Random intervals as a model for imprecise information *

Enrique Miranda†, Inés Couso ‡ and Pedro Gil ‡

Abstract

Random intervals constitute one of the classes of random sets with a greater number of applications. In this paper, we regard them as the imprecise observation of a random variable, and study how to model the information about the probability distribution of this random variable. Two possible models are the probability distributions of the measurable selections and those bounded by the upper probability. We prove that, under some hypotheses, the closures of these two sets in the topology of the weak convergence coincide, improving results from the literature. Moreover, we provide examples showing that the two models are not equivalent in general, and give sufficient conditions for the equality between them. Finally, we comment on the relationship between random intervals and fuzzy numbers.

Keywords: Random sets, random intervals, measurable selections, Dempster-Shafer upper and lower probabilities, fuzzy numbers, weak convergence.

1 Introduction

Random set theory has been applied in such different fields as economy ([20]), stochastic geometry ([27]) or when dealing with imprecise information ([26]). Within random sets, random intervals are especially interesting, as the works carried out in [9, 11, 24] show. One of their advantages respect to other types of random sets is their easy interpretation as a model for uncertainty and imprecision. Consider a probability space $(\Omega, \mathcal{A}, P)$ and a random variable $U_0 : \Omega \to \mathbb{R}$ modeling some behaviour of the elements of Ω. Due to some imprecision in the observation of the values $U_0(\omega)$, or to the existence of missing data, we may

*This is an updated and expanded version, with new results, of a paper [28] presented at SEIO'03, the 27th Spanish Conference on Statistics and Operations Research.

†Corresponding author. Rey Juan Carlos University, Dep. of Informatics, Statistics and Telematics. C-Tulipán, s/n, 28933 Móstoles (Madrid, Spain). Tel: (+34) 914888132, Fax: (+34) 914887338, e-mail: enrique.miranda@urjc.es

‡University of Oviedo, Dep. of Statistics and Operations Research. e-mail addresses: couso@pinon.ccu.uniovi.es (I. Couso), pedro@uniovi.es (P. Gil)

not know precisely the images of the elements from Ω by U_0. A possible model for this situation would be to give, for any ω in Ω, upper and lower bounds of its image by U_0 (i.e., a margin for error in the observation); we obtain then an interval $\Gamma(\omega) = [A(\omega), B(\omega)]$ which we assume is certain to include the value $U_0(\omega)$.

Given this model, we can study which is the information conveyed by the multi-valued mapping Γ about the probability distribution of the random variable U_0. On the one hand, we know that U_0 belongs to the class of the random variables whose values are included in the images of the multi-valued mapping. Thus, its probability distribution belongs to the class of the probability distributions of these random variables. We will denote this class by $P(\Gamma)$. On the other hand, the probability induced by U_0 is bounded between the *upper and lower probabilities* of Γ. These two functions were introduced by Dempster in 1967 ([8]; see also the previous work by Strassen in [36]). We will denote them by P^* and P_*, respectively, and will denote the class of the probabilities bounded between them by $M(P^*)$. Hence, we can consider two models of the available information: the class of probability distributions of the *measurable selections* of the random set, $P(\Gamma)$, and the set of probabilities bounded between the upper and the lower probability of the random set, $M(P^*)$. The first of these two models is the most precise we can consider with the available information, so $P(\Gamma) \subseteq M(P^*)$; however, the class $M(P^*)$ is more interesting from an operational point of view, because it is convex, closed in some cases, and is uniquely determined by the values of P^*. The goal of this paper is to study the relationship between these two models.

The paper is organized as follows: in Section 2, we introduce some concepts and notations that we will use in the rest of the paper. In Section 3, we recall some useful results from the literature and study the relationship between the classes $P(\Gamma)$ and $M(P^*)$. In Section 4, we establish sufficient conditions for the equality between these two sets of probabilities, first for random closed intervals and later for random open intervals. Section 5 contains some comments on the connection between random intervals and fuzzy numbers. Finally, in Section 6 we give our conclusions and open problems on the subject.

2 Preliminary concepts

Let us introduce the notation we will use throughout the paper. We will denote a probability space by $(\Omega, \mathcal{A}, P)$, a measurable space $(X, \mathcal{A}')$ and a multi-valued mapping, $\Gamma : \Omega \to \mathcal{P}(X)$. $\mathcal{N}_P$ will denote the class of null sets respect to a probability P, and δ_x will denote the degenerate probability distribution on a point x. Given a topological space (X,τ), β_X will denote its Borel σ-field, that is, the σ-field generated by the open sets. In particular, $\beta_{\mathbb{R}}$ will denote the Borel σ-field on $\mathbb{R}$, and given $A \in \beta_{\mathbb{R}}$, β_A will denote the relative σ-field on A. On the other hand, λ will denote the Lebesgue measure on $\beta_{\mathbb{R}}$, and λ_A will denote the restriction of λ to β_A. Given a random variable $U : \Omega \to \mathbb{R}$, $F_U : \mathbb{R} \to [0,1]$ will denote its distribution function, and $P_U : \beta_{\mathbb{R}} \to [0,1]$, its induced probability. A set of probabilities will be called $\mathcal{W}$-compact (resp. $\mathcal{W}$-closed) when it is compact (resp., closed) in the topology of the weak convergence. A multi-valued mapping will be called compact (resp., closed, open) when $\Gamma(\omega)$ is a compact (resp., closed, open) subset of X for every $\omega \in \Omega$. Most of the multi-valued mappings to appear in this paper will take values on $(\mathbb{R}, \beta_{\mathbb{R}})$; nevertheless, we will also consider the case where the final space is Polish.

Definition 2.1. *[25] A topological space* (X,τ) *is called* **Polish** *if it is metrizable for some metric* d *such that* (X,d) *is complete and separable.*

Formally, a random set is a multi-valued mapping satisfying some measurability condition. Most of the conditions appearing in the literature (see for instance [21]) use the concepts of upper and lower inverse:

Definition 2.2. *[34] Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* $(X, \mathcal{A}')$ *a measurable space and* $\Gamma : \Omega \to \mathcal{P}(X)$ *a multi-valued mapping. Given* $A \in \mathcal{A}'$*, its* **upper inverse** *by* Γ *is* $\Gamma^*(A) := \{\omega \in \Omega \mid \Gamma(\omega) \cap A \neq \emptyset\}$*, and its* **lower inverse***,* $\Gamma_*(A) := \{\omega \in \Omega \mid \emptyset \neq \Gamma(\omega) \subseteq A\}$*.*

In this paper, Γ will be a model of the imprecise observation of a random variable $U_0 : \Omega \to X$ (which we will call **original random variable**), in the sense that, given $\omega \in \Omega$, all we know about the value $U_0(\omega)$ is that it belongs to the set $\Gamma(\omega)$. We deduce from Definition 2.2 that $\Gamma_*(A) \subseteq U_0^{-1}(A) \subseteq \Gamma^*(A)$ for any A in the final σ-field. $\Gamma^*(A)$ is the smallest superset of $U_0^{-1}(A)$ we can give, taking into account the available information, whereas $\Gamma_*(A)$ is the greatest subset of $U_0^{-1}(A)$ that we can give. We will denote $A^* := \Gamma^*(A)$ and $A_* = \Gamma_*(A)$ when no confusion is possible.

Definition 2.3. *[34] Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* $(X, \mathcal{A}')$ *a measurable space and* $\Gamma : \Omega \to \mathcal{P}(X)$ *a multi-valued mapping.* Γ *is said to be* **strongly measurable** *when* $\Gamma^*(A) \in \mathcal{A}$ *for any* $A \in \mathcal{A}'$*.*

Taking into account that $\Gamma^*(A) = (\Gamma_*(A^c))^c \; \forall A \in \mathcal{A}'$, a strongly measurable multi-valued mapping also satisfies $\Gamma_*(A) \in \mathcal{A}$ for any A in the final σ-field.

Definition 2.4. *[8] Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(X, \mathcal{A}')$ a measurable space, and let $\Gamma : \Omega \to \mathcal{P}(X)$ be a strongly-measurable multi-valued mapping. The* **upper probability** *induced by Γ is given by*

$$\begin{array}{rcl} P^*_\Gamma : \mathcal{A}' & \longrightarrow & [0,1] \\ A & \hookrightarrow & \dfrac{P(\Gamma^*(A))}{P(\Gamma^*(X))}, \end{array}$$

and the **lower probability***, by*

$$\begin{array}{rcl} P_{*\Gamma} : \mathcal{A}' & \longrightarrow & [0,1] \\ A & \hookrightarrow & \dfrac{P(\Gamma_*(A))}{P(\Gamma^*(X))}. \end{array}$$

When Γ is not strongly measurable, it is not possible to define the upper and lower probabilities on the whole of the final σ-field. Since these functions are one of the main points of interest in this paper, we will assume that Γ is a strongly measurable multi-valued mapping, and call it then a **random set**. We will denote $P^* := P^*_\Gamma$ and $P_* := P_{*\Gamma}$ when there is no ambiguity about the random set inducing the upper and lower probabilities. These functions are conjugate (i.e., $P^*(A) = 1 - P_*(A^c)\ \forall A \in \mathcal{A}'$), because of the duality existing between the upper and lower inverses. On the other hand, we will assume throughout the paper that $\Gamma(\omega)$ is non-empty for all ω, because it includes at least the value $U_0(\omega)$. As a consequence, $P^*_\Gamma(A) = P(A^*), P_{*\Gamma}(A) = P(A_*)\ \forall A \in \mathcal{A}'$. As Nguyen proved in [34], the upper probability of a random set is lower continuous and ∞-alternating, while the lower probability is upper continuous and ∞-monotone.

We are now ready to introduce the problem that we will study in this paper. Let Γ be a random set modelling the imprecise observation of a random variable U_0. Then, all we know about this random variable is that it belongs to the class

$$S(\Gamma) := \{V : \Omega \to X \text{ measurable} \mid V(\omega) \in \Gamma(\omega)\ \forall\ \omega\}.$$

The elements of $S(\Gamma)$ are called **measurable selections** of the random set Γ[1]. In particular, the probability distribution of U_0 belongs to

$$P(\Gamma) := \{P_V \mid V \in S(\Gamma)\},$$

[1]Some authors ([1, 19]) prefer to work with *almost everywhere* selections, that is, measurable mappings V such that $V(\omega) \in \Gamma(\omega)$ almost surely; however, the interpretation we have given to Γ as a model for the imprecise observation of U_0 forces us to restrict our attention to measurable mappings whose images are included in those of Γ for all the elements of the initial space.

and the probability that the values of U_0 belong to $A \in \mathcal{A}'$ is an element of

$$P(\Gamma)(A) := \{P_V(A) \mid V \in S(\Gamma)\}.$$

On the other hand, the information given by Γ can also be modeled through the upper and lower probabilities. For any $V \in S(\Gamma)$ and for any $A \in \mathcal{A}'$, it is $\Gamma_*(A) \subseteq V^{-1}(A) \subseteq \Gamma^*(A)$. Hence, $P(\Gamma)(A)$ is included in the interval $[P_*(A), P^*(A)]$ for all $A \in \mathcal{A}'$, and if we define

$$M(P^*) := \{Q : \mathcal{A}' \to [0,1] \text{ probability} \mid Q(A) \leq P^*(A) \ \forall\ A \in \mathcal{A}'\},$$

we have $P(\Gamma) \subseteq M(P^*)$. We will refer to $M(P^*)$ as the class of probabilities **dominated** by the upper probability, or **credal set** generated by P^*. This is a convex set, and is uniquely determined by the upper probability.

If we are to use a random set as a model for imprecise information, it is interesting to see which is the best way of summarizing the probabilistic information it conveys. From an operational point of view, it is preferable to work with $M(P^*)$: the class $P(\Gamma)$ is not convex in general, and it doesn't have an easy representation in terms of a function. However, as we showed in [29], in some cases the class $P(\Gamma)$ can be way more precise than $M(P^*)$. Our goal in this paper is to study the relationship between these two sets of probabilities when Γ is a random interval. This problem has been studied for other types of random sets ([4, 5, 18, 19, 30]), but, as far as we are aware, never for random intervals.

3 Relationships between $P(\Gamma)$ and $M(P^*)$

In this section, we are going to study the relationships between the sets of probabilities $P(\Gamma)$ and $M(P^*)$ induced by a random interval Γ. Although the term 'random interval' usually means a multi-valued mapping whose images are intervals of the real line, in this paper it will refer to random sets of the type (A, B) or $[A, B]$, with $A, B : \Omega \to \mathbb{R}$. At the end of the paper we will give a brief account of the properties of random sets of the type $(A, B]$ or $[A, B)$, for $A, B : \Omega \to \mathbb{R}$.

Definition 3.1. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* $A, B : \Omega \to \mathbb{R}$. *The* **random closed interval** *of extremes* A *and* B *is given by*

$$\begin{aligned} \Gamma : \Omega &\to \mathcal{P}(\mathbb{R}) \\ \omega &\hookrightarrow [A(\omega), B(\omega)]. \end{aligned}$$

and the **random open interval** *of extremes A, B,*

$$\begin{aligned} \Gamma' : \Omega &\rightarrow \mathcal{P}(\mathbb{R}) \\ \omega &\hookrightarrow (A(\omega), B(\omega)). \end{aligned}$$

We are assuming in this paper that the multi-valued mapping Γ has non-empty images, because $U_0(\omega)$ belongs to $\Gamma(\omega)$ for all ω. As a consequence, given a random closed interval $[A, B]$, it should be $A(\omega) \leq B(\omega)$ for all ω in the initial space, and given a random open interval (A, B), we must assume $A(\omega) < B(\omega)\ \forall\omega$. Hence, there are choices of A, B such that the random closed interval $[A, B]$ is a possible model for our problem but the random open interval (A, B) is not. Although this seems to prevent us from making a simultaneous study of random open and random closed intervals, some of the properties we shall prove can be derived easily even if $\Gamma(\omega) = \emptyset$ for some ω; moreover, we find it simpler, from a mathematical point of view, to prove them simultaneously for both types of random intervals. This is for instance the case of our next result, where we characterize the strong measurability of a random interval. This characterization is interesting because, although the strong measurability is necessary to work with the upper and lower probabilities, it does not always have a straightforward interpretation in terms of the images of the multi-valued mapping. In the case of random intervals, we are going to show that it amounts to the measurability of the variables A and B determining the lower and upper bounds of the images of U_0[2]:

Theorem 3.1. *Consider a probability space $(\Omega, \mathcal{A}, P)$, and let $A, B : \Omega \rightarrow \mathbb{R}$ such that $A \leq B$. Let $\Gamma = [A, B]$, $\Gamma_1 = (A, B)$. The following statements are equivalent:*

(a) Γ *is strongly measurable.*

(b) A, B *are measurable.*

(c) Γ_1 *is strongly measurable.*

Proof:

$(a) \Rightarrow (b)$ Given $x \in \mathbb{R}$, $A^{-1}((-\infty, x]) = \Gamma^*((-\infty, x]) \in \mathcal{A}$ and $B^{-1}((-\infty, x]) = \Gamma_*((-\infty, x]) \in \mathcal{A}$, taking into account that Γ is strongly measurable. Hence, A and B are measurable.

[2] A similar result by Wasserman can be found in [17, Lemma 6.1], although his result applies to the composition of non-negative measurable mappings with random closed intervals.

$(b) \Rightarrow (c)$ Consider $C \in \beta_{\mathbb{R}}$. Then, (C, d) is a separable metric space, whence there exists a countable set $D \subseteq C$ dense in C. Given $\omega \in \Gamma_1^*(C)$, the intersection $\Gamma_1(\omega) \cap C \neq \emptyset$ is a non-empty open set in (C, β_C). Hence, $(\Gamma_1(\omega) \cap C) \cap D = \Gamma_1(\omega) \cap D \neq \emptyset$, because D is dense, and as a consequence $\omega \in \Gamma_1^*(D)$. Thus, $\Gamma_1^*(C) = \Gamma_1^*(D) = \cup_{x \in D} \Gamma_1^*(\{x\}) = \cup_{x \in D}(A^{-1}(-\infty, x) \cap B^{-1}(x, \infty))$. This last set is measurable, for it is a countable union of measurable sets, and this implies that Γ_1 is strongly measurable.

$(c) \Rightarrow (a)$ Assume finally that Γ_1 is strongly measurable. Then, A, B are also measurable: for any $x \in \mathbb{R}$, $A^{-1}(-\infty, x) = \Gamma_1^*(-\infty, x)$ and $B^{-1}(-\infty, x] = \Gamma_{1*}(-\infty, x)$. Now, for any $C \in \beta_{\mathbb{R}}$,

$$\begin{aligned}\Gamma^*(C) = \{\omega \mid \Gamma(\omega) \cap C \neq \emptyset\} &= \{\omega \mid (\Gamma_1 \cup \{A, B\})(\omega) \cap C \neq \emptyset\} \\ = \{\omega \mid \Gamma_1(\omega) \cap C \neq \emptyset\} &\cup \{\omega \mid A(\omega) \in C\} \cup \{\omega \mid B(\omega) \in C\} \\ &= \Gamma_1^*(C) \cup A^{-1}(C) \cup B^{-1}(C) \in \mathcal{A}.\end{aligned}$$

We conclude that Γ is strongly measurable. ■

Remark 3.1. *Although this is not relevant for the problem studied in the paper, this theorem implies the equivalence between a number of measurability conditions in the case of random intervals. The relationships between these conditions for other types of random sets were studied by Himmelberg and others in [21, 22]. Using Theorem 3.1, we can prove that, if Γ is a random (closed or open) interval, the strong measurability is equivalent to the so-called weak-measurability, $\mathcal{C}$-measurability and measurability.* ♦

Next, we recall the main results established in the literature about the sets $P(\Gamma)$ and $M(P^*)$ that hold in particular for random intervals. Most of them have been proven for more general types of random sets, and, as we will show, can be improved when the images of the random set are intervals of the real line. We start with random closed intervals. They constitute a particular case of compact random sets on Polish spaces, which satisfy the following properties:

Theorem 3.2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, (X, τ) a Polish space and $\Gamma : \Omega \to \mathcal{P}(X)$, a compact random set. Then,*

1. *P^* is continuous for decreasing sequences of compact sets.*
2. *$P^*(A) = \sup_{K \subseteq A compact} P^*(K) = \inf_{A \subseteq G open} P^*(G)\ \forall A \in \beta_X$.*
3. *$M(P^*)$ is $\mathcal{W}$-compact.*

4. $P^*(A) = \max P(\Gamma)(A)\ \forall A \in \beta_X$.

5. $M(P^*) = \overline{Conv(P(\Gamma))}$.

The first point of this theorem was proven in [5]. Together with some results from [23], it implies points 2 and 3. On the other hand, the equality $P^*(A) = \sup P(\Gamma)(A)\ \forall A \in \beta_X$ was given in [6]. In [29], we showed that $P(\Gamma)(A)$ has indeed a maximum and a minimum value. These two facts together imply the fourth point. Finally, point 5 was proven in [4]. Although this last point establishes a link between $P(\Gamma)$ and $M(P^*)$, this is not a very strong one, as the following example shows:

Example 3.1. *Consider* $\omega_0 \in \mathbb{R}$*, the probability space* $(\{\omega_0\}, \{\{\omega_0\}, \emptyset\}, \delta_{\omega_0})$ *and let us define* $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$ *by* $\Gamma(\omega_0) = [0,1]$*. Then,* $P(\Gamma) = \{\delta_x \mid x \in [0,1]\}$*. On the other hand,* $P^*(A) = 1\ \forall A \in \beta_{\mathbb{R}}$ *s.t.* $A \cap [0,1] \neq \emptyset$*, whence* $M(P^*) = \{Q : \beta_{\mathbb{R}} \to [0,1]$ *probability* $\mid Q([0,1]) = 1\}$*.* ♦

Let us summarize now the properties of random open intervals that have been established in the literature.

Theorem 3.3. *[29] Let* $(\Omega, \mathcal{A}, P)$ *be a probability space,* (X,d) *a separable metric space and* $\Gamma : \Omega \to \mathcal{P}(X)$*, an open random set. Then,*

1. $P^*(A) = \sup_{J \subseteq A \text{ finite}} P^*(J)\ \forall A \in \beta_X$.

2. $P^*(A) = \max P(\Gamma)(A)\ \forall A \in \beta_X$.

Although the results summarized in Theorems 3.2 and 3.3 are interesting in their own right, we would like to know if there exists a stronger relationship between $M(P^*)$ and $P(\Gamma)$: as Example 3.1 shows, the equality $M(P^*) = \overline{Conv(P(\Gamma))}$, which holds when Γ is a random closed interval, does not prevent the existence of an important difference of precision between $P(\Gamma)$ and $M(P^*)$; in the case of random open intervals, we do not even know whether that equality holds. We are going to prove that, given a random (closed or open) interval defined on a non-atomic probability space, the closures of $M(P^*)$ and $P(\Gamma)$ under the topology of the weak convergence coincide. We will use some ideas from [31].

Let $\mathbb{Q} := \{q_1, q_2, \dots\}$ denote the set of the rational numbers. Fix $n \in \mathbb{N}$, and let $\mathcal{Q}_n$ be the field generated by $\{(-\infty, q_1], \dots, (-\infty, q_n]\}$. Assume, for the sake of simplicity, that the elements $\{q_1, \dots, q_n\}$ satisfy $q_1 < q_2 < \dots < q_n$.[3]

[3] This is done merely to simplify the notation in our further development, but is not essential to the construction. Note also that we are only assuming it for the natural number n fixed.

Then, $\mathcal{Q}_n$ coincides with the field generated by

$$\mathcal{D}_n := \{(-\infty, q_1], (q_1, q_2], \ldots, (q_{n-1}, q_n], (q_n, \infty)\},$$

which is the class of the (finite) unions of elements from $\mathcal{D}_n$. Let us denote $\mathcal{D}_n = \{E_1, \ldots, E_{n+1}\}$, and let us define

$$P(\Gamma)_n := \{P_V \restriction_{\mathcal{Q}_n} | \; V \in S(\Gamma)\}$$

and

$$M(P^*)_n := \{Q \restriction_{\mathcal{Q}_n} | \; Q \in M(P^*)\},$$

the classes of the restrictions of the elements of $P(\Gamma)$ and $M(P^*)$ to the field $\mathcal{Q}_n$. Each of these restrictions is uniquely determined by its values in the class $\mathcal{D}_n$, because a probability measure is additive. We are going to prove that $P(\Gamma)_n$ and $M(P^*)_n$ coincide when the initial probability space is non-atomic. This will mean that, for every natural number n and for any $Q \in M(P^*)$, there exists $P_n \in P(\Gamma)$ such that Q and P_n coincide on the sets $(-\infty, q_1], \ldots, (-\infty, q_n]$. Our proof requires the following lemma:

Lemma 3.4. *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space, and let $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$ be a random interval. Then, $P(\Gamma)_n$ is convex.*

Proof: Consider $U_1, U_2 \in S(\Gamma), \alpha \in (0,1)$, and let us prove the existence of $U \in S(\Gamma)$ such that $P_U = \alpha P_{U_1} + (1-\alpha) P_{U_2}$ on $\mathcal{Q}_n$. Consider the measurable partition of Ω given by $\{C_{ij} \mid i, j = 1, \ldots, n+1\}$, where $C_{ij} = U_1^{-1}(E_i) \cap U_2^{-1}(E_j)$, and let $\beta_{ij} = P(C_{ij})$. The non-atomicity of $(\Omega, \mathcal{A}, P)$ implies the existence, for all $i, j = 1, \ldots, n+1$, of a measurable set $D_{ij} \subseteq C_{ij}$ s.t. $P(D_{ij}) = \alpha \beta_{ij}$. Take $D = \cup_{i=1}^{n+1} \cup_{j=1}^{n+1} D_{ij}$, and define

$$U := U_1 I_D + U_2 I_{D^c}.$$

- Taking into account that U_1, U_2 are measurable selections of Γ, we deduce that $U(\omega) \in \Gamma(\omega)$ for all $\omega \in \Omega$.
- Given $F \in \beta_{\mathbb{R}}, U^{-1}(F) = (U_1^{-1}(F) \cap D) \cup (U_2^{-1}(F) \cap D^c) \in \mathcal{A}$, because $D \in \mathcal{A}$ and U_1, U_2 are measurable. Hence, U belongs to $S(\Gamma)$.

- Fix $i \in \{1, \dots, n+1\}$. Then,

$$P_U(E_i) = P(U_1^{-1}(E_i) \cap D) + P(U_2^{-1}(E_i) \cap D^c) = \sum_{j=1}^{n+1} P(C_{ij} \cap D)$$
$$+ \sum_{l=1}^{n+1} P(C_{li} \cap D^c) = \sum_{j=1}^{n+1} P(D_{ij}) + \sum_{l=1}^{n+1} [P(C_{li}) - P(D_{li})] = \sum_{j=1}^{n+1} \alpha\beta_{ij}$$
$$+ \sum_{l=1}^{n+1} [\beta_{li} - \alpha\beta_{li}] = \alpha \sum_{j=1}^{n+1} \beta_{ij} + (1-\alpha) \sum_{l=1}^{n+1} \beta_{li} = \alpha P_{U_1}(E_i) + (1-\alpha) P_{U_2}(E_i).$$

  Using the additivity of P_U, we deduce that $P_U(G) = \alpha P_{U_1}(G) + (1-\alpha) P_{U_2}(G)$ for all $G \in \mathcal{Q}_n$.

We conclude that the class $P(\Gamma)_n$ is convex. ■

Let us define the multi-valued mapping

$$\begin{array}{rcl} \Gamma' : \Omega & \longrightarrow & \mathcal{P}(\mathcal{D}_n) \\ \omega & \hookrightarrow & \{E_i \mid \Gamma(\omega) \cap E_i \neq \emptyset\}. \end{array}$$

It is strongly measurable respect to the σ-fields $\mathcal{A}$ and $\mathcal{P}(\mathcal{D}_n)$: given $I \subseteq \{1, \dots, n+1\}$, it is ${\Gamma'}^*(\{E_i \mid i \in I\}) = \{\omega \mid \exists i \in I, E_i \in \Gamma'(\omega)\} = \{\omega \mid \exists i \in I, \Gamma(\omega) \cap E_i \neq \emptyset\} = \{\omega \mid \Gamma(\omega) \cap (\cup_{i \in I} E_i) \neq \emptyset\} = \Gamma^*(\cup_{i \in I} E_i) \in \mathcal{A}$. Moreover, it takes values on a space, $\mathcal{D}_n$, with a finite number of elements. Hence, we may apply the properties established in [8, 30] for random sets on finite spaces. Let $\pi \in S^{n+1}$ be a permutation, and let Q_π be the probability measure on $\mathcal{P}(\mathcal{D}_n)$ satisfying

$$Q_\pi(\{E_{\pi(1)}, \dots, E_{\pi(j)}\}) = P^*_{\Gamma'}(\{E_{\pi(1)}, \dots, E_{\pi(j)}\}) \ \forall j = 1, \dots, n+1. \quad (1)$$

(Q_π is completely determined on $\mathcal{P}(\mathcal{D}_n)$ by these equations because of its additivity). Then ([8, 30]), $Ext(M(P^*_{\Gamma'})) = \{Q_\pi \mid \pi \in S^{n+1}\}$ and $M(P^*_{\Gamma'}) = Conv(\{Q_\pi \mid \pi \in S^{n+1}\})$. These considerations allow us to establish the following result:

Lemma 3.5. *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space, and let $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$ be a random interval. Then, $M(P^*)_n = P(\Gamma)_n$.*

Proof: It is clear that $P(\Gamma)_n \subseteq M(P^*)_n$. Conversely, consider $Q \in M(P^*)$, and let us define a probability Q_1 on $\mathcal{P}(\mathcal{D}_n)$ by the equations

$$Q_1(\{E_j\}) = Q(E_j) \ \forall j = 1, \dots, n+1. \quad (2)$$

Q_1 belongs to $M(P^*_{\Gamma'})$: given $I \subseteq \{1,\dots,n+1\}$, $Q_1(\{E_i\}_{i\in I}) = Q(\cup_{i\in I} E_i) \leq P^*_\Gamma(\cup_{i\in I} E_i) = P^*_{\Gamma'}(\{E_i\}_{i\in I})$. Let us prove that there exists $P' \in P(\Gamma)_n$ s.t. $P'(E_j) = Q_1(\{E_j\})\ \forall j = 1,\dots,n+1$. Assume first that Q_1 is an extreme point of $M(P^*_{\Gamma'})$, i.e., that there exists some $\pi \in S^{n+1}$ s.t. Q_1 is equal to the probability Q_π defined by Equation (1). From Theorems 3.2 and 3.3, given $j \in \{1,\dots,n+1\}$, there exists $V_j \in S(\Gamma)$ such that $P_{V_j}(\cup_{i=1}^j E_{\pi(i)}) = P^*_\Gamma(\cup_{i=1}^j E_{\pi(i)}) = P^*_{\Gamma'}(\{E_{\pi(1)},\dots,E_{\pi(j)}\})$. Let us denote $F_j = V_j^{-1}(\cup_{i=1}^j E_{\pi(i)})$, and define $U_\pi : \Omega \to \mathbb{R}$ by

$$U_\pi = V_1 I_{F_1} + \sum_{i=2}^{n+1} V_i I_{F_i \setminus \cup_{j=1}^{i-1} F_j}.$$

- $F_{n+1} = V_{n+1}^{-1}(\cup_{i=1}^{n+1} E_{\pi(i)}) = V_{n+1}^{-1}(\mathbb{R}) = \Omega$. Hence, U_π is well-defined.
- It is a measurable selection of Γ, because it is a measurable finite combination of measurable selections.
- Consider $j \in \{1,\dots,n+1\}$. Then, $P_{U_\pi}(\cup_{i=1}^j E_{\pi(i)})$

$$= P(V_1^{-1}(\cup_{i=1}^j E_{\pi(i)}) \cap F_1) + \sum_{k=2}^{n+1} P(V_k^{-1}(\cup_{i=1}^j E_{\pi(i)}) \cap [F_k \setminus \cup_{l=1}^{k-1} F_l])$$

$$\geq P(V_1^{-1}(\cup_{i=1}^j E_{\pi(i)}) \cap F_1) + \sum_{k=2}^{j} P(V_k^{-1}(\cup_{i=1}^j E_{\pi(i)}) \cap [F_k \setminus \cup_{l=1}^{k-1} F_l])$$

$$= P(F_1) + \sum_{k=2}^{j} P(F_k \setminus \cup_{l=1}^{k-1} F_l) = P(F_1 \cup \dots \cup F_j) \geq P(F_j)$$

$$= P^*_\Gamma(E_{\pi(1)} \cup \dots \cup E_{\pi(j)}) = P^*_{\Gamma'}(\{E_{\pi(1)},\dots,E_{\pi(j)}\}).$$

  On the other hand, $P_{U_\pi}(\cup_{i=1}^j E_{\pi(i)}) \leq P^*_\Gamma(\cup_{i=1}^j E_{\pi(i)}) = P^*_{\Gamma'}(\{E_{\pi(i)}\}_{i=1}^j)$, because $U_\pi \in S(\Gamma)$. Therefore, $P_{U_\pi}(\cup_{i=1}^j E_{\pi(i)}) = P^*_{\Gamma'}(\{E_{\pi(i)}\}_{i=1}^j) = Q_1(\{E_{\pi(i)}\}_{i=1}^j)\ \forall j = 1,\dots,n+1$.

Now, if Q_1 is not an extreme point of $M(P^*_{\Gamma'})$, there exist $l \geq 2, \lambda_1,\dots,\lambda_l \geq 0, \sum_{i=1}^l \lambda_i = 1, \pi_1,\dots,\pi_l \in S^{n+1}$ such that $Q_1 = \sum_{i=1}^l \lambda_i Q_{\pi_i}$, where Q_{π_i} is the extreme point of $M(P^*_{\Gamma'})$ defined by (1). Then,

$$Q_1(\{E_j\}) = \sum_{i=1}^l \lambda_i Q_{\pi_i}(\{E_j\}) = \sum_{i=1}^l \lambda_i P_{U_{\pi_i}}(E_j) = \left(\sum_{i=1}^l \lambda_i P_{U_{\pi_i}}\right)(E_j)\ \forall j.$$

From Lemma 3.4, $P(\Gamma)_n$ is convex, whence there exists $W \in S(\Gamma)$ s.t.

$$P_W(E_j) = \left(\sum_{i=1}^l \lambda_i P_{U_{\pi_i}}\right)(E_j) = Q_1(\{E_j\}) = Q(E_j)\ \forall j = 1,\dots,n+1.$$

We deduce that $Q \in P(\Gamma)_n$, and this implies that $P(\Gamma)_n = M(P_n^*)$. ■

Let us prove now that, if the initial probability space is non-atomic, the $\mathcal{W}$-closures of $P(\Gamma)$ and $M(P^*)$ coincide.

Theorem 3.6. *Consider a non-atomic probability space* $(\Omega, \mathcal{A}, P)$ *and a random interval* $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$. *Then,* $\overline{M(P^*)} = \overline{P(\Gamma)}$.

Proof: It is clear that $\overline{P(\Gamma)} \subseteq \overline{M(P^*)}$. Conversely, consider $Q \in M(P^*)$. For any natural number n, the restriction of Q to the field $\mathcal{Q}_n$ belongs to $M(P^*)_n = P(\Gamma)_n$, whence there exists $P_n \in P(\Gamma)$ such that $P_n((-\infty, q_i]) = Q((-\infty, q_i])$ for all $i = 1, \ldots, n$. Consider the sequence $\{P_m\}_m \subseteq P(\Gamma)$. For any rational number q, the sequence $\{P_m((-\infty, q])\}_m$ converges to $Q((-\infty, q])$, because it is constant on this value after some natural m_q by construction. Hence ([2]), $\{P_m\}_m$ converges weakly to Q. We deduce that $M(P^*) \subseteq \overline{P(\Gamma)} \subseteq \overline{M(P^*)}$, whence $\overline{M(P^*)} = \overline{P(\Gamma)}$. ■

Remark 3.2. *The reader can find similar results in [18, 33]; nevertheless, these results assume either the completeness of the initial probability space or work with almost everywhere selections, something we do not consider in our case. Moreover, the work in [33] is related to the notion of* selectionable *distribution from [1], which allows to change the initial probability space while keeping the upper probability, something not possible in our context. Another interesting study on this subject can be found in [19], in that case with the canonical* σ*-field generated by* Γ *on the initial space, that is, the smallest* σ*-field that makes* Γ *strongly measurable.* ♦

If Γ is in particular a random closed interval, we deduce from the third point of Theorem 3.2 that $M(P^*) = \overline{P(\Gamma)}$. This is interesting because the difference between a set of probabilities and its closure will in general be much smaller than the one existing respect to its closed convex hull; hence, when the initial probability space is non-atomic, the difference of precision between $P(\Gamma)$ and $M(P^*)$ will not be too big, and in particular we will not have situations like the one given by Example 3.1. Next, we check that the previous theorem does not hold for arbitrary initial probability spaces:

Example 3.2. *Consider the random interval from Example 3.1, where* $\Omega = \{\omega_0\}$ *and* $\Gamma(\omega_0) = [0, 1]$. *We showed then that* $M(P^*) = \{Q : \beta_{\mathbb{R}} \to [0, 1] \text{ prob.} \mid Q([0, 1]) = 1\}$ *and* $P(\Gamma) = \{\delta_x \mid x \in [0, 1]\}$. *From Theorem 3.2,* $M(P^*)$ *is* $\mathcal{W}$*-closed. On the other hand, given a sequence of degenerate probabilities* $\{\delta_{x_n}\}_n$*, it can only converge weakly to another degenerate probability* δ_x*, with* $\lim_n x_n = x$. *Hence,* $P(\Gamma)$ *is also* $\mathcal{W}$*-closed and* $\overline{P(\Gamma)} \neq \overline{M(P^*)}$. ♦

Next, we prove that given a random open interval, the closure or $M(P^*)$ coincides with the closed convex hull of $P(\Gamma)$. We use the classes $\mathcal{Q}_n$ and $\mathcal{D}_n = \{E_1, \dots, E_{n+1}\}$ and the random set Γ' from our previous results.

Theorem 3.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, Γ a random open interval. Then, $\overline{M(P^*)} = \overline{Conv(P(\Gamma))}$.*

Proof: It is clear that $\overline{Conv(P(\Gamma))} \subseteq \overline{M(P^*)}$, because $M(P^*)$ is convex. Conversely, consider $Q \in M(P^*)$. Fix $n \in \mathbb{N}$, and let us define a probability measure Q_1^n on $\mathcal{P}(\mathcal{D}_n)$ by Equation (2). Then, Q_1^n belongs to $M(P^*_{\Gamma'}) = Conv(Ext(M(P^*_{\Gamma'})))$. From Lemma 3.5, given an extreme point Q_π of $M(P^*_{\Gamma'})$ there exists $P_{U_\pi} \in P(\Gamma)$ such that $P_{U_\pi}(E_j) = Q_\pi(\{E_j\}) \ \forall j = 1, \dots, n+1$[4]. Hence, given $Q_1^n \in Conv(Ext(M(P^*_{\Gamma'})))$, there exists $P_n \in Conv(P(\Gamma))$ such that $P_n(E_j) = Q_1^n(\{E_j\}) = Q(E_j) \ \forall j = 1, \dots, n+1$. Now, reasoning as in Theorem 3.6, it can be checked that the sequence of probability measures $\{P_n\}_n \subseteq Conv(P(\Gamma))$ converges weakly to Q. Therefore, $M(P^*) \subseteq \overline{Conv(P(\Gamma))} \subseteq \overline{M(P^*)}$, whence $\overline{M(P^*)} = \overline{Conv(P(\Gamma))}$. ∎

Summarizing, we have showed that, given a random interval Γ, the closure of $M(P^*)$ is the closed convex hull of $P(\Gamma)$ and, if the initial probability space is non-atomic, then it coincides with $\overline{P(\Gamma)}$. We are going to show next that, even in this last case, the sets $P(\Gamma)$ and $M(P^*)$ do not coincide in general.

Example 3.3. *Consider the probability space $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$, the random variables $A, B : [0,1] \to \mathbb{R}$ given by $A(\omega) = -\omega, B(\omega) = \omega \ \forall \omega$, and let $\Gamma = [A, B]$. Consider $Q = \frac{P_A + P_B}{2}$. It is clear that Q belongs to $M(P^*)$, because $P_A, P_B \in P(\Gamma) \subseteq M(P^*)$ and this set is convex. Let us show that it does not belong to $P(\Gamma)$. Assume ex-absurdo that there exists $V \in S(\Gamma)$ such that $P_V = Q$. Then, given $C = V^{-1}([0,1]) \in \beta_{[0,1]}$, it is $\lambda_{[0,1]}(C) = P_V([0,1]) = Q([0,1]) = 0.5$. Let us define $\mathcal{H} = \{D \in \beta_{[0,1]} \mid \lambda_{[0,1]}(D \cap C) = \frac{\lambda_{[0,1]}(D)}{2}\}$. We are going to prove that $\mathcal{H} = \beta_{[0,1]}$. For this, we are going to show that $\mathcal{H}$ contains the class $\mathcal{C} = \{[x,1] \mid x \in [0,1]\}$ and that it is a σ-field.*

- *It is clear that $\emptyset \in \mathcal{H}$. On the other hand, $\lambda_{[0,1]}(C) = 0.5 = \frac{\lambda_{[0,1]}([0,1])}{2}$, whence $[0,1] \in \mathcal{H}$.*
- *Given $x \in (0,1]$,*

$$V^{-1}([x,1]) \subseteq V^{-1}([0,1]) \cap \Gamma^*([x,1]) = C \cap [x,1] \Rightarrow$$
$$P_V([x,1]) = Q([x,1]) = \frac{\lambda_{[0,1]}([x,1])}{2} \leq \lambda_{[0,1]}(C \cap [x,1]).$$

[4]Although Lemma 3.5 assumes the non-atomicity of the initial probability space, it can be checked that this condition is not necessary for this particular property.

Similarly,

$$\begin{aligned} V^{-1}([-1,-x]) &\subseteq V^{-1}([-1,0)) \cap \Gamma^*([-1,-x]) = C^c \cap [x,1] \\ &\Rightarrow P_V([-1,-x]) = Q([-1,-x]) = \frac{\lambda_{[0,1]}([-1,-x])}{2} \\ &= \frac{\lambda_{[0,1]}([x,1])}{2} \leq \lambda_{[0,1]}(C^c \cap [x,1]). \end{aligned}$$

Now,

$$\begin{aligned} 1-x &= \lambda_{[0,1]}([x,1]) = \lambda_{[0,1]}([x,1] \cap C) + \lambda_{[0,1]}([x,1] \cap C^c) \\ &\geq \frac{\lambda_{[0,1]}([x,1])}{2} + \frac{\lambda_{[0,1]}([x,1])}{2} = 1 - x \Rightarrow \frac{\lambda_{[0,1]}([x,1])}{2} \\ &= \lambda_{[0,1]}([x,1] \cap C) = \lambda_{[0,1]}([x,1] \cap C^c). \end{aligned}$$

This shows that the class $\mathcal{C}$ is included in $\mathcal{H}$. Let us show now that $\mathcal{H}$ also includes the field generated by $\mathcal{C}$. Consider the following classes:

- $\mathcal{C}_1 := \{D, D^c \mid D \in \mathcal{C}\}$. *Given* $x \in [0,1]$,

  $$\begin{aligned} \lambda_{[0,1]}([0,x) \cap C) &= \lambda_{[0,1]}(C) - \lambda_{[0,1]}([x,1] \cap C) \\ &= 0.5 - \frac{1-x}{2} = \frac{x}{2} = \frac{\lambda_{[0,1]}([0,x))}{2}. \end{aligned}$$

  Hence, $\mathcal{C}_1 \subseteq \mathcal{H}$.

- $\mathcal{C}_2 = \{D_1 \cap \cdots \cap D_n \mid D_i \in \mathcal{C}_1\} = \{[0,x), [x,1], [x_1,x_2) \mid x, x_1, x_2 \in [0,1], x_1 < x_2\}$. *Given* $x_1 < x_2 \in [0,1]$,

  $$\begin{aligned} \lambda_{[0,1]}([x_1,x_2) \cap C) &= \lambda_{[0,1]}([x_1,1] \cap C) - \lambda_{[0,1]}([x_2,1] \cap C) \\ &= \frac{1-x_1}{2} - \frac{1-x_2}{2} = \frac{x_2 - x_1}{2}, \end{aligned}$$

  whence $[x_1,x_2) \in \mathcal{H}$. *Hence,* $\mathcal{C}_2 \subseteq \mathcal{H}$.

- $\mathcal{C}_3 := \{D_1 \cup \cdots \cup D_n \mid D_i \in \mathcal{C}_2 \ \forall i, D_i \cap D_j = \emptyset \ \forall i \neq j\}$. *Given* $D_1, \ldots, D_n$ *pairwise disjoint in* $\mathcal{H}$,

  $$\begin{aligned} \lambda_{[0,1]}((\cup_{i=1}^n D_i) \cap C) &= \sum_{i=1}^n \lambda_{[0,1]}(D_i \cap C) \\ &= \sum_{i=1}^n \frac{\lambda_{[0,1]}(D_i)}{2} = \frac{\lambda_{[0,1]}(\cup_{i=1}^n D_i)}{2}. \end{aligned}$$

  Hence, $\mathcal{C}_3 \subseteq \mathcal{H}$, *and it is ([3])* $\mathcal{C}_3 = \mathcal{Q}(\mathcal{C})$.

Now, taking into account that the Lebesgue measure is continuous, we can prove that given a monotone sequence of elements of $\mathcal{H}$, its limit also belongs to $\mathcal{H}$. Hence, $\mathcal{H}$ is a monotone class and contains the field generated by $\mathcal{C}$, whence it contains the σ-field generated by this class, i.e., $\beta_{[0,1]}$. Hence, for any $D \in \beta_{[0,1]}$, $\lambda_{[0,1]}(D \cap C) = \frac{\lambda_{[0,1]}(D)}{2}$. But this implies in particular that $0.5 = \lambda_{[0,1]}(C) = \frac{\lambda_{[0,1]}(C)}{2} = 0.25$. This is a contradiction. Hence, Q does not belong to $P(\Gamma)$ and this set is a proper subset of $M(P^*)$. ♦

This example shows that, even when the initial probability space is non-atomic, the upper probability does not necessarily keep all the available information about the probability distribution of P_{U_0}: the class $M(P^*)$ does not coincide in general with $P(\Gamma)$. Our next section will be devoted to the search of sufficient conditions for the equality between these two sets of probabilities. First, we will give conditions valid for random closed intervals, and later we will focus on random open intervals.

4 Sufficient conditions for the equality between $P(\Gamma)$ and $M(P^*)$

In this section, we are going to study under which conditions the sets of probabilities $P(\Gamma)$ and $M(P^*)$ coincide. The equality between them will mean that the available information about the probability induced by the original random variable (which is given by the class of the probability distributions of the measurable selections) can be modelled through the upper probability of the random interval. Indeed, taking into account that the upper probability of a random closed interval is completely determined by its values on the compact or the open sets (from the second point of Theorem 3.2) and that the upper probability of a random open interval is determined by its values on the finite sets (see the first point of Theorem 3.3), when $P(\Gamma)$ and $M(P^*)$ agree there is an even simpler way to represent the available information.

We are going to focus on random intervals defined on non-atomic probability spaces: as Theorem 3.6 shows, in that case the closures of $P(\Gamma)$ and $M(P^*)$ coincide, something that does not hold for arbitrary random intervals. Moreover, the non-atomicity of the initial probability space is not a very restrictive hypothesis: it holds for instance if we have the additional knowledge that P_{U_0}, the probability induced by the original random variable, is continuous (see for instance [15]).

4.1 Conditions on random closed intervals

Let us focus first on random closed intervals. As Example 3.3 shows, the sets of probabilities $P(\Gamma)$ and $M(P^*)$ do not necessarily coincide when the initial probability space is non-atomic. We shall prove that, under some additional conditions on the extremes of the random closed interval, it is $P(\Gamma) = M(P^*)$. The initial probability space of all the random intervals we shall consider in this section will be $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$. Although the use of the interval $[0,1]$ will simplify somewhat the proofs of the results we will establish, they can be easily generalized to the case where the initial space is an interval $[c,d]$ and we consider the probability measure $\frac{\lambda_{[c,d]}}{d-c}$ on $\beta_{[c,d]}$. It would be interesting to study if they can be further generalized to the case of an arbitrary non-atomic space, using that in this last case there exists a uniformly distributed random variable $V : \Omega \to [0,1]$.

We start showing that $P(\Gamma)$ and $M(P^*)$ coincide when the extremes A, B of the random closed interval are increasing. A random closed interval of increasing extremes can be used for instance when we know that U_0 is an increasing mapping. Then, given $\omega_1 \leq \omega_2$, it will be $U_0(\omega_2) \geq U_0(\omega_1) \geq A(\omega_1)$, whence, if $A(\omega_2)$ is the most precise lower bound of $U_0(\omega_2)$ we can give, it must be $A(\omega_2) \geq A(\omega_1)$. Similarly, it is $B(\omega_2) \geq U_0(\omega_2) \geq U_0(\omega_1)$, whence, if $B(\omega_1)$ is the most precise upper bound of $U_0(\omega_1)$ we can give, it is $B(\omega_1) \leq B(\omega_2)$. This means that, when U_0 is increasing, we may assume without loss of generality that A and B are also increasing.

Proposition 4.1. *Consider the probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$*, and let* $A, B : [0,1] \to \mathbb{R}$ *be increasing random variables. For the random closed interval* $\Gamma = [A, B]$*,* $P(\Gamma) = M(P^*)$*.*

Proof: It is clear that $P(\Gamma) \subseteq M(P^*)$. Conversely, consider $Q \in M(P^*)$, and let us show that there exists $U \in S(\Gamma)$ such that $P_U = Q$. Let

$$\begin{array}{rcl} V : (0,1) & \to & \mathbb{R} \\ \omega & \hookrightarrow & \inf\{y \mid \omega \leq Q((-\infty, y])\} \end{array}$$

be the quantile function ([3]) of Q. It is an increasing function (and, as a consequence, measurable) and satisfies the equality $P_V = Q$ when the probability measure considered on the initial space is $\lambda_{(0,1)}$. Taking into account that A is increasing, we deduce that there is a countable number of elements $\omega \in (0,1)$

such that $A(\omega) > \sup_{\omega'<\omega} A(\omega')$. Let N be the set of these points, and define

$$\begin{aligned} V' : [0,1] &\to \mathbb{R} \\ \omega &\hookrightarrow \begin{cases} V(\omega) & \text{if } \omega \in (0,1) \setminus N \\ A(\omega) & \text{otherwise.} \end{cases} \end{aligned}$$

Let us show that V' is a selection of Γ. Given $\omega \in \{0,1\} \cup N$, it is $V'(\omega) = A(\omega) \in \Gamma(\omega)$. Consider then $\omega \in (0,1) \setminus N$.

- $V'(\omega) = V(\omega) = \inf\{y \mid \omega \leq Q((-\infty, y])\} \geq \inf\{y \mid \omega \leq P^*((-\infty, y])\} = \inf\{y \mid \omega \leq F_A(y)\}$, where the first inequality holds because $Q \in M(P^*)$. Let us prove that $\inf\{y \mid \omega \leq F_A(y)\} \geq A(\omega)$: consider $y \in \mathbb{R}$ s.t. $F_A(y) \geq \omega$. Taking into account that A is increasing, it must be $[0, \omega) \subseteq A^{-1}((-\infty, y])$. On the other hand, $A(\omega) = \sup_{\omega'<\omega} A(\omega') \leq y$, because $\omega \notin N$. We deduce that $\inf\{y \mid \omega \leq F_A(y)\} \geq A(\omega)$ and as a consequence $V'(\omega) \geq A(\omega)$.

- On the other hand, $V'(\omega) = V(\omega) = \inf\{y \mid \omega \leq Q((-\infty, y])\} \leq \inf\{y \mid \omega \leq P_*((-\infty, y])\} = \inf\{y \mid \omega \leq F_B(y)\} \leq B(\omega)$, where the inequality follows from $Q \in M(P^*)$ and the second, from B being increasing.

Moreover, taking into account that the mappings V and A are measurable and that the sets $\{0,1\} \cup N$ and $(0,1) \setminus N$ belong to $\beta_{[0,1]}$, we deduce that V' is measurable. Finally, the restriction of V' to the interval $(0,1)$ coincides with V except for the null set N. As a consequence, the probability distributions of V and V' coincide and $P_{V'} = P_V = Q$. We deduce that $M(P^*) = P(\Gamma)$. ■

We can easily see that the equality between $P(\Gamma)$ and $M(P^*)$ also holds when the random variables $A, B : [0,1] \to \mathbb{R}$ are increasing except for a null subset of $[0,1]$. It also seems easy to show that $P(\Gamma)$ and $M(P^*)$ coincide when the variables A, B determining the random interval are decreasing functions. Such a random interval can be used when we know that U_0 is a decreasing function; following a reasoning similar to the one prior to Proposition 4.1, it can be checked that in that case the variables A and B can be assumed to be decreasing, too.

In some cases, a random closed interval can be transformed into a random closed interval with increasing extremes without modifying the upper probability. We will prove that in some of those situations the sets of probabilities $P(\Gamma)$ and $M(P^*)$ coincide. We need to establish the following lemma:

Lemma 4.2. *Consider the probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ *and a measurable mapping* $U : [0,1] \to \mathbb{R}$*. Let us define* $h : [0,1] \to [0,1]$ *by*

$$h(\omega) = F_U(U(\omega)^-) + \lambda_{[0,1]}([0,\omega] \cap U^{-1}(\{U(\omega)\})) \ \forall \omega.$$

1. $h(\omega) \leq F_U(U(\omega))$ $\forall \omega$.
2. *h is measurable and uniformly distributed on* $[0,1]$.
3. *If V is the quantile function of P_U, there exists a null set $N \in \mathcal{N}_{\lambda_{[0,1]}}$ s.t.* $(V \circ h)(\omega) = U(\omega)$ *for any* $\omega \notin N$.

Proof: Let us remark first that if $U(\omega)$ is a continuity point of the distribution function F_U, it is $h(\omega) = F_U(U(\omega)^-) = F_U(U(\omega))$.

1. Consider $\omega \in [0,1]$; then,

$$\begin{aligned} h(\omega) &= F_U(U(\omega)^-) + \lambda_{[0,1]}([0,\omega] \cap U^{-1}(\{U(\omega)\})) \\ &\leq F_U(U(\omega)^-) + \lambda_{[0,1]}(U^{-1}(\{U(\omega)\})) = F_U(U(\omega)). \end{aligned}$$

   Hence, $h \leq F_U \circ U$.

2. Let us show that h is a uniformly distributed random variable. Consider $x \in [0,1]$.

   - Assume first that there is some $\omega_0 \in [0,1]$ s.t. $F_U(U(\omega_0)^-) \leq x < F_U(U(\omega_0))$, and define $C = U^{-1}(\{U(\omega_0)\}) \cap h^{-1}([0,x])$. Given $\omega_1 \in C$, it is $[0,\omega_1] \cap U^{-1}(\{U(\omega_0)\}) \subseteq C$, because the restriction of h to $U^{-1}(\{U(\omega_0)\})$ is increasing. Let $\omega_2 = \sup_C$. Then, $C = [0,\omega_2] \cap U^{-1}(U(\omega_0))$ measurable. It is clear that $\sup_{\omega \in C} h(\omega) \leq x$. If $\sup_{\omega \in C} h(\omega) < x$, then there would exist some ω' such that $\sup_{\omega \in C} < h(\omega') < x$, because $\lambda_{[0,1]}$ is continuous and moreover $\sup_{\omega \in U^{-1}(\{U(\omega_0)\})} h(\omega) = F_U(U(\omega_0)) > x$. But then ω' would belong to C, a contradiction. Now,

$$\begin{aligned} &h^{-1}([0,x]) \\ &\quad = \{\omega \mid h(\omega) \leq F_U(U(\omega_0)^-)\} \cup \{\omega \mid F_U(U(\omega_0)^-) < h(\omega) \leq x\} \\ &\quad = \{\omega \mid F_U(U(\omega)) \leq F_U(U(\omega_0)^-)\} \ \cup ([0,\omega_2] \cap U^{-1}(\{U(\omega_0)\})), \end{aligned}$$

     is measurable, and consequently

$$\begin{aligned} &P_h([0,x]) = \lambda_{[0,1]}(\{\omega \mid F_U(U(\omega)) < F_U(U(\omega_0))\}) + \lambda_{[0,1]}(C) \\ &= F_U(U(\omega_0)^-) + \lambda_{[0,1]}([0,\omega_2] \cap U^{-1}(\{U(\omega_0)\})) = \sup_{\omega \in C} h(\omega) = x. \end{aligned}$$

- If there is not any ω_0 under the conditions of the previous point, then there exists $\omega_1 \in [0,1]$ such that $x = F_U(U(\omega_1))$. Then, $h^{-1}([0,x]) = \{\omega | F_U(U(\omega)) \leq x\} \in \beta_{[0,1]}$ and $P_h([0,x]) = F_U(U(\omega_1)) = x$, taking into account that $\{\omega \mid U(\omega) > U(\omega_1),\ F_U(U(\omega)) = F_U(U(\omega_1))\}$ is a null set.

3. Let $V : (0,1) \to \mathbb{R}$ be the quantile function associated to P_U, given by $V(\omega) = \inf\{y \mid \omega \leq P_U((-\infty, y])\}$, and let us define $V(0) = V(1) = 0$, to make the composition $V \circ h$ possible. We are going to show that $V \circ h = U$ except for a null subset of $[0,1]$. Consider $x \in \mathbb{R}$.

   - Assume that x is a discontinuity of F_U, and let us define the measurable set $N_x = U^{-1}(\{x\}) \cap h^{-1}(\{F_U(x)^-\})$. Then, taking into account that h is uniformly distributed, we deduce that N_x is null. Moreover, for any $\omega \in U^{-1}(\{x\}) \setminus N_x$, it is $F_U(x^-) < h(\omega) \leq F_U(x)$. As a consequence, $V(h(\omega)) = \inf\{y \mid h(\omega) \leq F_U(y)\} = x = U(\omega)$. Let us define $N^1 = \cup\{N_x \mid x \text{ discontinuity of } F_U\}$. This is a null subset of $[0,1]$, because every N_x is null and F_U has at most a countable number of discontinuities.

   - Let us assume now that x is a continuity point of F_U. If $F_U(x-\epsilon) < F_U(x)$ for all $\epsilon > 0$ then, given $\omega \in U^{-1}(\{x\})$,

   $$\begin{aligned} V(h(\omega)) &= V(F_U(U(\omega))) = V(F_U(x)) \\ &= \inf\{y \mid F_U(x) \leq F_U(y)\} = x = U(\omega). \end{aligned}$$

     If, on the contrary, there exists $\epsilon > 0$ such that $F_U(x-\epsilon) = F_U(x)$, let ϵ_x be the greatest ϵ under these conditions, and $\delta_x \geq 0$ the greatest real number such that $F_U(x) = F_U(x+\delta_x)$. Consider $N_x = U^{-1}((x-\epsilon_x, x+\delta_x]) \in \beta_{[0,1]}$. Then, the equality $F_U(x+\delta_x) = F_U(x-\epsilon_x)$ implies that N_x is null.

     There exists a countable number of disjoint intervals $(x-\epsilon_x, x+\delta_x]$ of this type, because any two different intervals are disjoint (each of them corresponds to a different value of F_U), and all the intervals have positive Lebesgue measure. As a consequence, the union N^2 of the inverse sets N_x of these intervals by U is a null subset of $[0,1]$.

   Now, the measurable set $N = N^1 \cup N^2 \cup \{0,1\}$ is null, and given $\omega \in [0,1] \setminus N$, it is $V(h(\omega)) = U(\omega)$. This completes the proof. ■

Next, we are going to use this lemma to prove that the equality $P(\Gamma) = M(P^*)$ holds when the random variable determining the lower bound is constant. This type of random closed intervals can be used when, due to the available information, we can only modify the upper bounds of the values $U_0(\omega)$, while the lower bound is invariably the minimum value that U_0 can achieve. Although we prove our result for random closed intervals of the type $\Gamma = [0, B]$, it can easily be generalized for random closed intervals of the type $\Gamma = [k, B]$, with $k \in \mathbb{R}$ and $B : [0,1] \to [k, \infty)$ measurable.

Theorem 4.3. *Consider the probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ *and a random variable* $B : [0,1] \to [0,\infty)$*. For the random closed interval* $\Gamma = [0, B]$*,* $P(\Gamma) = M(P^*)$*.*

Proof: Let V denote the quantile function of F_B, and let us extend it to $[0,1]$ with $V(0) = V(1) = 0$. Consider the random closed interval $\Gamma' = [0, V]$. The upper probabilities of Γ and Γ' coincide: given $C \in \beta_{\mathbb{R}}$, it is $P^*_\Gamma(C) = \lambda_{[0,1]}(\{\omega \mid [0, B(\omega)] \cap C \neq \emptyset\}) =$

$$\begin{cases} P_B((\inf_{C\cap[0,\infty)}, \infty)) & \text{if } \inf_{C\cap[0,\infty)} \notin C \\ P_B([\inf_{C\cap[0,\infty)}, \infty)) & \text{if } \inf_{C\cap[0,\infty)} \in C \end{cases}$$

Similarly, $P^*_{\Gamma'}(C) =$

$$\begin{cases} P_V((\inf_{C\cap[0,\infty)}, \infty)) & \text{if } \inf_{C\cap[0,\infty)} \notin C \\ P_V([\inf_{C\cap[0,\infty)}, \infty)) & \text{if } \inf_{C\cap[0,\infty)} \in C \end{cases}$$

Taking into account that $P_B = P_V$, we deduce that $P^*_\Gamma(C) = P^*_{\Gamma'}(C)\ \forall C \in \beta_{\mathbb{R}}$. The mapping $V : [0,1] \to [0,\infty)$ is increasing on $[0,1)$. Applying Proposition 4.1, we deduce that $M(P^*_\Gamma) = M(P^*_{\Gamma'}) = P(\Gamma')$. Consider $Q \in M(P^*_\Gamma)$. Then, there exists $U \in S(\Gamma')$ s.t. $P_U = Q$. On the other hand, Lemma 4.2 implies the existence of a uniformly distributed random variable $h : [0,1] \to [0,1]$ s.t. $h \leq F_B \circ B$, and a null set $N \in \beta_{[0,1]}$ with $V(h(\omega)) = B(\omega)\ \forall \omega \notin N$. Define

$$\begin{aligned} U_1 : [0,1] &\longrightarrow \mathbb{R} \\ \omega &\hookrightarrow \begin{cases} U(h(\omega)) & \text{if } \omega \in [0,1] \setminus N \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

- Let us show that U_1 is a selection of Γ. Given $\omega \in N$, $U_1(\omega) = 0 \in \Gamma(\omega)$. Consider now $\omega \notin N$. Then, $U_1(\omega) = U(h(\omega)) \geq 0$, and $U_1(\omega) = U(h(\omega)) \leq V(h(\omega)) = B(\omega)$.

- Taking into account that U and h are measurable and $N \in \beta_{[0,1]}$, we deduce that U_1 is measurable.

- Given $C \in \beta_{\mathbb{R}}$,

$$P_{U_1}(C) = \lambda_{[0,1]}(U_1^{-1}(C)) = \lambda_{[0,1]}(h^{-1}(U^{-1}(C))) = \lambda_{[0,1]}(U^{-1}(C)) = P_U(C) = Q(C),$$

because h is uniformly distributed on $[0,1]$ and N is a null set. Hence, U_1 is a measurable selection of Γ and $P_{U_1} = Q$. ■

We think that it could be proven similarly that $P(\Gamma) = M(P^*)$ when $\Gamma = [A,k]$, with $k \in \mathbb{R}$ and $A : [0,1] \to (-\infty,k]$ measurable. This would mean that whenever one of the bounds is constant, the upper probability keeps all the information about the probability distribution of the original random variable. On the other hand, we want to stress that, contrary to what might be expected, there is no relationship between the probabilistic information of a random interval $\Gamma_1 = [A,B]$ and that of $\Gamma_2 = [0,B-A]$, in the sense that in this last case $P(\Gamma_2) = M(P^*_{\Gamma_2})$, and, as Example 3.3 shows, $P(\Gamma_1)$ does not coincide with $M(P^*_{\Gamma_1})$ in general. Next, we consider the case where the functions A,B determining the random closed interval increase or decrease simultaneously. We call this type of functions strictly comonotonic [5].

Definition 4.1. *Two functions $A,B : [0,1] \to \mathbb{R}$ are said to be* **strictly comonotonic** *if and only if for every $\omega_1,\omega_2 \in [0,1]$, $A(\omega_1) \leq A(\omega_2) \Leftrightarrow B(\omega_1) \leq B(\omega_2)$.*

Random closed intervals with strictly comonotonic extremes can be used rather intuitively as a model of the imprecise observation of a random variable. If the observation made on ω_1 is greater than the one made on ω_2, the upper and lower bounds for the value $U_0(\omega_1)$ should intuitively be greater than those for $U_0(\omega_2)$. In particular, the following types of random closed intervals have strictly comonotonic extremes:

- Random closed intervals of fixed length, $\Gamma = [U-\epsilon, U+\epsilon]$, where the margin for imprecision is the same in all the observations. These can be used for instance when we observe the life time of some components, and we check their state (on/off) in intervals of 2ϵ units of time.

- Random closed intervals where the margin of imprecision increases with the lower bound, so $A(\omega_1) \leq A(\omega_2)$ yields $B(\omega_1) - A(\omega_1) \leq B(\omega_2) -$

[5] Denneberg ([10]) calls two functions A,B *comonotonic* when they satisfy $(A(\omega_2) - A(\omega_1))(B(\omega_2) - B(\omega_1)) \geq 0 \ \forall \omega_1,\omega_2$. The same concept is used by Dellacherie in [7]. For our next result, we need to introduce the following definition, which is slightly stronger.

$A(\omega_2)$. As a particular case, this model contains the intervals of the type $[(1-\delta)U, (1+\delta)U]$, where $U(\omega)$ is the observed value in the element ω and $\delta \in (0,1)$. In these cases, if the observed value $U(\omega)$ increases, so does the margin of error. Random closed intervals whose extreme functions satisfy the relationship $B = k_1 \cdot A + k_2$, with $k_1 \geq 1, k_2 \geq 0$ are also of this type.

On the other hand, the extremes of the random intervals considered in Proposition 4.1 and Theorem 4.3 are not strictly comonotonic in general: two increasing functions $A, B : [0,1] \to \mathbb{R}$ are strictly comonotonic if and only if $A(\omega_1) = A(\omega_2) \Leftrightarrow B(\omega_1) = B(\omega_2)$ for any $\omega_1, \omega_2 \in [0,1]$, and a non-negative random variable $B : [0,1] \to [0, \infty)$ is not strictly comonotonic with 0 unless B is constant.

For the purposes of this paper, the main advantage of random closed intervals with strictly comonotonic extremes is that there exists a random closed interval with increasing extremes with the same upper probability, which, from Proposition 4.1, satisfies $P(\Gamma) = M(P^*)$. In order to prove this, we need to establish first the following lemma:

Lemma 4.4. *Consider the probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$*, and let* $A, B : [0,1] \to \mathbb{R}$ *be two strictly comonotonic random variables. Then, for any* $\omega \in [0,1]$*,* $F_A(A(\omega)) = F_B(B(\omega))$ *and* $F_A(A(\omega)^-) = F_B(B(\omega)^-)$*.*

Proof: Take $\omega \in [0,1]$. Then, $F_A(A(\omega)) = \lambda_{[0,1]}(\{\omega' \in [0,1] \mid A(\omega') \leq A(\omega)\}) = \lambda_{[0,1]}(\{\omega' \in [0,1] \mid B(\omega') \leq B(\omega)\}) = F_B(B(\omega))$. On the other hand, $F_A(A(\omega)^-) = \lambda_{[0,1]}(\{\omega' \in [0,1] \mid A(\omega') < A(\omega)\}) = \lambda_{[0,1]}(\{\omega' \in [0,1] \mid B(\omega') < B(\omega)\}) = F_B(B(\omega)^-)$. ■

This lemma has an important consequence: if h_A, h_B denote the random variables defined applying Lemma 4.2 respect to A, B, we have $h_A = h_B$. This fact, together with Proposition 4.1, will allow us to prove the equality $P(\Gamma) = M(P^*)$ for random closed intervals of strictly comonotonic extremes.

Theorem 4.5. *Consider the probability space* $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$*, and let* $A, B : [0,1] \to \mathbb{R}$ *be two strictly comonotonic random variables. For the random closed interval* $\Gamma = [A, B]$*,* $P(\Gamma) = M(P^*)$*.*

Proof: Let V and W denote the quantile functions of F_A, F_B, and let us define $V = A, W = B$ in $\{0,1\}$. Then, $V, W : [0,1] \to \mathbb{R}$ are increasing functions (except for the null set $\{0,1\}$, which does not affect the result), and satisfy $F_V = F_A, F_W = F_B$. Let us define the random interval $\Gamma' = [V, W]$. Since $A \leq B \Rightarrow F_A \geq F_B$, we deduce that $V \leq W$ and Γ' is well-defined. Let $h : [0,1] \to [0,1]$ be the uniformly distributed random variable satisfying $h \leq$

$F_A \circ A = F_B \circ B$ defined on Lemma 4.2. Then, there exist two null sets N_1, N_2 such that $V(h(\omega)) = A(\omega)\ \forall \omega \in [0,1]\setminus N_1$ and $W(h(\omega)) = B(\omega)\ \forall \omega \in [0,1]\setminus N_2$. Consider $N = N_1 \cup N_2$, and let us show that $P^*_\Gamma = P^*_{\Gamma'}$.

From Theorem 3.2, it suffices to prove that $P^*_\Gamma(G) = P^*_{\Gamma'}(G)$ for every G open, and, taking into account that P^* is continuous for increasing sequences, it suffices to show the equality for finite unions of open intervals. Consider $C = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$, with $b_i \leq a_{i+1}$ for all $i = 1, \ldots, n-1$, and let us show that $P^*_\Gamma(C) = P^*_{\Gamma'}(C)$. We have

$$\begin{aligned} P^*_\Gamma(C) = \lambda_{[0,1]}(\Gamma^*(C)) = \lambda_{[0,1]}(A^{-1}(C)) \\ + \lambda_{[0,1]}(A^{-1}((-\infty, a_1]) \cap B^{-1}((a_1, \infty))) \\ + \sum_{i=1}^{n-1} \lambda_{[0,1]}(A^{-1}([b_i, a_{i+1}]) \cap B^{-1}((a_{i+1}, \infty))). \end{aligned}$$

Similarly,

$$\begin{aligned} P^*_{\Gamma'}(C) = \lambda_{[0,1]}(\Gamma'^*(C)) = \lambda_{[0,1]}(V^{-1}(C)) \\ + \lambda_{[0,1]}(V^{-1}((-\infty, a_1]) \cap W^{-1}((a_1, \infty))) \\ + \sum_{i=1}^{n-1} \lambda_{[0,1]}(V^{-1}([b_i, a_{i+1}]) \cap W^{-1}((a_{i+1}, \infty))). \end{aligned}$$

From the equality $F_V = F_A$, we deduce that $\lambda_{[0,1]}(V^{-1}(C)) = \lambda_{[0,1]}(A^{-1}(C))$. Consider now $i \in \{1, \ldots, n-1\}$. From Lemma 4.2, it is $\lambda_{[0,1]}(A^{-1}([b_i, a_{i+1}]) \cap B^{-1}((a_{i+1}, \infty))) = \lambda_{[0,1]}(h^{-1}[V^{-1}([b_i, a_{i+1}]) \cap W^{-1}((a_{i+1}, \infty))])$. Taking into account that h is uniformly distributed,

$$\begin{aligned} \lambda_{[0,1]}(h^{-1}[V^{-1}([b_i, a_{i+1}]) \cap W^{-1}((a_{i+1}, \infty))]) \\ = \lambda_{[0,1]}(V^{-1}([b_i, a_{i+1}]) \cap W^{-1}((a_{i+1}, \infty))). \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda_{[0,1]}(A^{-1}((-\infty, a_1]) \cap B^{-1}((a_1, \infty))) \\ = \lambda_{[0,1]}(h^{-1}[V^{-1}((-\infty, a_1]) \cap W^{-1}((a_1, \infty))]) \\ = \lambda_{[0,1]}(V^{-1}((-\infty, a_1]) \cap W^{-1}((a_1, \infty))). \end{aligned}$$

As a consequence, $P^*_\Gamma(C) = P^*_{\Gamma'}(C)$, and this implies that $P^*_\Gamma = P^*_{\Gamma'}$. We proceed now to prove that $P(\Gamma) = M(P^*_\Gamma)$. From Proposition 4.1, it is $M(P^*_{\Gamma'}) = P(\Gamma')$, because V and W are increasing except on a null subset of $[0,1]$. Consider

$Q \in M(P^*_\Gamma) = M(P^*_{\Gamma'})$. Then, there exists $U \in S(\Gamma')$ s.t. $P_U = Q$. Let us define

$$U_1 := (U \circ h)I_{[0,1]\setminus N} + AI_N.$$

- Given $\omega \in N$, $U(\omega) = A(\omega) \in \Gamma(\omega)$. Consider now $\omega \in [0,1] \setminus N$. Then, $U_1(\omega) = U(h(\omega)) \in [V(h(\omega)), W(h(\omega))] = [A(\omega), B(\omega)]$. Hence, $U(\omega) \in \Gamma(\omega)\ \forall\omega$.

- Taking into account that h, U and A are measurable mappings and $N \in \beta_{[0,1]}$, we deduce that U_1 is measurable.

- Finally, let us see that $P_{U_1} = Q$. Given $C \in \beta_{\mathbb{R}}$,

$$\begin{aligned} P_{U_1}(C) = \lambda_{[0,1]}(U_1^{-1}(C)) &= \lambda_{[0,1]}(U_1^{-1}(C) \cap N^c) \\ = \lambda_{[0,1]}(h^{-1}(U^{-1}(C)) \cap N^c) &= \lambda_{[0,1]}(h^{-1}(U^{-1}(C))) \\ &= \lambda_{[0,1]}(U^{-1}(C)) = Q(C), \end{aligned}$$

  where the third and the fifth equalities hold because h is uniformly distributed. Hence, U_1 is a measurable selection of Γ and satisfies $P_{U_1} = Q$. Therefore, $P(\Gamma) = M(P^*)$. ■

We conclude that, even if a random closed interval Γ does not satisfy in general the equality $P(\Gamma) = M(P^*)$, there are a number of interesting situations where the upper probability keeps all the information about the probability distribution of the original random variable.

4.2 Conditions on random open intervals

We focus now on random open intervals. We are going to establish necessary and sufficient conditions for the equality $P(\Gamma) = M(P^*)$. We start proving a relationship between the upper inverses of a set by the random intervals $(A, B), [A, B), (A, B]$ and $[A, B]$.

Proposition 4.6. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $A, B : \Omega \to \mathbb{R}$ two random variables s.t. $A(\omega) < B(\omega)\ \forall\omega$, and let us denote $\Gamma = [A, B], \Gamma_1 = (A, B), \Gamma_2 = [A, B)$ and $\Gamma_3 = (A, B]$. Then, for any $C \in \beta_{\mathbb{R}}$ there exists $D \subseteq C$ countable s.t. $\Gamma^*(C) = \Gamma_1^*(C) \cup A^{-1}(D) \cup B^{-1}(D), \Gamma_2^*(C) = \Gamma_1^*(C) \cup A^{-1}(D)$ and $\Gamma_3^*(C) = \Gamma_1^*(C) \cup B^{-1}(D)$.*

Proof: Consider $C \in \beta_{\mathbb{R}}$, and let us denote $E_1 = \Gamma_2^*(C) \setminus \Gamma_1^*(C)$. We are going to show first that there exists a countable set $D \subseteq C$ such that $E_1 \subseteq A^{-1}(D)$. Take $\omega_1 \in E_1$, and let $D_1 := A(\omega_1) = \Gamma_2(\omega_1) \cap C$. If $E_1 \subseteq$

$A^{-1}(D_1)$, then $\Gamma_2^*(C) = \Gamma_1^*(C) \cup E_1 \subseteq \Gamma_1^*(C) \cup A^{-1}(D_1) \subseteq \Gamma_1^*(C) \cup \Gamma_2^*(D_1) \subseteq \Gamma_2^*(C)$, and the result holds. Assume then that there exists $\omega_2 \in E_1 \setminus A^{-1}(D_1)$, and denote $D_2 = A(\omega_2) = \Gamma_2(\omega_2) \cap C$. Then, $D_2 \cap D_1 = \emptyset$, because $\omega_2 \notin A^{-1}(D_1)$. Moreover, $D_2 \cap \Gamma_1(\omega_1) = \emptyset$: otherwise, $\omega_1 \in \Gamma_1^*(D_2) \subseteq \Gamma_1^*(C)$, a contradiction. Hence, $D_2 \cap \Gamma_2(\omega_1) = \emptyset$, and as a consequence there is some $\epsilon > 0$ s.t. $[A(\omega_2), A(\omega_2)+\epsilon) \subseteq \Gamma_2(\omega_1)^c$. This implies that $\lambda(\Gamma_2(\omega_2) \setminus \Gamma_2(\omega_1)) \geq \lambda(\Gamma_2(\omega_2) \cap [A(\omega_2), A(\omega_2)+\epsilon)) > 0$, whence $\lambda(\Gamma_2(\omega_1) \cup \Gamma_2(\omega_2)) > \lambda(\Gamma_2(\omega_1))$. Again, if $E_1 \subseteq A^{-1}(D_1 \cup D_2)$, we deduce that $\Gamma_2^*(C) = \Gamma_1^*(C) \cup A^{-1}(D_1 \cup D_2)$, and the result holds because $D_1 \cup D_2$ is finite. Otherwise, we take $\omega_3 \in E_1 \setminus A^{-1}(D_1 \cup D_2)$ and repeat the process. This can be done at most a countable number of times, because $\lambda(\cup_{i=1}^n \Gamma_2(\omega_i)) > \lambda(\cup_{i=1}^{n-1} \Gamma_2(\omega_i)) \forall n \geq 2$. Hence, there exists $D = \cup_n D_n \subseteq C$ such that $E_1 \subseteq A^{-1}(D)$, whence $\Gamma_2^*(C) = \Gamma_1^*(C) \cup A^{-1}(D)$. Besides, the set D is countable, because $D_n = \{A(\omega_n)\}$ for all n.

Consider now $E_2 = \Gamma_3^*(C) \setminus \Gamma_1^*(C)$. Following a similar reasoning, we can deduce the existence of $D' \subseteq C$ countable such that $E_2 \subseteq B^{-1}(D')$, and as a consequence $\Gamma_3^*(C) = \Gamma_1^*(C) \cup B^{-1}(D')$. Finally, if we consider the countable set $D'' = D \cup D'$, it is $\Gamma^*(C) = \Gamma_1^*(C) \cup E_1 \cup E_2 = \Gamma_1^*(C) \cup A^{-1}(D) \cup B^{-1}(D') \subseteq \Gamma_1^*(C) \cup A^{-1}(D'') \cup B^{-1}(D'') \subseteq \Gamma^*(C)$. In particular, $\Gamma_2^*(C) = \Gamma_1^*(C) \cup A^{-1}(D'')$ and $\Gamma_3^*(C) = \Gamma_1^*(C) \cup B^{-1}(D'')$. ■

Using this result, we can establish a relationship between the upper probabilities induced by the random intervals $(A, B), [A, B), (A, B]$ and $[A, B]$.

Corollary 4.7. *Consider a probability space $(\Omega, \mathcal{A}, P)$, and let $A, B : \Omega \to \mathbb{R}$ be two random variables such that $A(\omega) < B(\omega)\ \forall \omega$. Let us denote $\Gamma = [A, B], \Gamma_1 = (A, B), \Gamma_2 = [A, B)$ and $\Gamma_3 = (A, B]$. Then,*

1. *If F_A is continuous, then $P^*_{\Gamma_1} = P^*_{\Gamma_2}$.*
2. *If F_B is continuous, then $P^*_{\Gamma_1} = P^*_{\Gamma_3}$.*
3. *If F_A and F_B are continuous, then $P^*_{\Gamma} = P^*_{\Gamma_1} = P^*_{\Gamma_2} = P^*_{\Gamma_3}$.*

Proof: Consider $C \in \beta_{\mathbb{R}}$. From the previous proposition, there exists a countable set $D \subseteq C$ such that $\Gamma^*(C) = \Gamma_1^*(C) \cup A^{-1}(D) \cup B^{-1}(D)$, $\Gamma_2^*(C) = \Gamma_1^*(C) \cap A^{-1}(D), \Gamma_3^*(C) = \Gamma_1^*(C) \cup B^{-1}(D)$.

1. If F_A is continuous, then $P^*_{\Gamma_2}(C) \leq P^*_{\Gamma_1}(C) + P_A(D) = P^*_{\Gamma_1}(C) \leq P^*_{\Gamma_2}(C)$, whence $P^*_{\Gamma_1} = P^*_{\Gamma_2}$.
2. If F_B is continuous, then $P^*_{\Gamma_3}(C) \leq P^*_{\Gamma_1}(C) + P_B(D) = P^*_{\Gamma_1}(C) \leq P^*_{\Gamma_3}(C)$, whence $P^*_{\Gamma_1} = P^*_{\Gamma_3}$.

3. If F_A and F_B are continuous, then $P^*_\Gamma(C) \leq P^*_{\Gamma_1}(C) + P_A(D) + P_B(D) = P^*_{\Gamma_1}(C) \leq P^*_\Gamma(C)$, whence $P^*_\Gamma = P^*_{\Gamma_1}$, and taking into account that $P^*_{\Gamma_1} \leq P^*_{\Gamma_2} \leq P^*_\Gamma$ and $P^*_{\Gamma_1} \leq P^*_{\Gamma_3} \leq P^*_\Gamma$, it is $P^*_\Gamma = P^*_{\Gamma_1} = P^*_{\Gamma_2} = P^*_{\Gamma_3}$. ■

In [9], Dempster claimed that the upper probabilities of the random intervals (A, B) and $[A, B]$ coincide when the joint distribution of A and B is absolutely continuous. Our corollary shows that it is only necessary that the (marginal) distribution functions of A and B are continuous.

We can use this result to prove that the sets of probabilities $P(\Gamma)$ and $M(P^*)$ associated to a random open interval do not coincide when either of the distribution functions of A and B is continuous. Hence, in those cases the use of the upper and lower probabilities will cause a loss of precision respect to the class of the probability distributions of the measurable selections.

Theorem 4.8. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $A, B : \Omega \to \mathbb{R}$ two random variables with $A(\omega) < B(\omega)\ \forall\omega$. Let us denote $\Gamma_1 = (A, B)$. If F_A or F_B is continuous, then $P(\Gamma_1) \subsetneq M(P^*)$.*

Proof: Assume for instance that the distribution function F_A is continuous (the proof when F_B is continuous is analogous). If we denote $\Gamma_2 = [A, B)$, then, applying the previous corollary, it is $P^*_{\Gamma_1} = P^*_{\Gamma_2}$. The random variable A is a measurable selection of Γ_2, whence $P_A \in P(\Gamma_2) \subseteq M(P^*_{\Gamma_2}) = M(P^*_{\Gamma_1})$. Let us show that P_A does not belong to $P(\Gamma_1)$. Assume ex-absurdo that U is a measurable selection of Γ_1 satisfying $P_U = P_A$. Then, given $C_n := \{\omega \in \Omega \mid U(\omega) - A(\omega) \geq \frac{1}{n}\}$, it is $\Omega = \cup_n C_n$, because $U(\omega) > A(\omega)\ \forall\omega$. Take $x \in \mathbb{R}$. Then,

$$\begin{aligned}
P(U > x) &= P(A > x) + P(A \leq x, U > x) \\
&\geq P(A > x) + P\left(x - \frac{1}{n} \leq A \leq x, U - A > \frac{1}{n}\right) \\
&= P(A > x) + P\left(A^{-1}\left(\left[x - \frac{1}{n}, x\right]\right) \cap C_n\right) \\
&\Rightarrow P\left(A^{-1}\left(\left[x - \frac{1}{n}, x\right]\right) \cap C_n\right) = 0\ \forall x,
\end{aligned}$$

whence $P(C_n) = 0$ for all n. But then it is $P(\Omega) = P(\cup_n C_n) = 0$, a contradiction. We conclude that P_A does not belong to $P(\Gamma_1)$, and as a consequence this set does not coincide with $M(P^*)$. ■

Note that, if either of the distribution functions F_A, F_B is continuous, the initial probability space must be non-atomic. As we said before, when the initial space has atoms, $P(\Gamma)$ will not coincide with $M(P^*)$ except for very particular situations. Hence, it remains to see if these two sets of probabilities coincide when the initial probability space is non-atomic and the variables A and B are discrete. In that respect, we have proven that $P(\Gamma) = M(P^*)$ whenever the random variables A and B are simple. Our proof requires the following lemma:

Lemma 4.9. *[5] Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space, $(X, \mathcal{P}(X))$ a measurable space, with $|X|$ finite, and $\Gamma : \Omega \to \mathcal{P}(X)$ a random set. Then, $P(\Gamma) = M(P^*)$.*

Let us show the aforementioned result.

Theorem 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a non-atomic probability space, and let $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$ be a simple random open interval. Then, $P(\Gamma) = M(P^*)$.*

Proof: We will prove first that the result holds when Γ is constant, and then we will use this fact, together with a relationship between simple random intervals and random sets on finite spaces to prove the general result.

- Assume first that Γ is a random set constant on some $B \in \beta_{\mathbb{R}}$. Then, $M(P^*) = \{Q \in \mathcal{P}_{\beta_{\mathbb{R}}} \mid Q(B) = 1\}$. Consider $Q \in M(P^*)$, and let $U : (0,1) \to \mathbb{R}$ be its quantile function. Then, $P_U = Q$, whence $P_U(B) = 1$. We can modify U in the null set $U^{-1}(B)^c$ so that $U(\omega) \in B \ \forall \omega \in (0,1)$, and without affecting the measurability of U. On the other hand, if $(\Omega, \mathcal{A}, P)$ is non-atomic, there is a uniformly distributed random variable $g : \Omega \to [0,1]$ (see for instance [15]). Consider $x \in B$, and let us define

$$\begin{aligned} V : \Omega &\to \mathcal{P}(\mathbb{R}) \\ \omega &\hookrightarrow \begin{cases} U(g(\omega)) & \text{if } g(\omega) \in (0,1) \\ x & \text{otherwise} \end{cases} \end{aligned}$$

  - Given $\omega \in \Omega$, $V(\omega) \in B = \Gamma(\omega)$, because $U(\omega) \in B \ \forall \omega \in [0,1]$ and $x \in B$.
  - V is measurable, because U and g are measurable.
  - Taking into account that g is uniformly distributed and $\{0,1\}$ is a null set, we deduce that $P_V = P_U = Q$.

  We conclude that $P(\Gamma) = M(P^*)$.

- Consider now the case where Γ is a simple random open interval. Then, Γ can be expressed in the form $\Gamma := \sum_{i=1}^{n} (a_i, b_i) I_{C_i}$, with $\{C_1, \ldots, C_n\}$ a partition of Ω. We can deduce from the strong measurability of Γ that $C_i \in \mathcal{A} \ \forall i = 1, \ldots, n$. Let us define the class $\mathcal{D} := \{H_1 \cap \cdots \cap H_n \mid H_i \in$

$\{(a_i,b_i),(a_i,b_i)^c\}\ \forall i\} = \{E_1,\ldots,E_m\}$. This is a finite and measurable partition of $\mathbb{R}$, and any interval (a_i,b_i) is a (finite) union of elements from $\mathcal{D}$. Let us define the bijection $f:\mathcal{D}\to\{1,\ldots,m\}$ by $f(E_i)=i\ \forall i$ and consider $\Gamma' = f\circ\Gamma:\Omega\to\mathcal{P}(\{1,\ldots,m\})$.

– Given $I\subseteq\{1,\ldots,m\}$,

$$\Gamma'^*(\{i\in I\}) = \Gamma^*(\cup_{i\in I}E_i) = \cup_{i\in I}\cup_{E_i\subseteq(a_j,b_j)} C_j \in \mathcal{A} \tag{3}$$

Hence, Γ' is strongly measurable.

– Γ' is defined between a non-atomic probability space and a finite space. Applying Lemma 4.9, $M(P^*_{\Gamma'}) = P(\Gamma')$.

Consider $Q\in M(P^*_\Gamma)$, and let us define $Q' = Q\circ f^{-1}:\mathcal{P}(\{1,\ldots,m\})\to[0,1]$. Then, Q' is a finitely additive probability. Besides, given $I\subseteq\{1,\ldots,m\}$,

$$\begin{aligned} Q'(I) = Q(f^{-1}(I)) = Q(\cup_{i\in I}E_i) &\leq P^*_\Gamma(\cup_{i\in I}E_i) \\ &= \sum_{(a_j,b_j)\cap(\cup_{i\in I}E_i)\neq\emptyset} P(C_j) = P^*_{\Gamma'}(I), \end{aligned}$$

also using Eq.(3). We deduce that $Q'\in M(P^*_{\Gamma'}) = P(\Gamma')$, and as a consequence there exists a measurable selection of Γ', $U_1:\Omega\to\{1,\ldots,m\}$, such that $P_{U_1}=Q'$. Denote $F_i = U_1^{-1}(\{i\})\in\mathcal{A}$ for $i=1,\ldots,m$, and let us define the multi-valued mapping

$$\begin{aligned} \Gamma_i:\Omega &\to \mathcal{P}(\mathbb{R}) \\ \omega &\hookrightarrow \begin{cases} E_i & \text{if } \omega\in F_i \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Consider the measure $Q_i:\beta_\mathbb{R}\to[0,1]$ given by $Q_i(A)=Q(A\cap E_i)$ for all $A\in\beta_\mathbb{R}$. Then, $Q_i(A)\leq P(\Gamma_i^*(A))\ \forall A\in\beta_{E_i}$, because $Q(E_i)=P(F_i)$ for all i. We can easily modify the first part of the proof [6] to show the existence of $W_i\in S(\Gamma_i)$ such that P_{W_i} coincides with Q_i. Let us define

$$\begin{aligned} W:\Omega &\to \mathbb{R} \\ \omega &\hookrightarrow W_i(\omega) \text{ if } \omega\in F_i. \end{aligned}$$

Let us show that W is a measurable selection of Γ and that $P_W=Q$.

[6] It would suffice to consider the quantile function of the finite measure Q_i, define $W_i(\omega)=\emptyset$ for all $\omega\notin F_i$, and proceed as in the first point of the proof.

– The class $\{F_1, \ldots, F_m\} = \{U_1^{-1}(\{1\}), \ldots, U_1^{-1}(\{m\})\}$ is a partition of Ω. Besides, $\Gamma_i(\omega) \neq \emptyset \ \forall \omega \in F_i$, whence W is well-defined.

– Consider $\omega \in F_i$ for some arbitrary $i = 1, \ldots, n$. Then,

$$\begin{aligned} W(\omega) = W_i(\omega) \in \Gamma_i(\omega) = E_i = f^{-1}(\{i\}) = f^{-1}(\{U_1(\omega)\}) \\ \in f^{-1}(\Gamma'(\omega)) = \{E_j \mid E_j \subseteq \Gamma(\omega)\} \Rightarrow W(\omega) \in \Gamma(\omega). \end{aligned}$$

– Given $G \in \beta_{\mathbb{R}}$, $W^{-1}(G) = \cup_{i=1}^m (W_i^{-1}(G) \cap F_i) \in \mathcal{A}$, taking into account that W_i is measurable for all i and U_1 is measurable. Hence, W is a measurable selection of Γ.

– Given $A \in \beta_{\mathbb{R}}$,

$$\begin{aligned} P_W(A) = \sum_{i=1}^m P(W^{-1}(A) \cap F_i) = \sum_{i=1}^m P(W_i^{-1}(A) \cap F_i) \\ = \sum_{i=1}^m Q_i(A) = \sum_{i=1}^m Q(A \cap E_i) = Q(A). \end{aligned}$$

Hence, $P_W = Q$ and as a consequence $M(P^*)$ is equal to $P(\Gamma)$. ■

We conclude that if the random variables A and B are simple and the initial probability space is non-atomic, the upper probability of the random open interval (A, B) keeps all the available information about the probability distribution of the original random variable. Taking into account that, from Theorem 3.3, this upper probability is determined by its values on the finite sets, these values would suffice to summarize all the information about P_{U_0}. Note also that the equality between $P(\Gamma)$ and $M(P^*)$ does not hold in general when we drop the hypothesis of non-atomicity from the initial probability space: to see this, it suffices to consider a probability space with only one element, ω_0, and the multi-valued mapping $\Gamma(\omega_0) = (0, 1)$. Then, it is $P(\Gamma) = \{\delta_x \mid x \in (0, 1)\}$, and this class does not coincide with $M(P^*) = \{Q : \beta_{\mathbb{R}} \to [0, 1] \text{ prob.} \mid Q((0, 1)) = 1\}$.

An open problem from this paper would be to determine whether $P(\Gamma)$ and $M(P^*)$ coincide when the random variables A and B are discrete but not simple. We conjecture that, if the initial probability space is non-atomic, we have $P(\Gamma) = M(P^*)$: in the same way that many results for random sets on finite spaces that can be extended to random sets on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, it might be possible to extend the result from the previous theorem (where Γ is simple) to the case of A, B discrete random variables (where, as a consequence, Γ has countable range except for a null subset of the initial space).

Remark 4.1. *In some situations, it may be useful to consider random intervals with one extreme open and the other one closed, such as* $\Gamma_2 = [A, B)$ *or* $\Gamma_3 = (A, B]$ *for some* $A, B : \Omega \to \mathbb{R}$ *with* $A < B$ *(see for instance [24]). Using the results from this paper, or straightforward adaptations of their proofs, we observe the following:*

- $\overline{M(P^*_{\Gamma_2})}$ *is the closed convex hull of* $P(\Gamma_2)$ *and, if the initial probability space is non-atomic, then* $\overline{M(P^*_{\Gamma_2})} = \overline{P(\Gamma_2)}$.
- *If* F_A *is continuous,* $P(\Gamma_2)$ *is a proper subset of* $M(P^*_{\Gamma_2})$.
- *If* A *and* B *are simple random variables and the initial probability space is non-atomic,* $P(\Gamma_2) = M(P^*_{\Gamma_2})$.

We can similarly derive properties for random intervals of the type $(A, B]$*, with* $A, B : \Omega \to \mathbb{R}$ *and* $A < B$. ♦

5 The connection with fuzzy numbers

As pointed our by several authors ([5, 26, 35]), random sets can be regarded as a special case of **fuzzy random variables**, that is, measurable mappings that point any element of the initial space to a fuzzy subset of the final space. Such mappings can be also interpreted as a model of the imprecise observation of a random variable. However, it is worth noting that particular instances of random intervals have also been connected to fuzzy numbers. A **fuzzy number** ([12]) is a normal fuzzy set $\tilde{X} : \mathbb{R} \to [0, 1]$ whose α-cuts are compact and convex subsets of the real line. Remember that these α-cuts are given by $\tilde{X}_\alpha = \{x \in \mathbb{R} : \tilde{X}(x) \geq \alpha\}$.

The connection between fuzzy numbers and random intervals is two-fold. On the one hand, it is easy to see ([11, 12]) that if $\tilde{X}$ is a fuzzy number, the multi-valued mapping $\Gamma : [0, 1] \to \mathcal{P}(\mathbb{R})$ given by $\Gamma(\alpha) = \tilde{X}_\alpha$ is a fuzzy (closed) interval. It satisfies moreover $P^*_\Gamma(\{x\}) = \tilde{X}(x)$, i.e., the one-point coverage function of Γ on x coincides with its image by the fuzzy number. This relationship has been studied further by Goodman ([14, 16]) and Gil ([13]). In particular, in [13] it is proven that, given a fuzzy number $\tilde{X}$ and a fixed probability space $(\Omega, \mathcal{A}, P)$, there exists a random interval on Ω whose one-point coverage function coincides with $\tilde{X}$.

Conversely, we may also study whether we can define a fuzzy number from a random interval. For instance, given an antitone random interval $\Gamma : [0, 1] \to \mathcal{P}(\mathbb{R})$ (i.e., such that $\omega_1 \leq \omega_2 \Rightarrow \Gamma(\omega_1) \supseteq \Gamma(\omega_2)$), the fuzzy set $\tilde{X} : \mathbb{R} \to [0, 1]$

given by $\tilde{X}(x) = P^*(\{x\})$ is a fuzzy number. This was extended by Dubois and Prade ([11]) to the case where the initial space is $([a,b], \beta_{[a,b]}, \frac{\lambda_{[a,b]}}{b-a})$. The next result shows that something similar holds if Γ is defined on an arbitrary probability space and is **consonant**, that is, if $\Gamma(\omega_1) \subseteq \Gamma(\omega_2)$ or viceversa for any ω_1, ω_2 in the initial space.

Proposition 5.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and consider a consonant random interval $\Gamma := [A, B]$. Let us define $\tilde{X} : \mathbb{R} \to [0,1]$ by $\tilde{X}(x) = P^*_\Gamma(\{x\})$. Then, $\tilde{X}$ is a fuzzy number.*

Proof: Let us show that the α-cuts of $\tilde{X}$ are compact subsets of $\mathbb{R}$.

- Consider $x_1 < x_2$ in $\tilde{X}_\alpha$. Then, $\min\{P^*(\{x_1\}), P^*(\{x_2\})\} \geq \alpha$. Since Γ is consonant, it is either $\{x_1\}^* \subseteq \{x_2\}^*$ or viceversa. Then, given $\omega \in \{x_1\}^* \cap \{x_2\}^*$, both x_1, x_2 belong to $\Gamma(\omega)$. Hence, $[x_1, x_2] \in \Gamma(\omega)$ and consequently $\{x_1\}^* \cap \{x_2\}^* \subseteq \{x\}^*$ for all $x \in [x_1, x_2]$. Hence, $[x_1, x_2] \subseteq \tilde{X}_\alpha$ and the α-cuts are convex.

- Let us prove now that these α-cuts are bounded. Assume for instance that $\sup \tilde{X}_\alpha = \infty$. From the previous point, we deduce that it is $P^*(\{x\}) \geq \alpha \ \forall x \geq k$ for some k. But this means that given $B = \cap_{n \geq k, n \in \mathbb{N}} \{n\}^*$, it is $P(B) = P(\cap_{n \geq k} \{n\}^*) \geq \alpha > 0$, because the consonancy of Γ implies that the class $\{\{x\}^* : x \in \mathbb{R}\}$ is totally ordered by set inclusion.

  Now, any $\omega \in B$ satisfies $x \in \Gamma(\omega) \ \forall x \geq k$, whence $[k, +\infty) \subseteq \Gamma(\omega)$, a contradiction. Since the same can be done respect to $\inf \tilde{X}_\alpha$, we deduce that this set is bounded.

- Let us show finally that these α-cuts are closed. Consider for instance a sequence $(x_n)_n$ in $\tilde{X}_\alpha$ s.t. $x_n \downarrow x$. Given $\omega \in \cap_n \{x_n\}^*$ it is $A(\omega) \leq x_n \leq B(\omega) \ \forall n$. Hence, it is $A(\omega) \leq x \leq B(\omega)$, whence $\cap_n \{x_n\}^* \subseteq \{x\}^*$. Hence, $P^*(\{x\}) \geq P(\cap_n \{x_n\}^*) \geq \alpha$, using again that $\{\{x\}^* : x \in \mathbb{R}\}$ is totally ordered. Since the same can be done respect to increasing sequences, we deduce that $\tilde{X}_\alpha$ is closed.

It remains to be proven only that the fuzzy set $\tilde{X}$ is normal: since P^*_Γ is a possibility measure ([32]), it is $\sup_{x \in \mathbb{R}} P^*(\{x\}) = 1$. Now, $\{\tilde{X}_\alpha : \alpha \in (0,1)\}$ is class of compact sets with the finite intersection property, whence there exists some $x_0 \in \cap_{\alpha \in (0,1)} \tilde{X}_\alpha = \tilde{X}_1$, and then it must be $P^*(\{x_0\}) = 1$. ■

The interpretation of this result would be the following: since the membership function of this fuzzy number coincides with the one-point coverage function of Γ, $\tilde{X}(x) = P^*_\Gamma(\{x\})$ would be the plausibility we give to the proposition 'x is the image of a point ω by the original random variable U_0'.

Since the upper probability of a consonant (in the sense defined above) random interval is a possibility measure[7], it is characterised by its one-point coverage function. Taking into account the duality between the upper and lower probabilities induced by a random set, we see that $\tilde{X}$ allows us to recover $P^*_\Gamma, P_{*\Gamma}$. On the other hand, since Theorem 3.2 guarantees that $P^*_\Gamma(A) = \max P(\Gamma)(A)$ and $P_{*\Gamma} = \min P(\Gamma)(A)\ \forall A \in \beta_{\mathbb{R}}$, we see that $\tilde{X}$ contains all the information about the values taken by P_{U_0} [8].

Unfortunately, from the point of view of the results established in this paper, consonant random intervals are not specially interesting: if we look at the sufficient conditions for the equality $P(\Gamma) = M(P^*)$ in Section 4, it is easy to check that a consonant random interval $\Gamma : [0,1] \to \mathcal{P}(\mathbb{R})$ does not satisfy any of them unless one of the extreme random variables A, B is constant.

If, on the other hand, we do not assume Γ to be consonant, we cannot assure the α-cuts of $\tilde{X}$ to be compact subsets of $\mathbb{R}$: it suffices to make $\Gamma = \{U\}$ for some simple random variable $U : \Omega \to X$. Then, the α-cuts of the fuzzy set $\tilde{X}$ defined in Proposition 5.1 would be finite sets (whence not necessarily convex). Moreover, for non-consonant random sets the upper probability will not be in general a possibility measure, whence a representation of its one-point coverage function in terms of a fuzzy set will produce in general a loss of information about the distribution of the original random variable.

6 Conclusions

In this paper, we have compared two different models of the probabilistic information of a random interval, interpreting this one as the result of the imprecise observation of a random variable. We have studied whether the class of the probability distributions of the measurable selections coincides with the class of probabilities bounded by the upper probability. This last set is easier to handle than the former, but it is less precise in general. We have focused our attention on random closed intervals and random open intervals. We have proven that the closures, in the topology of the weak convergence, of $M(P^*)$ and of $P(\Gamma)$ coincide when the initial probability space is non-atomic. Nevertheless, $P(\Gamma)$ can be a strict subset of $M(P^*)$. This means that, although the sets $P(\Gamma)$ and $M(P^*)$ are strongly related, the upper probability can cause a loss of precision respect to

[7] See a more complete study on this subject in [32].

[8] It is $P(\Gamma)(A) = [P_*(A), P^*(A)]\ \forall A \in \beta_{\mathbb{R}}$ whenever the initial probability space is non-atomic; see [29].

the class of probabilities induced by the measurable selections. Because of this, we have investigated if additional hypotheses on the random variables A and B determining the random interval guarantee the equality $P(\Gamma) = M(P^*)$. In the case of random closed intervals, we have obtained some sufficient conditions in terms of the relationships between the values of A and B. Concerning random open intervals, we have proven that if both these extremes are simple random variables and the initial probability space is non-atomic, then $P(\Gamma) = M(P^*)$. On the other hand, we have also shown that $P(\Gamma)$ is a proper subset of $M(P^*)$ when either of the distribution functions of the extremes of the random interval is continuous. All the conditions we have established for the equality between $P(\Gamma)$ and $M(P^*)$ require the initial probability space to be non-atomic. As we have already said, this hypothesis is not too strict, and holds for instance when we know that the probability distribution of the original random variable is absolutely continuous. On the other hand, when the initial probability space has atoms, we think that $P(\Gamma)$ does not coincide with $M(P^*)$ except in very particular cases.

These results lead us to conclude that the sets of probabilities $P(\Gamma)$ and $M(P^*)$ have a stronger relationship in the case of random intervals than in other types of random sets: in general, a random set does not necessarily have measurable selections, and even if it has, it may not satisfy the equalities $P^*(A) = \max P(\Gamma)(A)$ for all A in the final σ-field ([29]). Among the open problems from this paper, we want to point out the study of the properties of random rectangles (that is, random sets $\Gamma : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined as $\Gamma(\omega) = [A_1(\omega), B_1(\omega)] \times \cdots \times [A_n(\omega), B_n(\omega)]$ for $A_i, B_i : \Omega \to \mathbb{R}, A_i \leq B_i \ \forall i$). This type of random sets could be of interest when we are observing several characteristics of the elements of the same space. Concerning the connection between random intervals and fuzzy sets, we would like to study the relationship between random open intervals and fuzzy sets (not necessarily fuzzy numbers). This would give a different perspective to the problem studied.

Acknowledgements

The research in this paper has been partially supported by FEDER-MCYT, grant number BFM2001-3515, and MEC-DGI, grant number MTM2004-01269. We acknowledge the referees for their suggestions and comments.

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