F},\overline{F})}$ are the sets $E_0,\ldots,E_k$.

Finally, the masses of these focal sets follow from the equality
$$\overline{P}_{(\underline{F},\overline{F})}(E_l) = m(E_0) + m(E_1) + \cdots + m(E_l) = \overline{P}(A_{x_{i_l},y_{j-l}}) = \overline{F}(x_{i_l},y_{j_l}). \quad \square$$

Proof of Proposition 4. By Equation (10),

$$\begin{aligned}\underline{F}(x,y) &= N(A_{x,y}) = 1-\Pi(\{A_{x,y}\}^c)\\ &= 1-\sup\{\pi(u,v):(u,v)\in A^c_{x,y}\} = \begin{cases}0 & \text{if } (x,y)\neq(x_n,y_m).\\ 1 & \text{if } (x,y)=(x_n,y_m).\end{cases}\\ \overline{F}(x,y) &= \Pi(A_{x,y}) = \sup_{(u,v)\in A_{x,y}} \pi(x,y) = \pi(x,y),\end{aligned}$$

taking into account that π is component-wise increasing by assumption. Thus, $(\underline{F},\overline{F})$ is a type 1 bivariate p-box, whence we can apply Proposition 2(a). Hence, for every i,j, it holds that:

$$\overline{P}_{(\underline{F},\overline{F})}(\{(x_i,y_j)\}) = \overline{F}(x_i,y_j) = \pi(x_i,y_j) = \Pi(\{(x_i,y_j)\}).$$

Let us prove that for any $A\subseteq\mathcal{X}\times\mathcal{Y}$ it holds that $\overline{P}_{(\underline{F},\overline{F})}(A)=\Pi(A)$. Consider the following notation:

$$\begin{aligned}&x_A := \max\{x\in\mathcal{X}\mid (x,y)\in A \text{ for some } y\in\mathcal{Y}\}.\\ &y_A := \max\{y\in\mathcal{Y}\mid (x,y)\in A \text{ for some } x\in\mathcal{X}\}.\\ &x^* = \max\{x\in\mathcal{X}\mid (x,y_A)\in A\}.\\ &y^* = \max\{y\in\mathcal{Y}\mid (x_A,y)\in A\}.\end{aligned}$$

On the one hand, $A\subseteq A_{x_A,y_A}$ implies that:

$$\begin{aligned}\overline{P}_{(\underline{F},\overline{F})}(A) &\leq \overline{P}_{(\underline{F},\overline{F})}(A_{x_A,y_A}) = \overline{F}(x_A,y_A) = \pi(x_A,y_A)\\ &= \max\{\pi(x_A,y_1),\pi(x_1,y_A)\} = \max\{\overline{F}(x_A,y_1),\overline{F}(x_1,y_A)\}.\end{aligned}$$

On the other hand, since $(x_A,y^*)\in A$, it holds that:

$$\overline{F}(x_A,y_1)\leq\overline{F}(x_A,y^*) = \overline{P}_{(\underline{F},\overline{F})}(\{(x_A,y^*)\}) \leq \overline{P}_{(\underline{F},\overline{F})}(A).$$

Similarly, $\overline{F}(x^*,y_A)\leq\overline{P}_{(\underline{F},\overline{F})}(A)$, whence

$$\begin{aligned}\overline{P}_{(\underline{F},\overline{F})}(A) &= \max\{\overline{F}(x_A,y_1),\overline{F}(x_1,y_A)\}\\ &= \max\{\pi(x_A,y_1),\pi(x_1,y_A)\}) = \pi(x_A,y_A) = \Pi(A_{x_A,y_A}) \geq \Pi(A),\end{aligned}$$

where last equality follows because π is component-wise increasing. Finally:

$$\begin{aligned}\Pi(A) &= \max_{(x,y)\in A}\pi(x,y) \geq \max\{\pi(x_A,y^*),\pi(x^*,y_A)\}\\ &\geq \max\{\pi(x_A,y_1),\pi(x_1,y_A)\} = \overline{P}_{(\underline{F},\overline{F})}(A).\end{aligned}$$

Therefore, $\overline{P}_{(\underline{F},\overline{F})}(A)=\Pi(A)$ for any A. □

Proof of Proposition 5. Note that $E_1\subseteq\ldots\subseteq E_k$ and $m(E_1)+\ldots+m(E_k)=1$; thus, they define maxitive and minitive functions $\overline{P}$ and $\underline{P}$ by:

$$\overline{P}(A) = \sum_{E_i\cap A\neq\emptyset} m(E_i), \quad \underline{P}(A) = \sum_{E_i\subseteq A} m(E_i).$$

Let us see that $\underline{P}=\underline{P}_{(\underline{F},\overline{F})}$. Consider a set A and assume that $A_{u_i,v_i}\subseteq A\not\subseteq A_{u_{i+1},v_{i+1}}$. It is easy to check that:

$$\underline{P}(A) = m(E_1)+\ldots+m(E_i) = \underline{F}(u_i,v_i).$$

On the other hand, it follows by monotonicity that

$$\underline{P}_{(\underline{F},\overline{F})}(A) \geq \underline{P}_{(\underline{F},\overline{F})}(A_{u_i,v_i}) = \underline{F}(u_i,v_i).$$

Let us see that the inequality is in fact an equality. Since $A \not\subseteq A_{u_{i+1},v_{i+1}}$, there exists $(x^*,y^*) \in A_{u_{i+1},v_{i+1}} \backslash A$. Then, we can consider the probability P with mass function

$$\begin{aligned}
P(\{(u_1,v_1)\}) &= \underline{F}(u_1,v_1) \\
P(\{(u_2,v_2)\}) &= \underline{F}(u_2,v_2) - \underline{F}(u_1,v_1) \\
&\dots \\
P(\{(u_i,v_i)\}) &= \underline{F}(u_i,v_i) - \underline{F}(u_{i-1},v_{i-1}) \\
P(\{(x^*,y^*)\}) &= 1 - \underline{F}(u_1,v_1) - \ldots - \underline{F}(u_i,v_i).
\end{aligned}$$

Its associated cdf F_P is given by

$$F_P(x,y) = \begin{cases} \underline{F}(x,y) & \text{if } x < x^* \text{ or } y < y^*. \\ 1 & \text{if } x \geq x^* \text{ and } y \geq y^*. \end{cases}$$

Note that $\underline{F} \leq F_P \leq 1 = \overline{F}$, and as a consequence $P \in \mathcal{M}(\underline{P}_{(\underline{F},\overline{F})})$. Thus,

$$\underline{P}_{(\underline{F},\overline{F})}(A) \leq P(A) = \sum_{(x,y)\in A} P(A) = P(\{(u_1,v_1)\}) + \ldots + P(\{(u_i,v_i)\}) = \underline{F}(u_i,v_i).$$

We conclude that $\underline{P}(A) = \underline{P}_{(\underline{F},\overline{F})}(A)$. Thus, both minitive functions coincide and therefore $E_1,\ldots,E_k$ are the focal sets of $\underline{P}_{(\underline{F},\overline{F})}$ and $\overline{P}_{(\underline{F},\overline{F})}$. □

Proof of Theorem 4. First of all, let us see that the given conditions are necessary for the equality $\Pi = \overline{P}_{(\underline{F},\overline{F})}$. We shall establish that π is decreasing in the first component; the proof for the second component is analogous. According to Proposition 5, the focal sets of Π can be expressed as:

$$E_r = A_{u_r,v_r} \ \forall r = 1,\ldots,k.$$

In particular, this implies that the focal sets are nested. Then, in order to compute $\pi(x_i,y_j)$ we shall make the sum of the masses of the focal sets that include (x_i,y_j). Assume that $(x_i,y_j) \in E_s \subset E_{s+1} \subset \ldots \subseteq E_k$ but $(x_i,y_j) \notin E_{s-1}$. Then:

$$\pi(x_i,y_j) = m(E_s) + m(E_{s+1}) + \ldots + m(E_k).$$

Since the focal sets are cumulative rectangles, it means that $(x_{i-1},y_j) \in E_s \subset E_{s+1} \subset \ldots \subset E_k$, and therefore $\pi(x_{i-1},y_j) \geq \pi(x_i,y_j)$.

For the second condition, take (x_i,y_j) and let us compute $\pi(x_i,y_j)$. Since π is component-wise decreasing, $\pi(x_i,y_j) \leq \min\{\pi(x_1,y_j),\pi(x_i,y_1)\}$. Let us prove the converse inequality. Following the same reasoning as above, there is a focal set E_s such that $(x_i,y_j) \in E_s \subset E_{s+1} \subset \ldots \subset E_k$ but $(x_i,y_j) \notin E_{s-1}$. Since the focal sets are cumulative rectangles, this means that either $x_i > u_{s-1}$ or $y_j > v_{s-1}$, because otherwise $E_{s-1} \subseteq A_{x_i,y_j}$ and therefore $(x_i,y_j) \in E_{s-1}$, a contradiction.

- Assume that $x_i > u_{s-1}$. Then, $(x_i,y_1) \in E_s \subset E_{s+1} \subset \ldots \subset E_k$ but $(x_i,y_1) \notin E_{s-1}$. Therefore:

$$\pi(x_i,y_j) = m(E_s) + m(E_{s+1}) + \ldots + m(E_k) = \pi(x_i,y_1).$$

- Assume that $y_j > v_{s-1}$. Then, $(x_1, y_j) \in E_s \subset E_{s+1} \subset \ldots \subset E_k$ but $(x_1, y_j) \notin E_{s-1}$. Therefore:

$$\pi(x_i, y_j) = m(E_s) + m(E_{s+1}) + \ldots + m(E_k) = \pi(x_1, y_j).$$

Thus, $\pi(x_i, y_j) \leq \min\{\pi(x_1, y_j), \pi(x_i, y_1)\}$.

Finally, note that (x_1, y_1) belongs to all the focal sets, and therefore $\pi(x_1, y_1) = 1$.

Let us see next that the conditions are also sufficient.

We begin by proving that $\overline{P}_{(\underline{F},\overline{F})}$ is a maxitive function or, equivalently, that $\underline{P}_{(\underline{F},\overline{F})}$ is minitive:

$$\underline{P}_{(\underline{F},\overline{F})}(A \cap B) = \min\{\underline{P}_{(\underline{F},\overline{F})}(A), \underline{P}_{(\underline{F},\overline{F})}(B)\}. \tag{20}$$

By Lemma 1, $\underline{F}$ is a bivariate distribution function and moreover $\underline{F}(x_i, y_j) = \min\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\}$ for any i, j. Let S be the increasing set that is the support of $\underline{F}$. If $A_{(u_1,v_1)} \nsubseteq A$, then given $(x, y) \in A_{(u_1,v_1)} \setminus A$, the cdf F given by

$$F(x', y') = \begin{cases} 1 & \text{if } x' \geq x, y' \geq y \\ 0 & \text{otherwise} \end{cases}$$

belongs to $(\underline{F}, \overline{F})$, whence $\underline{P}_{(\underline{F},\overline{F})}(A) \leq P_F(A) = 0$. Thus, $0 = \underline{P}_{(\underline{F},\overline{F})}(A \cap B) \leq \min\{\underline{P}_{(\underline{F},\overline{F})}(A), \underline{P}_{(\underline{F},\overline{F})}(B)\} = 0$, and Equation (20) holds. The same applies if $A_{(u_1,v_1)} \nsubseteq B$.

Assume now that $A_{(u_1,v_1)} \subseteq A \cap B$. In this case, there is some $i \in \{1, \ldots, k\}$ such that $A_{u_i,v_i} \subseteq A$ but $A_{u_{i+1},v_{i+1}} \nsubseteq A$. Thus, $\underline{P}_{(\underline{F},\overline{F})}(A) \geq \underline{P}_{(\underline{F},\overline{F})}(A_{u_i,v_i}) = \underline{F}(u_i, v_i)$. To see the converse inequality, take $(x, y) \in A_{u_{i+1},v_{i+1}} \setminus A$ and let F be the distribution function associated with the probability mass function

$$P(\{(u_j, v_j)\}) = P_{\underline{F}}(\{(u_j, v_j)\}) \ \forall j = 1, \ldots, i, P(\{(x, y)\}) = 1 - \underline{F}(u_i, v_i);$$

then $F \in (\underline{F}, \overline{F})$, and as a consequence $\underline{P}_{(\underline{F},\overline{F})}(A) \leq P(A) = \underline{F}(u_i, v_i)$.

Similarly, given $j \in \{1, \ldots, k\}$ such that $A_{(u_j,v_j)} \subseteq B$ and $A_{(u_{j+1},v_{j+1})} \nsubseteq B$, it holds that $\underline{P}_{(\underline{F},\overline{F})}(B) = \underline{F}(u_j, v_j)$. Assume without loss of generality that $i \leq j$. Then, $A_{u_i,v_i} \subseteq A \cap B$ but $A_{u_{i+1},v_{i+1}} \nsubseteq A \cap B$. Thus,

$$\begin{aligned} \underline{P}_{(\underline{F},\overline{F})}(A \cap B) = \underline{F}(u_i, v_i) &= \min\{\underline{F}(u_i, v_i), \underline{F}(u_j, v_j)\} \\ &= \min\left\{\underline{P}_{(\underline{F},\overline{F})}(A), \underline{P}_{(\underline{F},\overline{F})}(B)\right\}. \end{aligned}$$

This implies that $\underline{P}_{(\underline{F},\overline{F})}$ is minitive and by conjugacy $\overline{P}_{(\underline{F},\overline{F})}$ is maxitive.

Let us now see that $\overline{P}_{(\underline{F},\overline{F})} = \Pi$, or equivalently that $N = \underline{P}_{(\underline{F},\overline{F})}$. To see this, note that the focal elements of Π are all cumulative rectangles: for any given $\alpha \in [0, 1]$, if we denote $x_\alpha = \max\{x : \pi(x, y_1) \geq \alpha\}$ and $y_\alpha = \max\{y : \pi(x_1, y) \geq \alpha\}$, then $\{(x, y) : \pi(x, y) \geq \alpha\} = A_{x_\alpha,y_\alpha}$, taking into account that π is componentwise decreasing and that $\pi(x, y) = \min\{\pi(x, y_1), \pi(x_1, y)\} \ \forall (x, y)$ by assumption.

Now, the maxitive (and therefore coherent) upper probability $\overline{P}_{(\underline{F},\overline{F})}$ is the natural extension of the restriction of Π to $\mathcal{K}_2$, which means in particular that $\overline{P}_{(\underline{F},\overline{F})}(A_{x,y}) = \Pi(A_{x,y})$ and $\underline{P}_{(\underline{F},\overline{F})}(A_{x,y}) = N(A_{x,y})$ for every (x, y). But two necessity measures that coincide on their focal elements coincide also on any other event, taking into account that, since the focal elements are nested,

$$\begin{aligned} N(A) &= \max\{N(B) : B \subseteq A, B \text{ focal}\} \\ &= \max\{\underline{P}_{(\underline{F},\overline{F})}(B) : B \subseteq A, B \text{ focal}\} = \underline{P}_{(\underline{F},\overline{F})}(A). \end{aligned}$$

Thus, $N = \underline{P}_{(\underline{F},\overline{F})}$ or, equivalently, $\overline{P}_{(\underline{F},\overline{F})} = \Pi$. □

Proof of Theorem 6. First of all, if $\overline{P}_{(\underline{F},\overline{F})}$ is maxitive, it follows from Proposition 2 that $\overline{F}(x_i, y_j) = \max\{\overline{F}(x_i, y_1), \overline{F}(x_1, y_j)\}$ for any i, j and either $\overline{F}_X = 1$ or $\overline{F}_Y = 1$.

Conversely, if $(\underline{F}, \overline{F})$ is a type 1 bivariate p-box we deduce from Proposition 2 that $\overline{P}_{(\underline{F},\overline{F})}(\{(x, y)\}) = \overline{F}(x, y)$ for every (x, y). Thus, if we consider the possibility distribution $\pi = \overline{F}$, then it satisfies the conditions of Proposition 4, meaning that the bivariate p-box $(\underline{F}', \overline{F}')$ it induces satisfies $\overline{P}_{(\underline{F}',\overline{F}')} = \Pi$, where Π is the maxitive function associated with π. But it follows from Equation (10) that

$$\overline{F}'(x, y) = \Pi(A_{(x,y)}) = \pi(x, y) = \overline{F}(x, y),$$

since π is component-wise increasing. Moreover, $\pi(x_n, y_m) = 1$ implies

$$\underline{F}'(x, y) = 1 - \Pi(A^c_{(x,y)}) = \begin{cases} 1 & \text{if } (x, y) = (x_n, y_m) \\ 1 - 1 = 0 & \text{otherwise,} \end{cases}$$

whence $\underline{F}' = \underline{F}$ vacuous. Thus, $(\underline{F}', \overline{F}') = (\underline{F}, \overline{F})$ and therefore $\overline{P}_{(\underline{F},\overline{F})} = \Pi$. □

Proof of Theorem 7. That this condition is necessary follows from Lemma 1. To see that it is also sufficient, let π be the possibility distribution given by

$$\pi(x_1, y_1) = 1,\ \pi(x_i, y_1) = 1 - \underline{F}(x_{i-1}, y_m),\ \pi(x_1, y_j) = 1 - \underline{F}(x_n, y_{j-1})\ \forall i, j > 2,$$
$$\text{and } \pi(x_i, y_j) = \min\{\pi(x_i, y_1), \pi(x_1, y_j)\}\ \forall i < n, j < m,$$

and let Π be its associated maxitive function.

By construction π satisfies the conditions of Theorem 4, whence, if we denote by $(\underline{F}', \overline{F}')$ the bivariate p-box it induces, $\overline{P}_{(\underline{F}',\overline{F}')} = \Pi$. Moreover,

$$\overline{F}'(x_1, y_1) = \pi(x_1, y_1) = 1 = \overline{F}(x_1, y_1),$$

whence $\overline{F}(x_i, y_j) = 1 = \overline{F}'(x_i, y_j)$ for any i, j, and

$$\underline{F}'(x_i, y_m) = 1 - \Pi(A^c_{x_i,y_m}) = 1 - \pi(x_{i+1}, y_1) = \underline{F}(x_i, y_m)\ \forall i < n$$
$$\underline{F}'(x_n, y_j) = 1 - \Pi(A^c_{x_n,y_j}) = 1 - \pi(x_1, y_{j+1}) = \underline{F}(x_n, y_j)\ \forall j < m$$

by construction. Now, given $i < n$ and $j < m$,

$$\begin{aligned} \underline{F}'(x_i, y_j) &= 1 - \Pi(A^c_{(x_i,y_j)}) = 1 - \max\{\pi(x_1, y_{j+1}), \pi(x_{i+1}, y_1)\} \\ &= \min\{1 - \pi(x_1, y_{j+1}), 1 - \pi(x_{i+1}, y_1)\} \\ &= \min\{\underline{F}(x_n, y_j), \underline{F}(x_i, y_m)\} = \underline{F}(x_i, y_j), \end{aligned}$$

where last equality follows by hypothesis. Thus, the maxitive function Π coincides with the upper probability $\overline{P}_{(\underline{F},\overline{F})}$. □

Proof of Proposition 6. Let us prove the first statement; the proof of the second is analogous. Assume that $(\underline{F}, \overline{F})$ can be expressed as in Equation (16) and that $\overline{F}_X = 1$. Then, there exists a non-empty set of copulas $\mathcal{C}$ such that:

$$\overline{F}(x, y) = \inf_{C \in \mathcal{C}} C(\overline{F}_X(x), \overline{F}_Y(y)) = \inf_{C \in \mathcal{C}} C(1, \overline{F}_Y(y)) = \overline{F}_Y(y).$$

Conversely, if $\overline{F}_X = 1$ and $\overline{F}(x_1, y_j) = \ldots = \overline{F}(x_n, y_j) = \overline{F}_Y(y_j)$, then:

$$\overline{F}(x_i, y_j) = \overline{F}_Y(y_j) = C(1, \overline{F}_Y(y_j)) = C(\overline{F}_X(x_i), \overline{F}_Y(y_j)),$$

for any copula C, in particular the upper Fréchet-Hoeffding bound C_M. Therefore, Equation (16) holds. □

Proof of Proposition 8. From Proposition 6, if $(\underline{F}, \overline{F})$ can be expressed as in Equation (16), we can assume without loss of generality that $\overline{F}_X$ is constantly 1 and $\overline{F}(x_i, y_j) = \overline{F}_Y(y_j)$ for any i, j. This means that:

$$\overline{F}(x, y) = \overline{F}_Y(y) = 1 \cdot \overline{F}_Y(y) = \overline{F}_X(x) \cdot \overline{F}_Y(y),$$

and therefore they are linked by the product copula. Similarly, $\underline{F}(x, y) = \underline{F}_X(x) \cdot \underline{F}_Y(y)$ since $\underline{F}$ is vacuous. Therefore, $(\underline{F}, \overline{F})$ is coupled by the product, and therefore $\overline{P}_{(\underline{F},\overline{F})}$ is the strong product of its marginals. □

Proof of Proposition 9. That $(\underline{F}, \overline{F})$ is a coherent bivariate p-box directly follows by Theorem 8. Moreover,

$$\overline{F}(x, y) = \min\{\overline{F}_X(x), \overline{F}_Y(y)\} = \min\{1, \overline{F}_Y(y)\} = \overline{F}_Y(y).$$

Therefore, for any (x, y) $\overline{F}(x, y) = \overline{F}_Y(y) = \overline{F}(x_1, y)$ and $\overline{F}(x, y) = \overline{F}_Y(y) \geq \overline{F}_Y(y_1)$. Thus, $\overline{F}(x, y) = \overline{F}_Y(y) = \max\{\overline{F}(x_1, y), \overline{F}(x, y_1)\}$, and using Theorem 6, we conclude that $(\underline{F}, \overline{F})$ is maxitive. □

Proof of Proposition 10. Let us consider a maxitive bivariate p-box $(\underline{F}, \overline{F})$ with these marginals. Consider $y \in \{y_{j^*+1}, \dots, y_m\}$, and let us see that $\overline{F}(x, y) = \overline{F}_Y(y)$ for any $x \in \mathcal{X}$. Assume that there is some $i \in \{1, \dots, n-1\}$ such that $\overline{F}(x_i, y) < \overline{F}_Y(y) = \overline{F}(x_n, y)$. Since $\overline{F}(x_1, y) \leq \overline{F}(x_i, y)$, we know that:

$$\overline{F}(x_1, y) < \overline{F}_Y(y) = \overline{F}(x_n, y) = \max\left\{\overline{F}(x_n, y_1), \overline{F}(x_1, y)\right\} = \overline{F}(x_n, y_1).$$

Therefore, $\overline{F}(y_1) = \overline{F}(y)$, a contradiction with $y > y_{j^*}$. We conclude that $\overline{F}(x, y) = \overline{F}_Y(y)$ for any x. Also, for $y \in \{y_1, \dots, y_{j^*}\}$, it follows from Theorem 6 that

$$\overline{F}(x, y) = \max\{\overline{F}(x_1, y), \overline{F}(x, y_1)\}.$$

Then, if we denote $\alpha_i = \overline{F}(x_i, y_1)$ and $\beta_j = \overline{F}(x_1, y_j)$, we obtain the bivariate cdf of Equation (18). □

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