n-Monotone lower previsions

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Abstract

We study n-monotone lower previsions, which constitute a generalisation of n-monotone lower probabilities. We investigate their relation with the concepts of coherence and natural extension in the behavioural theory of imprecise probabilities, and improve along the way upon a number of results from the literature.
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1 Introduction

Lower and upper previsions, mainly due to Walley [13], are among the more interesting uncertainty models in imprecise probability theory. They can be viewed as lower and upper expectations with respect to closed convex sets of probability measures (also called credal sets; see Levi [10]), and they provide a unifying framework for studying many other uncertainty models, such as probability charges (Bhaskara Rao and Bhaskara Rao [2]), 2- and n-monotone set functions (Choquet [3]), possibility measures ([4, 5, 6, 14]), and p-boxes (Ferson *et al* [9]). They have also been linked to various theories of integration, such as Choquet integration (Walley [12, p. 53]) and Lebesgue integration (Walley [13, p. 132]).

The goal of this paper is to investigate how n-monotonicity can be defined for lower previsions, and to study the properties of these n-monotone lower previsions. We start out from Choquet's [3] original and very general definition of n-monotonicity for functionals defined on arbitrary lattices.

The paper is structured as follows. Section 2 highlights the most important aspects of the theory of lower previsions that we shall need in the rest of the paper. Section 3 is concerned with the definition of n-monotonicity for lower previsions. In Section 4, we establish a number of interesting properties, and generalise a number of results in the literature, for n-monotone lower probabilities on fields of events. In Section 5, we relate n-monotone lower previsions to comonotone additive functionals and Choquet integrals. Finally, Section 6 contains some conclusions on the matter at hand.

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2 Coherent lower previsions

In this section, we introduce a few basic notions about lower previsions. We refer to Walley [13] for a more in-depth discussion, and for a behavioural interpretation of the notions discussed below.

Consider a non-empty set Ω. A *gamble* f on Ω is a bounded real-valued mapping on Ω. The set of all gambles on Ω is denoted by $\mathscr{L}$. It is a real linear space under the point-wise addition of gambles, and the point-wise scalar multiplication of gambles with real numbers. Hence, a linear space of gambles will mean in this paper a subset of $\mathscr{L}$ that is closed under these two operations.

Special gambles are the ones that only take values in $\{0,1\}$: let A be any subset of Ω, also called an *event*, then the gamble I_A, defined by $I_A(\omega) := 1$ if $\omega \in A$ and $I_A(\omega) := 0$ otherwise, is called the *indicator* of A. This establishes a correspondence between events and $\{0,1\}$-valued gambles. Often we shall also denote I_A by A.

A *lower prevision* $\underline{P}$ is defined as a real-valued map (a functional) defined on some subset $\operatorname{dom}\underline{P}$ of $\mathscr{L}$; we call $\operatorname{dom}\underline{P}$ the *domain* of $\underline{P}$. For any gamble f in $\operatorname{dom}\underline{P}$, $\underline{P}(f)$ is called the lower prevision of f. If the domain of $\underline{P}$ contains only (indicators of) events A, then we also call $\underline{P}$ a *lower probability*, and we write $\underline{P}(I_A)$ also as $\underline{P}(A)$, the lower probability of A.

Given a lower prevision $\underline{P}$, the *conjugate upper prevision* $\overline{P}$ of $\underline{P}$ is defined on $\operatorname{dom}\overline{P} = -\operatorname{dom}\underline{P}$ by $\overline{P}(f) := -\underline{P}(-f)$ for every $-f$ in the domain of $\underline{P}$. This conjugacy relationship allows us to focus on the study of lower previsions only.

A lower prevision $\underline{P}$ whose domain is a linear space is called *coherent* if the following three properties are satisfied for all f, g in $\operatorname{dom}\underline{P}$ and all non-negative real λ:

($\underline{P}$1) $\underline{P}(f) \geq \inf f$ (accepting sure gain);

($\underline{P}$2) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ (positive homogeinity);

($\underline{P}$3) $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ (superadditivity).

A coherent lower prevision on a linear space can always be extended to a coherent lower prevision on all gambles. A lower prevision $\underline{P}$ with a general domain (not necessarily a linear space) is then called *coherent* if it can be extended to a coherent lower prevision on all gambles. This is the case if and only if $\sup\left[\sum_{i=1}^n f_i - m f_0\right] \geq \sum_{i=1}^n \underline{P}(f_i) - m\underline{P}(f_0)$ for any natural numbers $n \geq 0$ and $m \geq 0$, and $f_0, f_1, \dots, f_n$ in the domain of $\underline{P}$.

There are a number of common consequences of coherence that we shall use throughout the paper. A lower envelope of coherent lower previsions (with the same domain) is again a coherent lower prevision. Moreover, consider a coherent lower prevision $\underline{P}$, let f and g be elements in $\operatorname{dom}\underline{P}$, and let μ and λ be real numbers, with $\lambda \geq 0$. Then whenever the relevant gambles belong to $\operatorname{dom}\underline{P}$, we have that $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$, $\underline{P}(\lambda f) = \lambda \underline{P}(f)$, $\underline{P}(\mu) = \mu$ and $\underline{P}(f+\mu) = \underline{P}(f) + \mu$. Moreover $\inf f \leq \underline{P}(f) \leq \overline{P}(f) \leq \sup f$ and consequently $0 \leq \underline{P}(|f|) \leq \overline{P}(|f|) \leq \sup|f|$. Also, $\underline{P}$ is monotone: if $f \leq g$ then $\underline{P}(f) \leq \underline{P}(g)$. Finally, both $|\underline{P}(f) - \underline{P}(g)| \leq \overline{P}(|f-g|)$ and $\left|\overline{P}(f) - \overline{P}(g)\right| \leq \overline{P}(|f-g|)$. As an immediate consequence of these properties, we see that if a sequence f_n of gambles converges uniformly to a gamble f, *i.e.*, $\sup|f_n - f| \to 0$, then also $\underline{P}(f_n) \to \underline{P}(f)$ and $\overline{P}(f_n) \to \overline{P}(f)$, so any coherent lower or upper prevision is continuous with respect to the supremum norm.

A lower prevision $\underline{Q}$ is said to *dominate* a lower prevision $\underline{P}$, if $\operatorname{dom}\underline{Q} \supseteq \operatorname{dom}\underline{P}$ and $\underline{Q}(f) \geq \underline{P}(f)$ for any f in $\operatorname{dom}\underline{P}$. We say that a lower prevision $\underline{P}$ *avoids sure loss* if it is dominated by some coherent lower prevision on $\mathscr{L}$. This is the case if and only if $\sup\left[\sum_{i=1}^n f_i\right] \geq \sum_{i=1}^n \underline{P}(f_i)$ for any natural number $n \geq 1$ and any $f_1, \dots, f_n$ in $\operatorname{dom}\underline{P}$. A lower prevision avoids sure loss if and only if there is a point-wise smallest coherent lower prevision $\underline{E}_{\underline{P}}$ on $\mathscr{L}$ that dominates $\underline{P}$, namely, the lower envelope of all the coherent lower previsions on $\mathscr{L}$ that dominate $\underline{P}$. $\underline{E}_{\underline{P}}$ is then called the *natural extension* of $\underline{P}$.

A *linear prevision* P is a real-valued functional defined on a set of gambles $\operatorname{dom}P$, that satisfies $\sup\left[\sum_{i=1}^n f_i - \sum_{j=1}^m g_j\right] \geq \sum_{i=1}^n P(f_i) - \sum_{j=1}^m P(g_j)$ for any natural numbers $n \geq 0$ and $m \geq 0$, and $f_1, \dots, f_n, g_1, \dots, g_m$ in $\operatorname{dom}P$. A linear prevision P is coherent, both when interpreted as a lower, and as an upper prevision; the former means that P is a coherent lower prevision on $\operatorname{dom}P$, the latter that $-P(-\cdot)$ is a coherent lower prevision on $-\operatorname{dom}P$. For any linear prevision P, it holds that $P(f) = -P(-f)$ whenever f and $-f$ belong to the domain of P. A lower prevision $\underline{P}$ whose domain is negation invariant (*i.e.*, $-\operatorname{dom}\underline{P} = \operatorname{dom}\underline{P}$), is a linear prevision if and only if it is coherent and *self-conjugate*, i.e., $\underline{P}(-f) = -\underline{P}(f)$ for all f in $\operatorname{dom}\underline{P}$. A linear prevision P on $\mathscr{L}$ is a non-negative, normed $[P(1) = 1]$, real-valued, linear functional on $\mathscr{L}$. Its restriction to (indicators of) events is then a *probability charge* (or finitely additive probability measure) on $\wp(\Omega)$.

Let us denote the set of linear previsions on $\mathscr{L}$ that dominate $\underline{P}$ by $\mathscr{M}(\underline{P})$. The following statements are equivalent: (i) $\underline{P}$ avoids sure loss, (ii) the natural extension of $\underline{P}$ exists; and (iii) $\mathscr{M}(\underline{P})$ is non-empty. The following statements are equivalent as well: (i) $\underline{P}$ is coherent; (ii) $\underline{P}$ coincides with its natural extension $\underline{E}_{\underline{P}}$ on $\operatorname{dom}\underline{P}$; and (iii) $\underline{P}$ coincides with the lower envelope of $\mathscr{M}(\underline{P})$ on $\operatorname{dom}\underline{P}$. The last statement simply follows from the important fact that the natural extension of $\underline{P}$ is equal to the lower envelope of $\mathscr{M}(\underline{P})$: $\underline{E}_{\underline{P}}(f) = \min_{Q \in \mathscr{M}(\underline{P})} Q(f)$, for any gamble f in $\mathscr{L}$. Often, as we shall see, this expression provides a convenient way to calculate the natural extension of a lower prevision that avoids sure loss. Finally it holds that $\mathscr{M}(\underline{P}) = \mathscr{M}(\underline{E}_{\underline{P}})$. This result can be used to prove the following "transitivity" property for natural extension: if we denote by $\underline{Q}$ the restriction of the natural extension $\underline{E}_{\underline{P}}$ of a lower prevision (that avoids sure loss) to some set of gambles $\mathscr{K} \supseteq \operatorname{dom}\underline{P}$, then $\mathscr{M}(\underline{P}) = \mathscr{M}(\underline{Q}) = \mathscr{M}(\underline{E}_{\underline{P}})$, and consequently $\underline{E}_{\underline{Q}}$ coincides with $\underline{E}_{\underline{P}}$ on all gambles.

3 n-Monotone lower previsions

Let us introduce our notion of n-monotonicity for lower previsions. A subset $\mathscr{S}$ of $\mathscr{L}$ is called a *lattice* if it is closed under point-wise maximum $\vee$ and point-wise minimum $\wedge$, *i.e.*, if for all f and g in $\mathscr{S}$, both $f \vee g$ and $f \wedge g$ also belong to $\mathscr{S}$. For instance, the set $\mathscr{L}$ of all gambles on Ω is a lattice. The set of natural numbers without zero is denoted by $\mathbb{N}$. By $\mathbb{N}^*$ we denote $\mathbb{N} \cup \{\infty\}$.

The following definition is a special case of Choquet's [3] general definition of n-monotonicity for functions from an Abelian semi-group to an Abelian group.

Definition 1. *Let $n \in \mathbb{N}^*$, and let $\underline{P}$ be a lower prevision whose domain* $\operatorname{dom}\underline{P}$ *is a lattice of gambles. Then we call* $\underline{P}$ n-monotone *if for all $p \in \mathbb{N}$, $p \leq n$, and all f, $f_1, \dots,$ f_p in* $\operatorname{dom}\underline{P}$*:*

$$\sum_{I \subseteq \{1,\dots,p\}} (-1)^{|I|} \underline{P}\left(f \wedge \bigwedge_{i \in I} f_i \right) \geq 0.$$

*The conjugate upper prevision of an n-monotone lower prevision is called n-*alternating*. An ∞-monotone lower prevision (i.e, a lower prevision which is p-monotone for all $p \in \mathbb{N}$) is also called* completely monotone, *and an ∞-alternating upper prevision* completely alternating.

We use the convention that for $I = \emptyset$, $\bigwedge_{i \in I} f_i$ simply drops out of the expressions (we could let it be equal to $+\infty$). Clearly, if a lower prevision $\underline{P}$ is n-monotone, it is also p-monotone for $1 \leq p \leq n$. The following proposition gives an immediate alternative characterisation for the n-monotonicity of lower previsions.

Proposition 1. *Let $n \in \mathbb{N}^*$, and consider a lower prevision $\underline{P}$ whose domain* $\operatorname{dom}\underline{P}$ *is a lattice of $\mathscr{L}$. Then $\underline{P}$ is n-monotone if and only if*

(i) $\underline{P}$ is monotone, i.e., for all f and g in $\operatorname{dom}\underline{P}$ *such that $f \leq g$, we have $\underline{P}(f) \leq \underline{P}(g)$; and*

(ii) for all $p \in \mathbb{N}$, $2 \leq p \leq n$, and all $f_1, \dots, f_p$ in $\operatorname{dom}\underline{P}$*:*

$$\underline{P}\left(\bigvee_{i=1}^{p} f_i \right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} \underline{P}\left(\bigwedge_{i \in I} f_i \right).$$

Coherence guarantees n-monotonicity only if $n = 1$: any coherent lower prevision on a lattice of gambles is monotone (or equivalently, 1-monotone) but not necessarily 2-monotone, as the following counterexample shows.

Counterexample 1. *Let $\Omega = \{a, b, c\}$, and consider the lower prevision $\underline{P}$ defined on the singleton $\{f\}$ by $\underline{P}(f) = 1$, where $f(a) = 0$, $f(b) = 1$, $f(c) = 2$. The natural extension $\underline{E}_{\underline{P}}$ of $\underline{P}$, defined on the set $\mathscr{L}$ of all gambles on Ω (obviously a lattice), is the coherent lower prevision given by*

$$\underline{E}_{\underline{P}}(g) = \min\left\{ g(b), g(c), \frac{g(a) + g(c)}{2} \right\}$$

for all gambles g on Ω. The restriction of $\underline{E}_{\underline{P}}$ to the lattice of $\{0,1\}$-valued gambles (i.e., indicators) on Ω, is a 2-monotone *coherent lower probability, simply because Ω contains only three elements (see Walley [12, p. 58]). However, $\underline{E}_{\underline{P}}$ is* not 2-*monotone: consider the gamble g defined by $g(a) = g(b) = g(c) = 1$, then $1 = \underline{E}_{\underline{P}}(f \vee g) < \underline{E}_{\underline{P}}(f) + \underline{E}_{\underline{P}}(g) - \underline{E}_{\underline{P}}(f \wedge g) = 1 + 1 - 0.5$, which violates the condition for 2-monotonicity.*

Theorem 2. *A linear prevision P on a lattice of gambles is completely monotone and completely alternating.*

Proof. Any linear prevision P is the restriction of some coherent prevision Q on $\mathscr{L}$ (see for instance [13, Theorem 3.4.2]). Now recall that Q is a real linear functional, and apply it to both sides of the following well-known identity (for indicators of events this is known as the *sieve formula*, or *inclusion-exclusion principle*, see [1])

$$\bigvee_{i=1}^{p} f_i = \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} \bigwedge_{i \in I} f_i.$$

to get

$$Q\left(\bigvee_{i=1}^{p} f_i \right) = \sum_{\emptyset \neq I \subseteq \{1,\dots,p\}} (-1)^{|I|+1} Q\left(\bigwedge_{i \in I} f_i \right).$$

So Q is completely monotone, and because in this case condition (ii) in Proposition 1 holds with equality, it is completely alternating as well. Now recall that Q and P coincide on the lattice of gambles $\operatorname{dom} P$, that contains all the suprema and infima in the above expression. $\square$

4 n-Monotone lower probabilities

4.1 Coherence, natural extension to events, and the inner set function

If a lattice of gambles contains only (indicators of) events, we call it a *lattice of events*. A lattice of events is therefore a collection of subsets of Ω that is closed under (finite) intersection and union. If it is also closed under set complementation and contains the empty set $\emptyset$, we call it a *field*. An n-monotone lower prevision on a lattice of events is called an *n-monotone lower probability*. A *completely monotone lower probability* is of course one that is ∞-monotone, or equivalently, p-monotone for all $p \in \mathbb{N}$.

In what follows, we shall assume that both $\emptyset$ and Ω belong to the domain. This simplifies the notation and the proofs of the results that follow. These results can be easily generalised to n-monotone lower probabilities whose domain does not contain Ω.

Let us first study the relationship between n-monotonicity and coherence. Recall that 1-monotonicity is necessary, but not sufficient, for coherence. We shall show in what follows that for $n \geq 2$, n-monotonicity is (up to normalisation) sufficient, but not necessary, for coherence. To this end, we consider the *inner set function* $\underline{P}_*$ associated with a monotone lower probability $\underline{P}$ whose domain $\operatorname{dom}\underline{P}$ is a lattice of events, containing $\emptyset$ and Ω. $\underline{P}_*$ is defined by

$$\underline{P}_*(A) = \sup\{\underline{P}(B) \colon B \in \operatorname{dom}\underline{P} \text{ and } B \subseteq A\},$$

for any $A \subseteq \Omega$. Clearly this inner set function $\underline{P}_*$ is monotone as well, and it coincides with $\underline{P}$ on its domain $\operatorname{dom}\underline{P}$.

Let's first mention some important known results for 2-monotone lower probabilities (recall that any n-monotone lower probability, for $n \geq 2$, is also 2-monotone). A coherent lower probability $\underline{P}$ defined on a lattice of events is 2-monotone if and only if for all A and B in $\operatorname{dom}\underline{P}$:

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B).$$

Walley showed that a 2-monotone lower probability $\underline{P}$ *on a field* is coherent if and only if $\underline{P}(\emptyset) = 0$ and $\underline{P}(\Omega) = 1$ (this is a consequence of [12, Theorem 6.1, p. 55–56]). He also showed that if $\underline{P}$ is a coherent 2-monotone lower probability *on a field*, then its inner set function $\underline{P}_*$ is 2-monotone as well and agrees with the natural extension $\underline{E}_{\underline{P}}$ of $\underline{P}$ on events (see [13, Theorem 3.1.5, p. 125]). In this section, we generalise these results to n-monotone lower probabilities defined on a lattice of events.

First, we prove that the inner set function preserves n-monotonicity; this result is actually due to Choquet [3, Chapter IV, Lemma 18.3] (once it is noted that Choquet's 'interior capacity' coincides with our inner set function). As the proof in Choquet's paper consists of no more than a hint [3, p. 186, ll. 6–9], we work out the details below.

Theorem 3. *Let $n \in \mathbb{N}^*$. If a lower probability $\underline{P}$ defined on a lattice of sets, containing $\emptyset$ and Ω, is n-monotone, its inner set function $\underline{P}_*$ is an n-monotone lower probability as well.*

Proof. Let $p \in \mathbb{N}$, $p \leq n$, and consider arbitrary subsets B, $B_1, \ldots, B_p$ of Ω. Fix $\varepsilon > 0$. Then for each $I \subseteq \{1, \ldots, p\}$ it follows from the definition of $\underline{P}_*$ that there is some D_I in $\operatorname{dom}\underline{P}$ such that $D_I \subseteq B \cap \bigcap_{i \in I} B_i$ and

$$\underline{P}_*\left(B \cap \bigcap_{i \in I} B_i\right) - \varepsilon \leq \underline{P}(D_I) \leq \underline{P}_*\left(B \cap \bigcap_{i \in I} B_i\right); \quad (1)$$

$\underline{P}_*$ is real-valued since $\underline{P}(\emptyset) \leq \underline{P}_* \leq \underline{P}(\Omega)$. Similarly as before, we use the convention that for $I = \emptyset$, the corresponding intersection drops out of the expressions (we let it be equal to Ω). We shall also let the union of an empty class be equal to $\emptyset$. Define, for any $I \subseteq \{1, \ldots, p\}$, $E_I = \bigcup_{I \subseteq J \subseteq \{1,\ldots,p\}} D_J$, then clearly $E_I \in \operatorname{dom}\underline{P}$ and $D_I \subseteq E_I \subseteq B \cap \bigcap_{i \in I} B_i$. Now let $F = E_\emptyset$ and $F_k = E_{\{k\}} \subseteq F$ for $k = 1, \ldots, p$. Then F and all the F_k belong to $\operatorname{dom}\underline{P}$, and we have for any $K \subseteq \{1, \ldots, p\}$ and any $k \in K$ that $E_K \subseteq E_{\{k\}} = F_k \subseteq B \cap B_k$, whence

$$E_K \subseteq \bigcap_{k \in K} F_k = F \cap \bigcap_{k \in K} F_k \subseteq B \cap \bigcap_{k \in K} B_k.$$

Summarising, we find that for every given $\varepsilon > 0$, there are F and F_k in $\operatorname{dom}\underline{P}$, such that for all $I \subseteq \{1, \ldots, p\}$

$$D_I \subseteq F \cap \bigcap_{i \in I} F_i \subseteq B \cap \bigcap_{i \in I} B_i \quad (2)$$

and, using the monotonicity of $\underline{P}_*$ and the fact that it coincides with $\underline{P}$ on its domain $\operatorname{dom}\underline{P}$, since $\underline{P}$ is monotone, we deduce from Eqs. (1) and (2) that

$$\underline{P}_*\left(B \cap \bigcap_{i \in I} B_i\right) - \varepsilon \leq \underline{P}\left(F \cap \bigcap_{i \in I} F_i\right) \leq \underline{P}_*\left(B \cap \bigcap_{i \in I} B_i\right).$$

Consequently, for every $\varepsilon > 0$ we find that

$$\begin{aligned}
&\sum_{I\subseteq\{1,\dots,p\}} (-1)^{|I|}\underline{P}_*\left(B\cap\bigcap_{i\in I}B_i\right)\\
&= \sum_{\substack{I\subseteq\{1,\dots,p\}\\ I \text{ even}}} \underline{P}_*\left(B\cap\bigcap_{i\in I}B_i\right) - \sum_{\substack{I\subseteq\{1,\dots,p\}\\ I \text{ odd}}} \underline{P}_*\left(B\cap\bigcap_{i\in I}B_i\right)\\
&\geq \sum_{\substack{I\subseteq\{1,\dots,p\}\\ I \text{ even}}} \underline{P}\left(F\cap\bigcap_{i\in I}F_i\right)\\
&\qquad - \sum_{\substack{I\subseteq\{1,\dots,p\}\\ I \text{ odd}}} \left[\underline{P}\left(F\cap\bigcap_{i\in I}F_i\right)+\varepsilon\right]\\
&= \sum_{I\subseteq\{1,\dots,p\}} (-1)^{|I|}\underline{P}\left(F\cap\bigcap_{i\in I}F_i\right) - N_p\varepsilon \geq -N_p\varepsilon,
\end{aligned}$$

where $N_p = 2^{p-1}$ is the number of subsets of $\{1,\dots,p\}$ with an odd number of elements, and the last inequality follows from the n-monotonicity of $\underline{P}$. Since this holds for all $\varepsilon > 0$, we find that $\underline{P}_*$ is n-monotone on the lattice of events $\wp(\Omega)$. $\square$

We mentioned in Section 2 that a coherent lower probability on a lattice of events is always (1-)monotone. In Counterexample 1, we showed that a coherent lower prevision that is 2-monotone on all events need not be 2-monotone on all gambles. But at the same time, a lower probability defined on a field of events can be coherent without necessarily being 2-monotone, as Walley shows in [12, p. 51]. Conversely, a 2-monotone lower probability defined on a lattice of events need not be coherent: it suffices to consider any constant lower probability $\underline{P}$ on $\wp(\Omega)$. Below, we give simple necessary and sufficient conditions for the coherence of an n-monotone lower probability, we characterise its natural extension, and we prove that the natural extension of an n-monotone lower probability to all events is still n-monotone.

Proposition 4. *Let $\underline{P}$ be an n-monotone lower probability ($n \in \mathbb{N}^*$, $n \geq 2$) defined on a lattice $\mathscr{S}$ that contains $\emptyset$ and Ω. Then $\underline{P}$ is coherent if and only if $\underline{P}(\emptyset) = 0$ and $\underline{P}(\Omega) = 1$.*

Proof. The conditions are clearly necessary for coherence. Conversely, Theorem 3 implies that the inner set function $\underline{P}_*$ of $\underline{P}$ is also n-monotone, and hence 2-monotone. Now, by Delbaen [7, p. 213], this lower probability is coherent, and consequently so is $\underline{P}$. $\square$

The following proposition relates the natural extension $\underline{E}_{\underline{P}}$ of an n-monotone lower probability $\underline{P}$ with the inner set function $\underline{P}_*$.

Proposition 5. *Let $\underline{P}$ be a coherent n-monotone lower probability ($n \in \mathbb{N}^*$, $n \geq 2$) defined on a lattice of events $\mathscr{S}$ that contains $\emptyset$ and Ω. Then its natural extension $\underline{E}_{\underline{P}}$ restricted to events is an n-monotone lower probability as well, and coincides with the inner set function $\underline{P}_*$ of $\underline{P}$.*

Proof. Consider any $A \subseteq \Omega$. Then for any P in $\mathscr{M}(\underline{P})$,

$$P(A) \geq \sup_{B\subseteq A, B\in\mathscr{S}} P(B) \geq \sup_{B\subseteq A, B\in\mathscr{S}} \underline{P}(B) = \underline{P}_*(A).$$

Since we know that $\underline{E}_{\underline{P}}(A) = \min\{Q(A)\colon Q \in \mathscr{M}(\underline{P})\}$, we deduce that $\underline{E}_{\underline{P}}(A) \geq \underline{P}_*(A)$ for all $A \subseteq \Omega$.

Conversely, let $\underline{P}$ be a coherent n-monotone lower probability on $\mathscr{S}$. From Theorem 3, $\underline{P}_*$ is n-monotone if $\underline{P}$ is, and applying Proposition 4, $\underline{P}_*$ is a coherent extension of $\underline{P}$ to all events. It therefore dominates the natural extension $\underline{E}_{\underline{P}}$ of $\underline{P}$, whence $\underline{E}_{\underline{P}}(A) \leq \underline{P}_*(A)$ for all $A \subseteq \Omega$. $\square$

In particular, the natural extension to all events of a coherent and n-monotone lower probability is also n-monotone. This result will be generalised further on.

4.2 Natural extension to all gambles, and the Choquet integral

Walley [12, p. 56] has shown that the natural extension $\underline{E}_{\underline{P}}$ to all gambles of a coherent 2-monotone lower probability $\underline{P}$ defined on the set $\wp(\Omega)$ of all events, is given by the Choquet functional with respect to $\underline{P}$.

$$\underline{E}_{\underline{P}}(f) = (C)\int f\,\mathrm{d}\underline{P} = \inf f + (R)\int_{\inf f}^{\sup f} G_f^{\underline{P}}(x)\,\mathrm{d}x, \quad (3)$$

where the integral on the right-hand side is a Riemann integral, and the function $G_f^{\underline{P}}$ defined by $G_f^{\underline{P}}(x) = \underline{P}(\{f \geq x\})$, is the *decreasing distribution function of* f with respect to $\underline{P}$; $G_f^{\underline{P}}$ is always bounded and non-increasing, and therefore always Riemann integrable. We have used the common notation $\{f \geq x\}$ for the set

$\{\omega \in \Omega\colon\ f(\omega) \geq x\})$. This tells us that this natural extension is *comonotone additive* on $\mathscr{L}$, because that is a property of any Choquet functional associated with a monotone set function on a field (see [8, Proposition 5.1]): if two gambles f and g are *comonotone*, i.e.,

$$(\forall \omega_1, \omega_2 \in \Omega)(f(\omega_1) < f(\omega_2) \implies g(\omega_1) \leq g(\omega_2)),$$

then $\underline{E}_{\underline{P}}(f+g) = \underline{E}_{\underline{P}}(f) + \underline{E}_{\underline{P}}(g)$.

By Proposition 5, we may assume that a coherent 2-monotone lower probability defined on a lattice of events that contains $\emptyset$ and Ω, is actually defined on all of $\wp(\Omega)$, since we can extend it from the lattice of events $\operatorname{dom}\underline{P}$ to $\wp(\Omega)$ using the inner set function (natural extension) $\underline{P}_*$ associated with $\underline{P}$, which is still 2-monotone. Moreover, the natural extension of $\underline{P}$ to all gambles coincides with the natural extension of $\underline{P}_*$ to all gambles, because of the transitivity property mentioned in Section 2. So Eq. (3) also holds for 2-monotone coherent lower probabilities defined on a lattice of events. We conclude:

Theorem 6. *Let $n \in \mathbb{N}^*$, $n \geq 2$, and let $\underline{P}$ be a coherent n-monotone lower probability defined on a lattice of events that contains both $\emptyset$ and Ω. Then its natural extension $\underline{E}_{\underline{P}}$ to the set $\mathscr{L}$ of all gambles is given by*

$$\underline{E}_{\underline{P}}(f) = (C)\int f\,\mathrm{d}\underline{P}_* = \inf f + (R)\int_{\inf f}^{\sup f} \underline{P}_*(\{f \geq x\})\,\mathrm{d}x.$$

We already know from Theorem 3 that the natural extension $\underline{P}_*$ of $\underline{P}$ to the set of all events is 2-monotone (or more generally n-monotone) as well. This result holds also for the natural extension to gambles.

Theorem 7. *Let $n \in \mathbb{N}^*$, $n \geq 2$. If a coherent lower probability $\underline{P}$ on a lattice of events $\mathscr{S}$, containing $\emptyset$ and Ω, is n-monotone, then its natural extension $\underline{E}_{\underline{P}}$ is n-monotone on the lattice of gambles $\mathscr{L}$.*

Proof. Let $p \in \mathbb{N}$, $p \leq n$, and let f, $f_1, \dots, f_p$ be arbitrary gambles on Ω. Let

$$a = \min\{\inf f, \min_{k=1}^{p} \inf f_k\}\ ,\ b = \max\{\sup f, \max_{k=1}^{p} \sup f_k\}.$$

Consider $I \subseteq \{1,\dots,p\}$ then $a \leq \inf(f \wedge \bigwedge_{i\in I} f_i)$ and $b \geq \sup(f \wedge \bigwedge_{i\in I} f_i)$. It is easily verified that

$$\underline{E}_{\underline{P}}\left(f \wedge \bigwedge_{i\in I} f_i\right) = a + (R)\int_a^b G^{\underline{P}_*}_{f\wedge\bigwedge_{i\in I} f_i}(x)\,\mathrm{d}x.$$

Since it is obvious that for any x in $\mathbb{R}$

$$G^{\underline{P}_*}_{f\wedge\bigwedge_{i\in I} f_i}(x) = \underline{P}_*\left(\{f \geq x\} \cap \bigcap_{i\in I}\{f_i \geq x\}\right),$$

it follows from the n-monotonicity of $\underline{P}_*$ (see Theorem 3) that for all real x

$$\sum_{I\subseteq\{1,\dots,p\}} (-1)^{|I|} G^{\underline{P}_*}_{f\wedge\bigwedge_{i\in I} f_i}(x) \geq 0.$$

If we take the Riemann integral over $[a,b]$ on both sides of this inequality, and recall that $\sum_{I\subseteq\{1,\dots,p\}}(-1)^{|I|} = 0$, we get

$$\sum_{I\subseteq\{1,\dots,p\}} (-1)^{|I|}\underline{E}_{\underline{P}}\left(f \wedge \bigwedge_{i\in I} f_i\right) \geq 0.$$

This tells us that $\underline{E}_{\underline{P}}$ is n-monotone. $\square$

We deduce in particular from this result that given a coherent n-monotone lower probability defined on $\wp(\Omega)$, the lower prevision that we can define on $\mathscr{L}$ by means of its Choquet functional is also n-monotone. Since trivially the converse also holds, we deduce that the Choquet functional respect to a lower probability $\underline{P}$ on $\wp(\Omega)$ is n-monotone if and only if $\underline{P}$ is. This generalises a result by Walley [12, Theorem 6.4].

Corollary 8. *Let $\underline{P}$ be any coherent lower probability defined on a lattice containing both $\emptyset$ and Ω. Let $n \in \mathbb{N}^*$, $n \geq 2$. Then $\underline{P}$ is n-monotone, if and only if $\underline{E}_{\underline{P}}$ is n-monotone, if and only if $(C)\int\cdot\,\mathrm{d}\underline{P}_*$ is n-monotone.*

Proof. If $\underline{P}$ is n-monotone, then $\underline{E}_{\underline{P}}$ is n-monotone by Theorem 7.

If $\underline{E}_{\underline{P}}$ is n-monotone, then $\underline{P}$ is n-monotone since $\underline{E}_{\underline{P}}$ is an extension of $\underline{P}$ (because $\underline{P}$ is coherent), and so, by Theorem 6, $\underline{E}_{\underline{P}}$ must coincide with $(C)\int\cdot\,\mathrm{d}\underline{P}_*$, which must be therefore n-monotone as well.

Finally, if $(C)\int\cdot\,\mathrm{d}\underline{P}_*$ is n-monotone, then $\underline{P}_*$ must be n-monotone since $(C)\int\cdot\,\mathrm{d}\underline{P}_*$ is an extension of $\underline{P}_*$. But, $\underline{P}_*$ is also an extension of $\underline{P}$ (because $\underline{P}$ is coherent), so, $\underline{P}$ is n-monotone as well. This completes the chain. $\square$

5 Representation results

Let us now focus on the notion of n-monotonicity we have given for lower previsions. If $\underline{P}$ is a monotone lower prevision on a lattice of gambles that contains all constant gambles, then its *inner extension* $\underline{P}_*$ is given by

$$\underline{P}_*(f) = \sup\{\underline{P}(g) \colon g \in \operatorname{dom}\underline{P} \text{ and } g \leq f\}. \qquad (4)$$

for all gambles f on Ω. Clearly this inner extension is monotone as well, and it coincides with $\underline{P}$ on its domain $\operatorname{dom}\underline{P}$. The following result generalises Theorem 3; their proofs are completely analogous.

Theorem 9. *Let $n \in \mathbb{N}^*$. If a lower prevision $\underline{P}$ defined on a lattice of gambles that contains all constant gambles is n-monotone, its inner extension $\underline{P}_*$ is n-monotone on $\mathscr{L}$.*

We now investigate whether a result akin to Theorem 7 holds for *n*-monotone lower previsions: when will the natural extension of a coherent *n*-monotone lower prevision be *n*-monotone? For Theorem 7, we needed the domain of the lower probability to be a lattice of events containing $\emptyset$ and Ω. It turns out that for our generalisation we also have to impose a similar condition on the domain: it will have to be a linear lattice containing all constant gambles. Recall that a subset $\mathscr{K}$ of $\mathscr{L}$ is called a *linear lattice* if $\mathscr{K}$ is closed under point-wise addition and scalar multiplication with real numbers, and moreover closed under point-wise minimum $\wedge$ and point-wise maximum $\vee$.

Consider a coherent lower prevision whose domain is a linear lattice of gambles $\mathscr{K}$ that contains all constant gambles. Then its natural extension to the set of all gambles $\mathscr{L}$ is precisely its inner extension $\underline{P}_*$ (see Walley [13, Theorem 3.1.4]). This leads at once to the following theorem, which is a counterpart of Theorem 7 for *n*-monotone lower previsions.

Theorem 10. *Let $n \in \mathbb{N}^*$. If a coherent lower prevision $\underline{P}$ defined on a linear lattice of gambles that contains all constant gambles is n-monotone, then its natural extension $\underline{E}_{\underline{P}}$ to $\mathscr{L}$ is equal to its inner extension $\underline{P}_*$, and is therefore n-monotone on the lattice of gambles $\mathscr{L}$.*

Counterexample 1 tells us that this result cannot be extended to lattices that are not linear spaces.

We have not made any mention yet of the Choquet integral in relation to the natural extension. It turns out that there is also a relationship between both concepts. Consider a linear lattice of gambles $\mathscr{K}$ that contains all constant gambles. Then the set $\mathscr{F}_{\mathscr{K}} = \{A \subseteq \Omega \colon I_A \in \mathscr{K}\}$ of events that belong to $\mathscr{K}$ is a field of subsets of Ω. Let us denote by $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$ the uniformly closed linear lattice

$$\mathscr{L}_{\mathscr{F}_{\mathscr{K}}} = \operatorname{cl}(\operatorname{span}(I_{\mathscr{F}_{\mathscr{K}}})),$$

where of course $I_{\mathscr{F}_{\mathscr{K}}} = \{I_A \colon I_A \in \mathscr{K}\}$, 'cl' denotes uniform closure, and 'span' the linear span. $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$ contains all constant gambles as well. We call its elements *$\mathscr{F}_{\mathscr{K}}$-measurable gambles*. Every $\mathscr{F}_{\mathscr{K}}$-measurable gamble is a uniform limit of *$\mathscr{F}_{\mathscr{K}}$-simple gambles*, *i.e.*, elements of $\operatorname{span}(I_{\mathscr{F}_{\mathscr{K}}})$. Moreover, $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}} \subseteq \operatorname{cl}(\mathscr{K})$.

Theorem 11. *Let $\underline{P}$ be an n-monotone coherent lower prevision on a linear lattice of gambles $\mathscr{K}$ that contains all constant gambles. This lower prevision has a unique coherent extension (its natural, or inner, extension) $\underline{E}_{\underline{P}}$ to $\operatorname{cl}(\mathscr{K})$, and this extension is n-monotone as well. Denote by $\underline{Q}$ the restriction of $\underline{P}$ to $\mathscr{F}_{\mathscr{K}}$. Then for all f in $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$,*

$$\underline{E}_{\underline{P}}(f) = \underline{E}_{\underline{Q}}(f) = (C)\int f \,\mathrm{d}\underline{Q}_* \\ = \inf f + (R)\int_{\inf f}^{\sup f} \underline{Q}_*(\{f \geq x\})\,\mathrm{d}x.$$

Consequently, $\underline{E}_{\underline{P}}$ is both n-monotone and comonotone additive on $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$.

Proof. Let us first show that $\underline{P}$ has a unique coherent extension to $\operatorname{cl}(\mathscr{K})$. Let $\underline{P}'$ be any coherent extension. There is at least one coherent extension, namely its natural extension, which we denote by $\underline{E}_{\underline{P}}$. We show that $\underline{P}'$ and $\underline{E}_{\underline{P}}$ coincide on $\operatorname{cl}(\mathscr{K})$. Consider any element h in $\operatorname{cl}(\mathscr{K})$. Then there is a sequence g_n of gambles in $\mathscr{K}$ that converges uniformly to h. Since both $\underline{P}'$ and $\underline{E}_{\underline{P}}$ coincide with $\underline{P}$ on $\mathscr{K}$, and are continuous on their domain $\operatorname{cl}(\mathscr{K})$, because they are coherent, we find that

$$\underline{P}'(h) = \lim_{n\to\infty} \underline{P}'(g_n) = \lim_{n\to\infty} \underline{P}(g_n) = \lim_{n\to\infty} \underline{E}_{\underline{P}}(g_n) = \underline{E}_{\underline{P}}(h).$$

Since $\underline{P}$ is *n*-monotone and coherent, its restriction $\underline{Q}$ to the field $\mathscr{F}_{\mathscr{K}}$ is an *n*-monotone and coherent lower probability. By Theorem 6, the natural extension $\underline{E}_{\underline{Q}}$ of $\underline{Q}$ to the set $\mathscr{L}$ of all gambles is the Choquet functional associated with the *n*-monotone inner set function $\underline{Q}_*$ of $\underline{Q}$: for any gamble f on Ω,

$$\underline{E}_{\underline{Q}}(f) = (C)\int f \,\mathrm{d}\underline{Q}_* = \inf f + (R)\int_{\inf f}^{\sup f} \underline{Q}_*(\{f \geq x\})\,\mathrm{d}x.$$

To prove that the coherent lower previsions $\underline{E}_{\underline{Q}}$ and $\underline{E}_{\underline{P}}$ coincide on the subset $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$ of $\operatorname{cl}(\mathscr{K})$, it suffices to prove that $\underline{E}_{\underline{Q}}$ and $\underline{P}$ coincide on $\operatorname{span}(I_{\mathscr{F}_{\mathscr{K}}})$, since the lower previsions $\underline{E}_Q$ and $\underline{E}_P$ are guaranteed by coherence to be continuous, and since $\underline{E}_P$ and $\underline{P}$ coincide on $\mathrm{span}(I_{\mathscr{F}_{\mathscr{K}}}) \subseteq \mathscr{K}$, because $\underline{P}$ is coherent on $\mathscr{K}$. Let therefore h be any element of $\mathrm{span}(I_{\mathscr{F}_{\mathscr{K}}})$, *i.e.*, h is an $\mathscr{F}_{\mathscr{K}}$-simple gamble. Then we can always find $n \geq 1$, real μ_1, real non-negative $\mu_2, \dots, \mu_n$, and nested $F_2 \supseteq \cdots \supseteq F_n$ such that $h = \mu_1 + \sum_{k=2}^n \mu_k I_{F_k}$. It then follows from the comonotone additivity of the Choquet integral that $\underline{E}_Q(h) = \mu_1 + \sum_{k=2}^n \mu_k \underline{Q}(F_k)$. On the other hand, it follows from the coherence and the 2-monotonicity of $\underline{P}$ that

$$\begin{aligned}\underline{P}(h) &= \mu_1 + \underline{P}\left(\sum_{k=2}^n \mu_k I_{F_k}\right)\\ &= \mu_1 - \mu_2 + \underline{P}\left(\sum_{k=2}^n \mu_k I_{F_k}\right) + \underline{P}(\mu_2)\\ &\leq \mu_1 - \mu_2 + \underline{P}\left(\mu_2 \vee \sum_{k=2}^n \mu_k I_{F_k}\right) + \underline{P}\left(\mu_2 \wedge \sum_{k=2}^n \mu_k I_{F_k}\right).\end{aligned}$$

Now it is easily verified that $\mu_2 \vee \sum_{k=2}^n \mu_k I_{F_k} = \mu_2 + \sum_{k=3}^n \mu_k I_{F_k}$ and $\mu_2 \wedge \sum_{k=2}^n \mu_k I_{F_k} = \mu_2 I_{F_2}$, and consequently, again using the coherence and the 2-monotonicity of $\underline{P}$, the fact that $\underline{Q}$ coincides with $\underline{P}$ on $\mathscr{F}_{\mathscr{K}}$, and continuing in the same fashion,

$$\begin{aligned}\underline{P}(h) &\leq \mu_1 - \mu_2 + \underline{P}\left(\mu_2 + \sum_{k=3}^n \mu_k I_{F_k}\right) + \underline{P}(\mu_2 I_{F_2})\\ &= \mu_1 + \mu_2 \underline{Q}(F_2) + \underline{P}\left(\sum_{k=3}^n \mu_k I_{F_k}\right)\\ &\leq \mu_1 + \mu_2 \underline{Q}(F_2) + \mu_3 \underline{Q}(F_3) + \underline{P}\left(\sum_{k=4}^n \mu_k I_{F_k}\right)\\ &\ \ \vdots\\ &\leq \mu_1 + \sum_{k=2}^n \mu_k \underline{Q}(F_k).\end{aligned}$$

This tells us that $\underline{E}_Q(h) \geq \underline{P}(h)$. On the other hand, since $\underline{P}$ is a coherent extension of $\underline{Q}$, and since the natural extension $\underline{E}_Q$ is the point-wise smallest coherent extension of $\underline{Q}$, we also find that $\underline{E}_Q(h) \leq \underline{P}(h)$. This tells us that $\underline{P}$ and $\underline{E}_Q$ indeed coincide on $\mathrm{span}(I_{\mathscr{F}_{\mathscr{K}}})$. □

Walley has shown in [13] that in general a coherent lower prevision is not determined by the values it assumes on events. But the preceding theorem tells us that for coherent lower previsions that are 2-monotone and defined on a sufficiently rich domain, we can somewhat improve upon this negative result: on $\mathscr{F}_{\mathscr{K}}$-measurable gambles, at least the natural extension $\underline{E}_P$ of the n-monotone coherent $\underline{P}$ is completely determined by the values that $\underline{P}$ assumes on the events in $\mathscr{F}_{\mathscr{K}}$. Nevertheless, the following counterexample tells us that we cannot expect to take this result beyond the set $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$ of $\mathscr{F}_{\mathscr{K}}$-measurable gambles.

Counterexample 2. *Let Ω be the closed unit interval $[0,1]$ in $\mathbb{R}$, and let $\underline{P}$ be the lower prevision on the lattice $\mathscr{K}$ of all continuous gambles on Ω, defined by $\underline{P}(f) = f(0)$ for any f in $\mathscr{K}$. Since $\underline{P}$ is actually a linear prevision, it must be completely monotone (see Theorem 2). Observe that $\mathscr{K}$ is a uniformly closed linear lattice that contains all constant gambles. Moreover, $\mathscr{F}_{\mathscr{K}} = \{\emptyset, \Omega\}$, so $\mathscr{L}_{\mathscr{F}_{\mathscr{K}}}$ is the set of all constant gambles, and the natural extension $\underline{E}_Q$ of the restriction $\underline{Q}$ of $\underline{P}$ to $\mathscr{F}_{\mathscr{K}}$ is the vacuous lower prevision on $\mathscr{L}$: $\underline{E}_Q(f) = \inf f$ for all gambles f on Ω. Therefore, for any g in $\mathscr{K}$ such that $g(0) > \inf g$, it follows that $\underline{E}_Q(g) < \underline{P}(g)$: the equality in Theorem 11 holds only for those gambles in $\mathscr{K}$ that satisfy $g(0) = \inf g$.*

So we conclude that an n-monotone and coherent lower prevision $\underline{P}$ defined on a linear lattice of gambles that contains the constant gambles, cannot generally be written (on its entire domain) as a Choquet functional associated with its restriction $\underline{Q}$ to events. The following theorem is therefore quite surprising, as it tells us that, for a lower prevision is defined on a sufficiently rich domain, 2-monotonicity and comonotone additivity of a lower prevision are equivalent under coherence. As a consequence, 2-monotone coherent lower previsions $\underline{P}$ on such domains can indeed *always* be represented on their entire domain by a Choquet integral (but not necessarily with respect to the inner set function of the restriction $\underline{Q}$ of $\underline{P}$ to events).

Theorem 12. *Let $\underline{P}$ be a coherent lower prevision on a linear lattice of gambles that contains all constant gambles. Then $\underline{P}$ is comonotone additive if and only if it is 2-monotone, and in both cases we have for all f in $\mathrm{dom}\,\underline{P}$*

$$\underline{P}(f) = (C)\int f \,\mathrm{d}\underline{P}_* = \inf f + (R)\int_{\inf f}^{\sup f} \underline{P}_*(\{f \geq x\})\,\mathrm{d}x.$$

Proof. Let us first prove the direct implication. Assume that $\underline{P}$ is comonotone additive. Let us define $\mathscr{K}_+ := \{f \in \operatorname{dom}\underline{P} \colon f \geq 0\}$, and let $\underline{P}_+$ be the restriction of $\underline{P}$ to $\mathscr{K}_+$. This lower prevision is also coherent and comonotone additive, and it is defined on a class of non-negative gambles. Moreover, given f in $\mathscr{K}_+$ and $a \geq 0$, the gambles af, $f \wedge a$ and $f - f \wedge a$ belong to $\mathscr{K}_+$ because $\operatorname{dom}\underline{P}$ is a linear lattice that contains the constant gambles and all the above gambles are trivially non-negative. Hence, we may apply Greco's representation theorem (see [8, Theorem 13.2], the conditions (iv) and (v) there are trivially satisfied because all elements in $\mathscr{K}_+$ are bounded), and conclude that there is a monotone set function μ on $\wp(\Omega)$ with $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$ such that for all f in $\mathscr{K}_+$:

$$\underline{P}_+(f) = (C)\int f\,\mathrm{d}\mu.$$

Consider now any f in $\operatorname{dom}\underline{P}$. Since f is bounded, and coherence implies that $\underline{P}(f+a) = \underline{P}(f) + a$ for all a in $\mathbb{R}$, this also implies that $\inf f + \underline{P}_+(f - \inf f) = \underline{P}(f)$, whence

$$\underline{P}(f) = \inf f + (C)\int [f - \inf f]\,\mathrm{d}\mu = (C)\int f\,\mathrm{d}\mu. \quad (5)$$

It follows from the proof of Greco's representation theorem (see [8, Theorem 13.2]) that we can actually assume μ to be defined as the restriction of $\underline{P}_*$ to events:

$$\mu(A) = \underline{P}_*(A) = \sup\{\underline{P}(f) \colon f \leq I_A \text{ and } f \in \operatorname{dom}\underline{P}\} \quad (6)$$

for all $A \subseteq \Omega$. By Theorem 10, μ is also equal to the restriction to events of the natural extension $\underline{E}_{\underline{P}} = \underline{P}_*$ of $\underline{P}$. Let us consider $A \subseteq B \subseteq \Omega$, and show that $\underline{E}_{\underline{P}}(I_A + I_B) = \underline{E}_{\underline{P}}(I_A) + \underline{E}_{\underline{P}}(I_B) = \mu(A) + \mu(B)$. Since the coherence of $\underline{E}_{\underline{P}}$ implies that it is superadditive, we only need to prove that $\underline{E}_{\underline{P}}(I_A + I_B) \leq \mu(A) + \mu(B)$. Given $\varepsilon > 0$, we deduce from Eq. (4) that there is some f in $\operatorname{dom}\underline{P}$ such that $f \leq I_A + I_B$ and $\underline{E}_{\underline{P}}(I_A + I_B) \leq \underline{P}(f) + \varepsilon$. We may assume without loss of generality that f is non-negative [because $f \vee 0$ belongs to $\operatorname{dom}\underline{P}$ and satisfies the same inequality]. Let us define $g_1 = f \wedge 1$ and $g_2 = f - f \wedge 1$. These gambles belong to the linear lattice $\operatorname{dom}\underline{P}$. Moreover, $g_1 + g_2 = f$. Let us show that $g_1 \leq I_B$ and $g_2 \leq I_A$. If $\omega \notin B$, we have $0 \leq f(\omega) \leq (I_A + I_B)(\omega) = 0$ whence $g_1(\omega) = g_2(\omega) = 0$. If on the other hand $\omega \in A$, there are two possibilities: if $f(\omega) \leq 1$, then $g_2(\omega) = 0$ and $g_1(\omega) = f(\omega) \leq 1$. If on the other hand $f(\omega) > 1$, then $g_1(\omega) = 1$ and $g_2(\omega) = f(\omega) - 1 \leq 2 - 1 = 1$. Finally, if $\omega \in B \setminus A$, we have $f(\omega) \leq 1$, whence $g_1(\omega) = f(\omega) \leq 1$ and $g_2(\omega) = 0$.

Moreover, g_1 and g_2 are comonotone: consider any ω_1 and ω_2 in Ω, and assume that $g_2(\omega_1) < g_2(\omega_2)$. Then $g_2(\omega_2) > 0$ and consequently $\omega_2 \in A$ and $f(\omega_2) > 1$. This implies in turn that indeed $g_1(\omega_2) = 1 \geq g_1(\omega_1)$. Hence, since $\underline{P}$ is assumed to be comonotone additive,

$$\begin{aligned}\underline{E}_{\underline{P}}(I_A + I_B) &\leq \underline{P}(f) + \varepsilon = \underline{P}(g_1 + g_2) + \varepsilon \\ &= \underline{P}(g_1) + \underline{P}(g_2) + \varepsilon \leq \underline{E}_{\underline{P}}(A) + \underline{E}_{\underline{P}}(B) + \varepsilon,\end{aligned}$$

and since this holds for all $\varepsilon > 0$ we deduce that indeed $\underline{E}_{\underline{P}}(I_A + I_B) \leq \underline{E}_{\underline{P}}(A) + \underline{E}_{\underline{P}}(B) = \mu(A) + \mu(B)$.

Now consider two arbitrary subsets C and D of Ω. Then $C \cap D \subseteq C \cup D$, and consequently

$$\begin{aligned}&\mu(C \cup D) + \mu(C \cap D) = \underline{E}_{\underline{P}}(I_{C\cup D} + I_{C\cap D}) \\ &\quad = \underline{E}_{\underline{P}}(I_C + I_D) \geq \underline{E}_{\underline{P}}(I_C) + \underline{E}_{\underline{P}}(I_D) = \mu(C) + \mu(D),\end{aligned}$$

taking into account that $\underline{E}_{\underline{P}}$ is superadditive (because it is coherent). We conclude that μ is 2-monotone on $\wp(\Omega)$. From Proposition 4, we conclude that μ is a coherent lower probability on $\wp(\Omega)$, so by Theorem 6, its natural extension is the Choquet functional associated with μ, and is therefore equal to $\underline{P}$, by Eq. (5). If we now apply Theorem 7, we see that the coherent lower prevision $\underline{P}$ given by $\underline{P}(f) = (C)\int f\,\mathrm{d}\mu$ for all f in $\operatorname{dom}\underline{P}$ is also 2-monotone.

We now prove the converse implication. Assume that $\underline{P}$ is 2-monotone. Then, applying Theorems 9 and 10, its natural extension $\underline{E}_{\underline{P}} = \underline{P}_*$ to all gambles is also 2-monotone, and consequently so is its restriction μ to events. Moreover, $\mathscr{L}_{\wp(\Omega)} = \mathscr{L}$, because any gamble is the uniform limit of some sequence of simple gambles. If we now apply Theorem 11, we see that $\underline{E}_{\underline{P}}(f) = (C)\int f\,\mathrm{d}\mu$ for all f in $\mathscr{L}$. Consequently, $\underline{E}_{\underline{P}}$ is comonotone additive, because the Choquet functional associated with a monotone lower probability is, and so is therefore $\underline{P}$. $\square$

Hence, the natural extension of an n-monotone ($n \geq 2$) and coherent lower prevision defined on a linear lattice of gambles that contains the constant gambles is always comonotone additive. Indeed, this natural extension is the Choquet functional associated to its restriction to events.

Corollary 13. *Let $n \in \mathbb{N}^*$, $n \geq 2$, and let $\underline{P}$ be an n-monotone coherent lower prevision defined on a linear lattice that contains all constant gambles. Then $\underline{E}_{\underline{P}}$ is n-monotone, is comonotone additive, and is equal to the Choquet integral with respect to $\underline{P}_*$ (restricted to events).*

Moreover, such a lower prevision is generally not uniquely determined by its restriction to events, but it is uniquely determined by the values that its natural extension $\underline{E}_{\underline{P}} = \underline{P}_*$ assumes on events. Of course, this natural extension also depends in general on the values that $\underline{P}$ assumes on gambles, as is evident from Eq. (6).

An n-monotone ($n \geq 2$) coherent lower probability $\underline{P}$ on $\wp(\Omega)$, which usually has many coherent extensions to $\mathscr{L}$, has actually *only one* 2-*monotone* coherent extension to $\mathscr{L}$. Of course, this unique 2-monotone coherent extension coincides with the natural extension of $\underline{P}$.

Corollary 14. *Let $n \in \mathbb{N}^*$, $n \geq 2$. An n-monotone coherent lower probability defined on all events has a* unique 2-*monotone (or equivalently, comonotone additive) coherent extension to all gambles, that is furthermore automatically also n-monotone.*

Proof. Let $\underline{P}$ be an n-monotone lower probability defined on all events. By Theorem 7, its natural extension $\underline{E}_{\underline{P}}$ to $\mathscr{L}$ is an n-monotone, and hence, 2-monotone coherent extension of $\underline{P}$. The proof is complete if we can show that $\underline{E}_{\underline{P}}$ is the only 2-monotone coherent extension of $\underline{P}$.

So, let $\underline{Q}$ be any 2-monotone coherent extension of $\underline{P}$. We show that $\underline{Q} = \underline{E}_{\underline{P}}$. Let f be any gamble on Ω, then

$$\underline{Q}(f) = (C)\int f \, \mathrm{d}\underline{Q} = (C)\int f \, \mathrm{d}\underline{P} = \underline{E}_{\underline{P}}(f),$$

where the first equality follows from Corollary 13, the second equality holds because $\underline{Q}$ coincides with $\underline{P}$ on events, and the third one follows by applying Theorem 6. This establishes uniqueness. □

Next, we give a couple of properties that relate comonotone additivity (or, equivalently, 2-monotonicity) of coherent lower previsions to properties of their sets of dominating linear previsions.

Proposition 15. *Let $\underline{P}$ be a coherent lower prevision on a linear lattice of gambles. Consider its set of dominating linear previsions $\mathscr{M}(\underline{P})$.*

(a) If $\underline{P}$ is comonotone additive on its domain then for all comonotone f and g in $\operatorname{dom}\underline{P}$, *there is some P in $\mathscr{M}(\underline{P})$ such that $P(f) = \underline{P}(f)$ and $P(g) = \underline{P}(g)$.*

(b) Assume in addition that $\operatorname{dom}\underline{P}$ *contains all constant gambles. Then $\underline{P}$ is comonotone additive (or equivalently* 2-*monotone) on its domain if and only if for all comonotone f and g in* $\operatorname{dom}\underline{P}$, *there is some P in $\mathscr{M}(\underline{P})$ such that $P(f) = \underline{P}(f)$ and $P(g) = \underline{P}(g)$.*

Proof. To prove the first statement, assume that $\underline{P}$ is comonotone additive on its domain, and consider f and g in $\operatorname{dom}\underline{P}$ that are comonotone. Then $f+g$ also belongs to $\operatorname{dom}\underline{P}$, so we know that $\underline{P}(f+g) = \underline{P}(f) + \underline{P}(g)$. On the other hand, since $\underline{P}$ is coherent, there is some P in $\mathscr{M}(\underline{P})$ such that $\underline{P}(f+g) = P(f+g) = P(f) + P(g)$. So $P(f) + P(g) = \underline{P}(f) + \underline{P}(g)$ and since we know that $\underline{P}(f) \leq P(f)$ and $\underline{P}(g) \leq P(g)$, this implies that $\underline{P}(f) = P(f)$ and $\underline{P}(g) = P(g)$.

The 'only if' part of the second statement is an immediate consequence of the first. To prove the 'if' part, consider arbitrary comonotone f and g in $\operatorname{dom}\underline{P}$. Then $f \vee g$ and $f \wedge g$ are comonotone as well, and belong to $\operatorname{dom}\underline{P}$, so by assumption there is a P in $\mathscr{M}(\underline{P})$ such that $\underline{P}(f \wedge g) = P(f \wedge g)$ and $\underline{P}(f \vee g) = P(f \vee g)$. Then

$$\begin{aligned}\underline{P}(f \vee g) + \underline{P}(f \wedge g) &= P(f \vee g) + P(f \wedge g) \\ &= P(f) + P(g) \geq \underline{P}(f) + \underline{P}(g).\end{aligned}$$

This tells us that $\underline{P}$ is 2-monotone, and by Theorem 12 also comonotone additive, on its domain. □

As a corollary, we deduce the following result, apparently first proven by Walley [12, Cors. 6.4 and 6.5, p. 57].

Corollary 16. *Let $\underline{P}$ be a coherent lower probability on a lattice of events. Consider its set of dominating linear previsions $\mathscr{M}(\underline{P})$. Then $\underline{P}$ is* 2-*monotone if and only if for all A and B in* $\operatorname{dom}\underline{P}$ *such that $A \subseteq B$, there is some P in $\mathscr{M}(\underline{P})$ such that $P(A) = \underline{P}(A)$ and $P(B) = \underline{P}(B)$.*

Proof. We just show that the direct implication is a consequence of the previous results; the converse follows easily by applying the condition to $A \cap B \subseteq A \cup B$, for A and B in $\operatorname{dom}\underline{P}$. Let $\underline{P}$ be a coherent lower prevision defined on a lattice of events, that is moreover 2-monotone. By Theorem 7, the natural extension $\underline{E}_{\underline{P}}$ of $\underline{P}$ to all gambles is 2-monotone and coherent. Hence, given

$A \subseteq B \in \operatorname{dom} \underline{P}$, since I_A and I_B are comonotone, Proposition 15 implies the existence of a P in $\mathscr{M}(\underline{E}_{\underline{P}}) = \mathscr{M}(\underline{P})$ such that $P(A) = \underline{E}_{\underline{P}}(A) = \underline{P}_*(A) = \underline{P}(A)$ and $P(B) = \underline{E}_{\underline{P}}(B) = \underline{P}_*(B) = \underline{P}(B)$. $\square$

6 Conclusions

The results in this paper show that there is no real reason to restrict the notion of n-monotonicity to lower probabilities. In fact, it turns out that it is fairly easy, and completely within the spirit of Choquet's original definition, to define and study this property for lower previsions. And in fact, we have shown above that doing this does not lead to just another generalisation of something that existed before, but that it leads to genuinely new insights. One important conclusion that may be drawn from our results is that, under coherence, 2-monotonicity of a lower prevision is actually equivalent to comonotone additivity, and therefore to being representable as a Choquet functional (see Theorem 12 for a precise formulation).

We have presented our results for coherent lower previsions, which are positively homogeneous, super-additive functionals that satisfy a normalisation condition. Our results can be easily generalised to situations where normalisation isn't important, which is the case, for instance, with Maaß's so-called *exact functionals* [11]. Moreover, the material presented above allows us to claim that most (if not all) of the lower integrals defined in the literature are actually completely monotone, and therefore representable as Choquet functionals. Due to limitations of space, we could not discuss these additional results here, but we do intend to report on them elsewhere.

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