# Probability and time

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**Abstract**

Probabilistic reasoning is often attributed a temporal meaning, in which conditioning is regarded as a normative rule to compute future beliefs out of current beliefs and observations. However, the well-established 'updating interpretation' of conditioning is not concerned with beliefs that evolve in time, and in particular with future beliefs. On the other hand, a temporal justification of conditioning was proposed already by De Moivre and Bayes, by requiring that current and future beliefs be consistent. We reconsider the latter proposal while dealing with a generalised version of the problem, using a behavioural theory of imprecise probability in the form of coherent lower previsions as well as of coherent sets of desirable gambles, and letting the possibility space be finite or infinite. We obtain that using conditioning is normative, in the imprecise case, only if one establishes future behavioural commitments at the same time of current beliefs. In this case it is also normative that present beliefs be *conglomerable*, which is a result that touches on a long-term controversy at the foundations of probability. In the remaining case, where one commits to some future behaviour after establishing present beliefs, we characterise the several possibilities to define consistent future assessments; this shows in particular that temporal consistency does not preclude changes of mind. And yet, our analysis does not support that rationality requires consistency in general, even though pursuing consistency makes sense and is useful, at least as a way to guide and evaluate the assessment process. These considerations narrow down in the special case of precise probability, because this formalism cannot distinguish the two different situations illustrated above: it turns out that the only consistent rule is conditioning and moreover that it is not rational to be willing to stick to precise probability while using a rule different from conditioning to compute future beliefs; rationality requires in addition the *disintegrability* of the present-time probability.

*Keywords:* Temporal reasoning, imprecise probabilities, conditioning, lower previsions, sets of desirable gambles, coherence, conglomerability.

## 1. Introduction

*What has time to do with probability?*

We are interested in probability understood in the subjective tradition: as an uncertainty formalism that allows you[1] to express beliefs and do rational reasoning. Conditioning is an important component to reason with probability. In fact, the computation of conditional beliefs (i.e., expectations or probabilities) is taken by some researchers as 'the' procedure to obtain future rational beliefs out of current beliefs and observations (i.e., some evidence), as if the Bayesian calculus—and Bayes' rule in particular—had captured the essence of the reasoning process itself through time.

Is this view justified? To see whether this is the case, it is useful to go back at the foundations of probability. As it has been well documented by Shafer [55, 56], De Moivre and Bayes provided, already in the 18$^{\text{th}}$ century, an argument for the temporal use of conditioning: it relies on constructing two bets, at present and future times, that jointly yield

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[1]We follow Good, de Finetti and Walley in referring to 'you' as the subject that holds some probabilistic assessments.

you a sure loss if you do not use conditioning to compute your future beliefs. This is, in other words, a (Dutch) book argument applied through time. The approach is not uncontroversial[2] and it may well clash with one's intuition: in fact, from the temporal-book argument it follows that once you have established your initial beliefs, your future rational behaviour will be 'mechanically' determined. Should this be the case, should you not be allowed to change your mind?

Nowadays, well-established approaches to probability seem to have taken a more cautious approach to defining the role of conditioning; this caution de facto corresponds to eliminating time from the picture. The so-called *updating interpretation* of conditioning reads as follows: "your expectation of a gamble (i.e., a bounded random variable) $f : \Omega \to \mathbb{R}$, conditional on event $B$ from a partition $\mathcal{B}$ of the possibility space $\Omega$, represents your *current* beliefs about $f$ under the assumption that $B$ occurs and that you obtain no other relevant information about $\Omega$". The crucial word in the previous phrase is 'current': it means that under the updating interpretation, conditional beliefs are beliefs that you hold now; moreover, there is nothing in that phrase that relates your current conditional beliefs with the behaviour you will adopt once, and if, $B$ occurs. In this view, Bayes' rule loses its temporal flavor and reveals a simpler nature, that of a consistency requirement between your current conditional and unconditional beliefs: in fact, Bayes' rule can be made to follow from the traditional book argument, the one that is applied to beliefs held at the same point in time.

Yet, part of the literature has kept on exploring the relationship between probability and time, in the spirit of De Moivre and Bayes' original intuition: this is the case, for instance, of the philosophical work on 'dynamic coherence' started in the seventies with Teller (who credited David Lewis for having originated the argument, see [63, note 1 to Section 1.3]) and that continued in the eighties with a number of papers [2, 3, 59, 60, 61]; Shafer's work, we have already mentioned, was also concerned to some degree with temporal considerations [55, 56]. More recent work by Shafer et al. [57] stresses such an aspect even more: among other things, it shows that Walley's generalisation of Bayes' rule to sets of probabilities [68, Section 6.4] is temporally consistent in a game-theoretic sense [58].

Some other tightly connected approach is the statistical work on 'temporal coherence' by Goldstein [21, 22, 23, 24], and the related one in philosophy by van Fraassen [66, 67]. In our view the aim here is different, however, as the focus does not appear to be on relating present and future behaviour, but rather on widening present beliefs so as to encompass also beliefs about future beliefs. The field of 'belief revision', originated in the work of Gärdenfors and colleagues [1, 18], attempts also to deal with temporal considerations in probability, besides logic. Its connection with the temporal-book idea is weaker, though.

*Contributions*

We aim at making a thorough analysis about the extent to which De Moivre and Bayes' intuition can be made to provide a firm foundation for a temporal interpretation of probabilistic reasoning. To this end, we consider a framework made of two time points: now, and a future one determined by the occurrence of an event $B \in \mathcal{B}$. Accordingly, we consider two uncertainty models: one that you hold at present time, that is, your *current beliefs* (we also call them your *current commitments*[3]), and another one that you will hold after $B$ occurs. We call the latter your *future commitments*.

Our approach to the problem initially makes no assumptions on the relationship between current and future commitments. We do not even force the analysis to focus on conditional beliefs: present beliefs are allowed to be generically made both of conditional and unconditional information. Rather, we let the relationship between current and future assessments emerge by itself by characterising what it means that current and future commitments are consistent. This will also reveal whether and when it is actually rational (or normative) for you to be self-consistent in time.

We shall pursue our aims within the framework of imprecise probability, and in particular start our work using Walley's behavioural theory of coherent lower previsions [68]: this is an extension of de Finetti's theory [12] to sets of probabilities that is close to robust Bayesianism. De Finetti's theory is based on the concept of a (linear) *prevision*, which is another name for an expectation functional; a *coherent lower prevision* is a lower envelope of linear previsions, which is in one-to-one correspondence with a closed and convex set of finitely additive probabilities. These tools enable us to deal uniformly with precise and imprecise probability, as well as with any cardinality of the possibility space $\Omega$—which is then allowed to be infinite. Section 2 provides an introduction to the theory that is conceived to make the work as self-contained as it is possible in a research paper. It also discusses the alternative representation of coherent lower previsions in terms of a *set of desirable gambles*: this is the set of gambles that you find desirable (i.e., that you

[2] See also philosopher Levi's fierce opposition to the idea of the temporal-book argument in his 'demons of decision' [37].

[3] Probabilistic assessments can be interpreted as commitments to engage in special types of bets.

would accept if they were offered to you) as a logical consequence of your probabilistic assessments. This helps us to convey the intuition behind the concepts and the results we present. Section 3 describes our temporal framework in detail, and introduces your uncertainty models in the form of two coherent lower previsions for your present and future commitments, respectively.

The core of our work starts in Section 4. We define a number of consistency notions for your current and future lower previsions, and show that each of these notions is appropriate in a different scenario, depending on the time when your future commitments are declared. For each of these notions, we give a number of characterisations, and establish a connection with other notions from the theory of coherent lower previsions.

One of the most interesting is the equivalence between one of our consistency notions and the *conglomerability* of your present beliefs. Loosely speaking, the notion of conglomerability is what allows us, in precise probability, to represent a prevision (i.e., an expectation) as an infinite mixture of conditional previsions. This notion was originally introduced by de Finetti in 1930 [9, 11] as a property that a finitely additive—but not countably additive—probability may not satisfy. Since then, the debate concerning whether or not conglomerability should be a rationality requirement in probability has never had an end (e.g, see [68, Section 6.8], [46, Section 3.4], but also [28, 29, 49, 50, 54]). Here we explicitly relate conglomerability to temporal considerations; and it is this very connection that allows us to make a clear point: that conglomerability should in fact be a rationality requirement when present and future commitments are established together (and moreover that future commitments should be equal to present conditional beliefs). Let us stress that we achieve this without strengthening the assumptions commonly employed in these cases,[4] like those in de Finetti's theory, but rather just coupling them with temporal considerations.

When present and future commitments are established at different times, the situation is more open: there are many ways to define future commitments that are consistent with current beliefs (and hence may be constrained by them to some extent). This means, in particular, that in the imprecise framework changing mind is compatible with temporal consistency. On the other hand, we do not see the possibility to argue in general that it should be normative for you to be consistent in time. The situation changes if we restrict the attention to the special case of precise probability, as in Section 4.5, and especially if we assume in addition that conditioning events have positive probability. We argue that in this case rationality requires that your present probability be *disintegrable* (disintegrability is a special case of conglomerability, see [14]) and that future commitments be defined by Bayes' rule. Stated differently, it appears to be a specificity of the Bayesian setup to disallow you to change your mind in order to keep consistency, and hence to regard future beliefs as predetermined once you have established your present beliefs. The framework is less rigid in case we allow conditioning events to be assigned zero probability.

Although we can say much about the consistency of your uncertainty models when these are represented by means of coherent lower previsions, there are situations when these are not expressive enough; this is for instance the case when we want to condition on sets of probability zero, as it is common in infinite spaces, or if we want to give a meaning to gambles with prevision equal to zero, which is important when we wish to model preferences. A more informative model for those cases are just *sets of desirable gambles*, when we use them in their full generality. We review the model in this light in Section 5, and discuss in addition how desirable gambles can be regarded as a particularly natural and powerful generalisation of propositional logic to uncertainty.

In Section 6 we take desirable gambles as our primitive model (from which one can actually derive coherent lower previsions, in case), and show how the consistency notions we have introduced in Section 4 can be extended to such a generalised setup, and which are the properties that hold in this case. In particular, we consider the important special case where your present beliefs are constructed in a hierarchical way through *marginal extension* [41, 44, 68], which is a generalisation of the law of iterated expectations to imprecise probability. We reconsider also the Bayesian case in the light of desirability: thanks to the notion of precision that is tailored to sets of desirable gambles, we show that in this case conditioning is the only rational rule to compute future commitments even when the probability of a conditioning event is zero.

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[4] Technically speaking, this has to do with a finitary feature of these uncertainty representations, including the ones we use in this paper, according to which it is never assumed that you should be willing to accept infinitely many gambles that are desirable to you, in case they are offered to you. It is controversial that the opposite should be assumed (e.g., de Finetti does not find it rationally justified while Walley does), and in addition it is a very strong assumption: one that makes it trivial to derive conglomerability, so that the latter is deprived of its own meaning—which lies entirely in the assumption—, and even that probabilities should be $\sigma$-additive. But $\sigma$-additivity leads us into measurability problems, which we are instead dispensed of in case we stay with finitely additive models.

In Section 7 we comment on the connections between our work and some related approaches: dynamic coherence; Jeffreys' rule and probability kinematics; the work by Goldstein and van Fraassen; Shafer et al.'s game-theoretic reinterpretation of the theory of coherent lower previsions; belief revision. In Section 8 we discuss at some length our updated point of view on these matters after the analysis we have done. Some additional technical results are given in Appendix A.

## 2. Coherent lower previsions

### 2.1. An introduction to the theory

Let us denote by $\Omega$ the possibility space, that is, the set of possible outcomes of an experiment, intended in a broad sense. Let us make this more concrete with the help of a running example that will be developed throughout the text.

*Example* 1 (Running example). You are a physician confronted with a situation of uncertainty originated by two different viruses causing flu: the usual seasonal virus (i.e., virus *s*) and a more serious atypical variant (virus *a*). Your experience tells you something about the likelihood of each of these two viruses on a person; there is also a blood test (with *p*ositive or *n*egative outcomes) that allows you to discriminate the viruses to some good degree. However, the information in these two cases may be uncertain, and this may render impossible the task of modelling it by means of precise probabilities. For the time being, we shall only model this problem by the variables $V$ and $T$, for virus and test, with possible values $\{a, s\}$ and $\{p, n\}$, respectively. The possibility space in this example is just the product space $\Omega \coloneqq \{(a, p), (a, n), (s, p), (s, n)\}$. We shall use the convention to denote by $V = v$ and $T = t$, for generic values $(v, t) \in \Omega$, the events $\{(v, p), (v, n)\}$ and $\{(a, t), (s, t)\}$, respectively. ⧫

In this paper we let $\Omega$ be general in the sense that we do not impose any restriction on its cardinality. We call a *gamble* any bounded function from $\Omega$ to the real numbers. A gamble is interpreted as an uncertain reward that depends on the unknown outcome of the experiment; we assume that the rewards are expressed in a utility scale that is linear for you.

We denote by $\mathcal{L}(\Omega)$ the set of all gambles on $\Omega$. $\mathcal{L}^+(\Omega)$ denotes the set of *positive* gambles on $\Omega$: that is, all gambles $f$ such that $f(\omega) \geq 0$ for all $\omega \in \Omega$ and $f(\omega) > 0$ for some $\omega \in \Omega$; we rewrite this notation for short as $f \gneq 0$ (similarly, the *negative* gambles $\{f \lneq 0\}$ are such that $f(\omega) \leq 0$ for all $\omega \in \Omega$ and $f(\omega) < 0$ for some $\omega \in \Omega$). We shall often use the symbol $B$ to denote a subset $B$ of $\Omega$, as well as to denote the indicator function $\mathbb{I}_B$ of subset $B$; this means for instance that $Bf$ shall denote the gamble given by

$$Bf(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in B \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$\mathcal{L}(B)$ denotes the set of all gambles on $B$, and $\mathcal{L}^+(B)$ is its subset of positive gambles. Sometimes we shall also use the shorter notation $\mathcal{L}$ to refer to the set of all gambles on a certain possibility space, when this is clear from the context or when we want to establish some result for a generic possibility space. We shall also use sometimes the short notation $\mathcal{L}^+$.

It is convenient to also introduce the following notation: when $f \in \mathcal{L}(\Omega)$ and $B$ is an element of a partition $\mathcal{B}$ of $\Omega$, we shall denote by $f_B$ the restriction of $f$ to $B$. Hence, $f_B \in \mathcal{L}(B)$. More generally speaking, we shall use subscript $B$ for gambles in $\mathcal{L}(B)$. On the other hand, if $f_B$ is a gamble on $B$, we shall denote by $Bf_B$ its extension to a gamble on $\Omega$, given by Eq. (1). Thus, $f = \sum_{B \in \mathcal{B}} Bf_B$.

Operations on gambles are understood point-wise. We shall focus in particular on the multiplication of a gamble $f$ with a constant $\lambda$, giving rise to gamble $\lambda f$, and on the sum of two gambles $f$ and $g$, giving rise to gamble $f + g$. Constant gambles are denoted by the corresponding real value: the distinction between the two cases will be clear from the context; for example, if $\alpha$ is a real number, then $f + \alpha$ denotes the sum of $f$ and the gamble constant on $\alpha$. Also, when a gamble is constant on the elements of a partition $\mathcal{B}$ of $\Omega$, we call it *$\mathcal{B}$-measurable*. Comparisons of gambles, such as $f \geq g$, are to be intended point-wise too (although $f \gneq 0$ is an exception to this rule as it is different from $f > 0$).

The theory of coherent lower previsions generalises probability theory (in the sense of de Finetti [12]) to the case where beliefs are specified imprecisely via sets of (finitely additive) probabilities. To see this, we need to define expectation, which de Finetti calls *prevision*. We focus in particular on lower and upper previsions, which arise naturally when your set of beliefs is consistent with a range of previsions; in addition we directly focus on the conditional case, and discuss the unconditional one as a special case.

*Definition* 1 (**Coherent conditional lower previsions**). Let $\mathcal{B}$ be a partition of $\Omega$, and for every $B \in \mathcal{B}$ let $\underline{P}(\cdot|B)$ be a real-valued functional on $\mathcal{L}(\Omega)$. Then $\underline{P}(f|B)$ is called the *lower prevision of $f$ conditional on $B$*. The $\mathcal{B}$-measurable gamble $\underline{P}(f|\mathcal{B}) \coloneqq \sum_{B\in\mathcal{B}} B\underline{P}(f|B)$ is called the *lower prevision of $f$ conditional on $\mathcal{B}$*. The functional $\underline{P}(\cdot|\mathcal{B}) : \mathcal{L}(\Omega) \to \mathcal{L}(\Omega)$ is called a *separately coherent conditional lower prevision* when the following conditions hold for every $f, g \in \mathcal{L}$, $B \in \mathcal{B}$ and $\lambda > 0$:

(SC1) $\underline{P}(f|B) \geq \inf_{\omega\in B} f(\omega)$;

(SC2) $\underline{P}(\lambda f|B) = \lambda\underline{P}(f|B)$;

(SC3) $\underline{P}(f + g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$.

In that case $\underline{P}(\cdot|B)$ is called a *coherent* lower prevision.

The behavioural interpretation of the conditional lower prevision $\underline{P}(f|B)$ is that of your current supremum price for buying gamble $f$ under the assumption that $\omega \in B$. When the focus is on selling rather than buying gambles, we come to conditional *upper* previsions: $\overline{P}(f|B)$ is your current infimum price to sell gamble $f$ under the assumption that $\omega \in B$. $\overline{P}(\cdot|B)$ and the functional $\overline{P}(\cdot|\mathcal{B})$ are then defined analogously to the case of lower previsions, and are called coherent and, respectively, separately coherent conditional upper previsions. The definition of conditional lower and upper previsions makes it clear that the following conjugacy relationship holds: $\overline{P}(f|B) = -\underline{P}(-f|B)$ for all $f \in \mathcal{L}$ and $B \in \mathcal{B}$; this allows us to focus our development on conditional lower previsions only. The occasional use we do of conditional upper previsions will be mostly motivated by mathematical convenience.

As we have mentioned already, coherent lower previsions represent lower expectation functionals (note that they are applied to gambles, that is, to bounded random variables). In fact, it is important to be aware right from the beginning that the theory of imprecise probability we have just started to describe, regards expectation as the primitive concept rather than probability. The idea is that you model your uncertainty by providing conditional and unconditional coherent lower previsions. Accordingly, when we speak of ‘beliefs’, we technically mean the uncertainty model, which in this case are lower previsions.[5]

Of course, you may still want to model your uncertainty using probabilities; this is possible by providing your lower previsions of the indicator functions. Thus, in this theory, an event $A \subseteq \Omega$ is represented by gamble $\mathbb{I}_A$, and your lower probability for $A$ is just the lower prevision $\underline{P}(\mathbb{I}_A)$. This can be written more simply as $\underline{P}(A)$, thanks to the convention that allows us to use $A$, besides $\mathbb{I}_A$, to denote the indicator function of set $A$.

*Example* 2 (Running example). Assume for instance that you have imprecise information about the likelihood of the two different types of flu depending on the outcome of the test. You may know for instance that if the test is positive, the atypical virus is at least three times as likely as the seasonal one, while if the test is negative, the seasonal virus is at least four times as likely as the atypical one. This leads to the probabilistic assessments

$$P(V = a|T = p) \geq 3P(V = s|T = p) \text{ and } P(V = s|T = n) \geq 4P(V = a|T = n).$$

These assessments can equivalently be represented by the following lower and upper conditional probabilities:

$$\underline{P}(V = a|T = p) = 0.75 \text{ and } \overline{P}(V = a|T = n) = 0.2,$$

which, in turn, determine the following coherent conditional lower previsions:

$$\underline{P}(f|T = p) = \min\{f(a, p), 0.75f(a, p) + 0.25f(s, p)\}$$
$$\underline{P}(f|T = n) = \min\{f(s, n), 0.8f(s, n) + 0.2f(a, n)\},$$

where $f$ is any gamble on the product space $\{(a, p),(s, p),(a, n),(s, n)\}$. ♦

[5] Later, when we introduce a more general theory than coherent lower previsions in Section 5, it will be a set of gambles that you desire, i.e., which you are prepared to accept.

However, modelling uncertainty only with probabilities (that is, specifying buying and selling prices only for indicator functions) limits expressiveness in the imprecise case: it is well known that lower probabilities do not determine coherent lower previsions in general, in the sense that there may be more than one coherent lower prevision that is consistent with the specified probabilities (see, e.g., [68, Section 2.7.3]). Therefore, to take full advantage of the theory, it is necessary to move from events to the richer language of gambles.

Precise probability is obtained when conditional lower and upper previsions coincide:

*Definition* 2 (**Conditional linear previsions**). If for a coherent lower prevision it holds that $\underline{P}(f|B) = \overline{P}(f|B)$ for all $f \in \mathcal{L}$, then we denote the common value by $P(f|B)$ and call it the *linear prevision of $f$ conditional on $B$*. When this holds for all $B \in \mathcal{B}$, we define $P(\cdot|\mathcal{B})$ in analogy with the case of conditional lower previsions, and call it a *conditional linear prevision*.

*Example* 3 (Running example). In case you had precise information, stating that if the test is positive, the atypical virus is *exactly* three times as likely as the seasonal one, while if the test is negative, the seasonal virus is exactly four times as likely as the atypical one, we would end up with the conditional linear prevision $P(V|T)$ given by

$$P(f|T = p) = 0.75f(a, p) + 0.25f(s, p) \text{ and } P(f|T = n) = 0.8f(s, n) + 0.2f(a, n)$$

for any gamble $f$ on $\{(a, p), (s, p), (a, n), (s, n)\}$. ♦

The case of (unconditional) lower and linear previsions follows as a special case from the above definitions by considering the trivial partition $\{\Omega\}$. In that case we simplify the notation by writing $\underline{P} \coloneqq \underline{P}(\cdot|\{\Omega\})$ as well as $P \coloneqq P(\cdot|\{\Omega\})$.

A linear prevision $P$ is in one-to-one correspondence with the finitely additive probability that is its restriction to (indicator functions of) events. A coherent lower prevision $\underline{P}$ is in one-to-one correspondence with the *credal set*[6] $\mathcal{M}(\underline{P})$ of linear previsions given by $\mathcal{M}(\underline{P}) \coloneqq \{P : P(f) \geq \underline{P}(f)\ \forall f \in \mathcal{L}\}$. This shows that $\underline{P}$ is in correspondence with a set of finitely additive probabilities; in addition, $\underline{P}$ corresponds to the lower envelope of the previsions in $\mathcal{M}(\underline{P})$: $\underline{P}(f) = \inf\{P(f) : P \in \mathcal{M}(\underline{P})\}$ for every $f \in \mathcal{L}$. These relationships immediately extend to the conditional case.

In this paper we are going to work with beliefs established at different points in time. When these beliefs are represented by means of lower previsions, we may end up with an unconditional coherent lower prevision $\underline{P}$ and with a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$. The consistency between an unconditional and a conditional lower prevision in Walley's theory is verified by means of a notion of (joint) coherence. In order to define it, it is convenient to consider some special gambles: given a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ and a gamble $f \in \mathcal{L}$, we let

$$G(f|B) \coloneqq B(f - \underline{P}(f|B)) \text{ and } G(f|\mathcal{B}) \coloneqq f - \underline{P}(f|\mathcal{B}) = \sum_{B \in \mathcal{B}} G(f|B).$$

*Definition* 3 (**Coherence for lower previsions**). Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of $\Omega$ and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent conditional lower prevision on $\mathcal{L}$. We say that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are *coherent* when they satisfy the following conditions:

GBR. $\underline{P}(G(f|B)) = 0$ for every $f \in \mathcal{L}, B \in \mathcal{B}$. [Generalised Bayes rule]

CNG. $\underline{P}(G(f|\mathcal{B})) \geq 0$ for every $f \in \mathcal{L}$. [Conglomerability]

Condition GBR is called *generalised Bayes rule* because it amounts to applying Bayes' rule to all the elements of a credal set—if that is possible—in order to obtain the conditional credal set. Therefore it becomes Bayes' rule in the precise case. Condition CNG refers to the *conglomerability* of an unconditional lower prevision with a conditional one, and is tightly related to de Finetti's original formulation of conglomerability [9] (a related definition of conglomerability, in this case of a single coherent lower prevision, is provided in Definition 5 later on). Note moreover that CNG follows from GBR and the coherence of $\underline{P}$ when the partition $\mathcal{B}$ is finite. Together, GBR and CNG can also be given a behavioral interpretation [68, Chapter 6]: they mean that a finite combination of gambles whose desirability follows from the lower previsions $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ should still be desirable, and moreover that a gamble you have not deemed desirable should not become desirable by considering the implications of the assessments in $\underline{P}, \underline{P}(\cdot|\mathcal{B})$.

[6]By a credal set we mean a set of linear previsions that is closed in the *weak* topology* and convex. The weak* topology is the smallest topology for which all the evaluation functionals given by $f(P) \coloneqq P(f)$, where $f \in \mathcal{L}$, are continuous.

*Example* 4 (Running example). Assume for instance that you have unconditional information, stating that the prevalence of the atypical virus in that region is of at most 5%, while that of the seasonal virus is of at least 95%. This corresponds to the probabilistic assessments

$$P(a) \leq 0.05 \text{ and } P(s) \geq 0.95;$$

if we consider the set of probabilities on $\{(a,p),(s,p),(a,n),(s,n)\}$ that are compatible with this information, we end up with the coherent lower prevision $\underline{P}$ given by

$$\underline{P}(f) = \min\{\min\{f(s,n), f(s,p)\}, 0.95 \min\{f(s,n), f(s,p)\} + 0.05 \min\{f(a,n), f(a,p)\}\} \tag{2}$$

for any gamble $f$. To see if this lower prevision satisfies coherence with respect to the conditional model from Example 2, note that given the gamble $f := \mathbb{I}_{(a,p)}$, it holds that

$$\underline{P}(f|T=p) = 0.75, \text{ whence } G(f|T=p) = 0.25\mathbb{I}_{(a,p)} - 0.75\mathbb{I}_{(s,p)},$$

and this means that

$$\underline{P}(G(f|T=p)) = \min\{-0.75, 0.05 \cdot 0 - 0.95 \cdot 0.75\} = -0.75 < 0.$$

Therefore, the two lower previsions are not coherent, because they do not satisfy GBR. Note that in this specific example GBR is equivalent to coherence, because condition CNG follows from it since the partition is finite. ⧫

Generalised Bayes enables us to define the least-committal extension of a coherent lower prevision to the conditional case:

*Definition* 4 (**Conditional natural extension for lower previsions**). Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$. The *natural extension of $\underline{P}$ conditional on $B \in \mathcal{B}$* is given by

$$\underline{E}(f|B) := \begin{cases} \sup\{\mu : \underline{P}(B(f-\mu)) \geq 0\} & \text{if } \underline{P}(B) > 0 \\ \inf_{\omega \in B} f(\omega) & \text{otherwise.} \end{cases} \tag{3}$$

This is a separately coherent conditional lower prevision determined through generalised Bayes rule when the conditioning event has positive lower probability: in that case, $\underline{E}(f|B)$ is the only value for which GBR is satisfied with respect to $\underline{P}$; and it is vacuous, which means completely uninformative, in the remaining case. In fact, when the conditioning event has zero lower probability there may be many conditional lower previsions satisfying generalised Bayes rule with $\underline{P}$ [68, Section 6.10]; the vacuous one corresponds to the smallest [68, Theorem 8.1.6]. The conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ is always coherent with $\underline{P}$ when $\mathcal{B}$ is finite.

*Example* 5 (Running example). If we consider the coherent lower prevision $\underline{P}$ from Example 4, then it satisfies $\underline{P}(V=a) = 0$ and $\underline{P}(V=s) = 0.95$. Hence, the conditional natural extension $\underline{P}(T|V=a)$ is vacuous: we have $\underline{P}(f|V=a) = \min\{f(a,p), f(a,n)\}$. On the other hand, $\underline{P}(f|V=s)$ is determined by the generalised Bayes rule, which in this case produces a vacuous model too:

$$\underline{P}(T|V=s) = \min\{f(s,p), f(s,n)\};$$

this is because if we consider any $\mu > \min\{f(s,p), f(s,n)\}$, then

$$\underline{P}(\mathbb{I}_s(f-\mu)) = \min\{f(s,n), f(s,p)\} - \mu < 0,$$

and therefore GBR is not satisfied. ⧫

When the conditioning partition $\mathcal{B}$ is infinite, a lower prevision need not be coherent with its conditional natural extension; the latter is rather a lower bound of any conditional lower prevision that is coherent with the unconditional model. Hence, the acceptable buying prices encoded by the conditional natural extension should be acceptable under any coherent extension to the conditional case. On the other hand, the coherence of a coherent lower prevision with its conditional natural extension is characterised by the notion of conglomerability:

*Definition* 5 (**Conglomerability for lower previsions**). A coherent lower prevision $\underline{P}$ is called $\mathcal{B}$-*conglomerable* when there is a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ such that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent.

In this paper we always focus on a single partition $\mathcal{B}$ of $\Omega$; for this reason we shall often say simply that a lower prevision is conglomerable rather than $\mathcal{B}$-conglomerable. In particular, $\underline{P}$ is conglomerable if and only if it is coherent with its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ given by Eq. (3) [68, Theorem 6.8.2]. Moreover, $\underline{P}$ is always conglomerable when $\underline{P}(B) = 0$ for every $B \in \mathcal{B}$, or when the partition $\mathcal{B}$ is finite.

A related notion is that of disintegrability:

*Definition* 6 (**Disintegrability**). A linear prevision $P$ is called $\mathcal{B}$-*disintegrable* (or just disintegrable) when there is a linear conditional prevision $P(\cdot|\mathcal{B})$ such that $P, P(\cdot|\mathcal{B})$ are coherent.

Given a linear prevision $P$ and a conditional linear prevision $P(\cdot|\mathcal{B})$, conditions GBR and CNG together are equivalent to the equality $P = P(P(\cdot|\mathcal{B}))$. Trivially, if a linear prevision $P$ is disintegrable, then it is also conglomerable. However, the converse is not true: there are linear previsions $P$ satisfying conditions GBR and CNG with respect to a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ but not with respect to any conditional linear prevision $P(\cdot|\mathcal{B})$ (for an example, see [14] and [68, Example 6.6.10]).

There is also a weaker consistency notion for lower previsions that is called avoiding sure loss:

*Definition* 7 (**Avoiding sure loss for previsions**). Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, $\mathcal{B}$ a partition of $\Omega$ and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent conditional lower prevision on $\mathcal{L}$. We say that $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ *avoid sure loss* when for every $f, g \in \mathcal{L}$, it holds that

ASL. $\sup[G(f) + G(g|\mathcal{B})] \geq 0$.

The behavioural interpretation of this condition is that by accepting a finite combination of gambles whose desirability follows from the definition of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ you should never be subject to a sure loss. If $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent then they also avoid sure loss; both conditions are equivalent when $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are linear (unconditional and conditional) previsions.

*Example* 6 (Running example). Consider again the unconditional and conditional lower previsions $\underline{P}, \underline{P}(V|T)$ from Example 4. We already saw there that they are not coherent. To see that they avoid sure loss, note that for any gamble $f$ it follows from Eq. (2) that $\underline{P}(f) \leq \min\{f(s,n), f(s,p)\}$, whence $G(f)(s,n) \geq 0$ and $G(f)(s,p) \geq 0$. Since on the other hand it also follows from the definition of $\underline{P}(V|T)$ that $\underline{P}(g|T = n) \leq g(s,n)$ for any gamble $g$, we deduce that

$$[G(f) + G(g|T)](s,n) \geq 0$$

for any pair of gambles $f, g$. As a consequence, $\underline{P}, \underline{P}(V|T)$ avoid sure loss. ♦

In particular, if we apply condition ASL to an unconditional lower prevision $\underline{P}$ on $\mathcal{L}$, it turns out that $\underline{P}$ avoids sure loss if and only if its associated credal set $\mathcal{M}(\underline{P})$ is non-empty. This holds in particular when $\underline{P}$ is coherent, although both conditions are not equivalent.

### 2.2. *Correspondence with a set of gambles*

A coherent lower prevision can be expressed equivalently by determining the set of gambles whose acceptability it encompasses.

Given a coherent lower prevision $\underline{P}$ on $\mathcal{L}$, its associated set of gambles is given by

$$\mathcal{R} := \mathcal{L}^+ \cup \{f \in \mathcal{L} : \underline{P}(f) > 0\}. \tag{4}$$

If we interpret the values $\underline{P}(f), f \in \mathcal{L}$, as your supremum acceptable buying prices for the gambles $f \in \mathcal{L}$, then the set $\mathcal{R}$ represents those that you are sure to find acceptable: those non-zero gambles that may only make you subject to a gain ($\mathcal{L}^+$) and those for which you are disposed to pay a positive amount in order to buy them ($\{f \in \mathcal{L} : \underline{P}(f) > 0\}$).

Similarly, given a conditional lower prevision $\underline{P}(\cdot|B)$ on $\mathcal{L}(\Omega)$, its associated set of gambles is given by

$$\mathcal{R}^{|B} := \{f \in \mathcal{L} : f = Bf \text{ and } (Bf \in \mathcal{L}^+ \text{ or } \underline{P}(f|B) > 0)\}. \tag{5}$$

Note also that we can recover $\underline{P}(\cdot|B)$ from $\mathcal{R}^{|B}$ by means of

$$\underline{P}(f|B) = \sup\{\mu : B(f-\mu) \in \mathcal{R}^{|B}\}. \tag{6}$$

In this paper, we shall make a connection between a number of notions of temporal consistency for coherent lower previsions and associated properties of the sets of desirable gambles they induce. In particular, we shall use the following:

*Definition* 8 (**Coherence for gambles**). A subset $\mathcal{R} \subseteq \mathcal{L}$ is called *coherent* when it satisfies the following conditions:

D1. $\mathcal{L}^+ \subseteq \mathcal{R}$; [Accepting Partial Gains]

D2. $0 \notin \mathcal{R}$; [Avoiding Null Gain]

D3. $f \in \mathcal{R}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{R}$; [Positive Homogeneity]

D4. $f, g \in \mathcal{R} \Rightarrow f + g \in \mathcal{R}$, [Additivity]

and it is said to *avoid partial loss* when it is included in a coherent set.

Coherence means that you should be willing to accept a transaction represented by a finite number of desirable gambles (D4), and to make a linear change in the utility scale without affecting the desirability of the gambles (D3). Moreover, it implies that a non-zero gamble that can never give you a negative reward should be desirable (D1), while a gamble that can never give you a positive reward should not (D2). Avoiding partial loss suffices to extend your assessments while 'correcting' them into coherent ones. Geometrically, a coherent set is a convex cone, that is, a set closed with respect to finite positive linear combinations (as from D3–D4). The smallest, and hence the least committal, of the coherent cones that include $\mathcal{R}$, is called its natural extension:

*Definition* 9 (**Natural extension for gambles**). If a set $\mathcal{R}$ avoids partial loss, then the intersection of all its coherent supersets, denoted by $\mathcal{E}_{\mathcal{R}}$, is coherent and it is called its *natural extension*.

Then $\mathcal{R}$ is coherent if and only if it coincides with its natural extension. An example of a coherent set is the one induced by a coherent lower prevision, as in (4).

*Remark* 1. Note that $\mathcal{E}_{\mathcal{R}}$ does not contain any gamble $g \leq 0$. The case $g = 0$ is excluded by D2. In the case $g \lneq 0$, we should have that $0 = -g + g \in \mathcal{E}_{\mathcal{R}}$, by D1 and D4, and this contradicts D2 again.

On the other hand, it is not difficult to show that given two coherent sets of gambles $\mathcal{R}_1, \mathcal{R}_2$, their union $\mathcal{R}_1 \cup \mathcal{R}_2$ avoids partial loss if and only if the set

$$\mathcal{R}_1 \oplus \mathcal{R}_2 := \{f + g : f \in \mathcal{R}_1 \cup \{0\}, g \in \mathcal{R}_2 \cup \{0\}, f \neq 0 \text{ or } g \neq 0\}$$

satisfies D2, and it is coherent if and only if $\mathcal{R}_1 \cup \mathcal{R}_2 = \mathcal{R}_1 \oplus \mathcal{R}_2$ (see Proposition 7 in Secton 6 and Lemma 3 in Appendix A). ♦

Axioms D1–D4 show that the coherence notion we have introduced is equivalent to that proposed by Peter Williams in [69]; in particular, it should not be confused with the strongest definition proposed by Walley in [68, Appendix F1]. Walley's definition includes in addition a requirement of conglomerability as expressed by axiom D5 below:[7]

*Definition* 10 (**Conglomerability for gambles**). Given a set of desirable gambles $\mathcal{R}$ that satisfies D1–D4 and a partition $\mathcal{B}$ of $\Omega$, $\mathcal{R}$ is called *$\mathcal{B}$-conglomerable* (or just *conglomerable*) when it also satisfies:

D5. $f \in \mathcal{L} \setminus \{0\}, Bf \in \mathcal{R} \cup \{0\}\ \forall B \in \mathcal{B} \Rightarrow f \in \mathcal{R}$,

where $Bf$ is given by Eq. (1), and of course $f = \sum_{B \in \mathcal{B}} Bf$.

[7]However, Walley requires axiom D5 to hold for all the partitions of $\Omega$; this is called 'full conglomerability'. In contradistinction, our axiom D5 is only used with respect to the single partition $\mathcal{B}$. This is called 'partial conglomerability' and bears no implications on the much stronger requirement of full conglomerability. Therefore, whenever we talk of conglomerability in this paper, we mean partial conglomerability.

Conglomerability follows from D4 in case the partition is finite; hence Williams' and Walley's coherence notions are equivalent in that case (and in particular when $\Omega$ is finite). The treatment of the infinite case is a controversial matter: Walley claims that conglomerability should be imposed as a rationality requirement [68, Section 6.9.8] while other authors, such as de Finetti [9, 11] and Williams [69], reject this idea. This question is important, especially because conglomerability can be related to $\sigma$-additivity: this means that what de Finetti rejects, by rejecting conglomerability, is that $\sigma$-additivity should be a rationality requirement; to him, rationality only constrains us to stay with finitely additive probabilities.

Conglomerability can be used to define a special type of natural extension [44]:

*Definition* 11 (**Conglomerable natural extension for gambles**). Given a set of desirable gambles $\mathcal{R}$ and a partition $\mathcal{B}$ of $\Omega$, the *$\mathcal{B}$-conglomerable natural extension* of $\mathcal{R}$, if it exists, is the smallest set $\mathcal{F}$ that contains $\mathcal{R}$ and satisfies D1–D5.

Let us move on now to consider conditioning for a set of desirable gambles. Conditioning is made with respect to an element $B$ of a partition $\mathcal{B}$ of $\Omega$.

*Definition* 12 (**Conditioning for gambles**). Given a coherent set of desirable gambles $\mathcal{R}$ and a partition $\mathcal{B}$ of $\Omega$, we define beliefs conditional on an element $B$ of $\mathcal{B}$ as the set

$$\mathcal{R}^{\rfloor B} := \{f \in \mathcal{L} \colon f = Bf \in \mathcal{R}\}. \tag{7}$$

Sometimes we need to represent this set through gambles defined on $\mathcal{L}(B)$; then we use the equivalent representation given by $\mathcal{R}_{\rfloor B} := \left\{f_B \in \mathcal{L}(B) \colon Bf_B \in \mathcal{R}^{\rfloor B}\right\}$.

One interesting property is that this conditional set of gambles induces the conditional natural extension of $\underline{P}$, given by Eq. (3). If we define

$$\underline{P}(f|B) := \sup\{\mu : B(f-\mu) \in \mathcal{R}\} = \sup\{\mu : B(f-\mu) \in \mathcal{R}^{\rfloor B}\}, \tag{8}$$

where $\mathcal{R}$ is the set of gambles induced by $\underline{P}$ through (4) and $\mathcal{R}^{\rfloor B}$ is derived from $\mathcal{R}$ by means of (7), then we have the following:

**Lemma 1.** *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$, and let $\underline{P}(\cdot|\mathcal{B})$ be the conditional lower prevision it induces by means of Eq.* (8)*. Then:*

(a) *$\underline{P}, \underline{P}(\cdot|\mathcal{B})$ satisfy GBR.*

(b) *$\underline{P}(\cdot|\mathcal{B})$ coincides with the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$.*

*Proof.*

(a) Consider a gamble $f$ on $\Omega$ and $B \in \mathcal{B}$. From Eq. (8), for every $\delta > 0$ the gamble $B(f - \underline{P}(f|B) + \delta) \in \mathcal{R}$, whence $\underline{P}(B(f - \underline{P}(f|B) + \delta)) \geq 0$ by Eq. (4). Since this holds for every $\delta > 0$ we deduce that $\underline{P}(G(f|B)) \geq 0$. On the other hand, if $\underline{P}(G(f|B)) > 0$, then there is some $\delta > 0$ such that $\underline{P}(B(f - \underline{P}(f|B) - \delta)) > 0$, whence $B(f - \underline{P}(f|B) - \delta) \in \mathcal{R}$ and as a consequence $\underline{P}(f|B) \geq \underline{P}(f|B) + \delta$, a contradiction.

(b) Taking point (a) into account, now we must show only that $\underline{P}(f|B) = \inf_B f$ whenever $\underline{P}(B) = 0$. We have that

$$\begin{aligned}\underline{P}(f|B) = \sup\{\mu : B(f-\mu) \in \mathcal{R}\} &= \sup\{\mu : B(f-\mu) \geq 0 \text{ or } \underline{P}(B(f-\mu)) > 0\} \\ &= \sup\{\mu : B(f-\mu) \geq 0\} = \inf_B f,\end{aligned}$$

because of Eq. (4), and because $\underline{P}(B(f-\mu)) \leq \underline{P}(B(\sup_B f - \mu)) = \underline{P}(B)(\sup_B f - \mu) = 0$, taking into account that $\mu \leq \underline{P}(f|B) \leq \sup_B f$. □

The definition of conditioning for a set of desirable gambles is simply based on restricting the attention to the desirable gambles that are zero outside $B$. This corresponds to the so-called *contingent interpretation* of conditioning: we can think of it as a way of modelling your present attitudes towards gambles that are called off if the outcome $\omega \in \Omega$ of the experiment, for which you are accepting gambles, does not belong to $B$. On the other hand, the *updating interpretation* of conditioning understands set $\mathcal{R}^{\rfloor B}$ as the gambles you are disposed to accept *now* if you assume that $B$

occurs and that you obtain no other relevant information about $\Omega$. Walley discusses the agreement of the contingent and the updating interpretations in [68, Section 6.1.5].

A question of particular importance is that the updating interpretation makes no claim whatsoever concerning the gambles you will be committed to accept once (and if) $B$ actually obtains. Since this is a frequent source of confusion in subjective probability, let us stress once again that conditional beliefs refer only to your current beliefs and bear no implications (in absence of further conditions or justifications) on the future. To enforce this distinction, in the following we shall use the terminology *conditional* beliefs to talk of (current) updated beliefs, and *future commitments* for the probability model you will endorse after $B$ actually obtains. See Section 3 for a more detailed discussion.

## 3. Introducing the temporal setup

### *3.1. On the relationship of probability with time*

Let us recall that you are interested in the outcome of an experiment that is known to belong to the set $\Omega$. Section 2 described, among other things, how to model your uncertainty about $\Omega$ through a coherent lower prevision $\underline{P}$. This lower prevision has to be intended as *commitments* on your side: by providing $\underline{P}$, you commit yourself to accept any gamble you are offered from the set $\mathcal{R}$ originated by $\underline{P}$ through Eq. (4). This implies also that you will not accept the zero gamble, because of D2, nor any gamble $-f \in -\mathcal{R}$, for you are committed to accept $f$ and $f - f = 0$. On the other hand, establishing $\mathcal{R}$ has no implications on the gambles in $\mathcal{L} \setminus (\mathcal{R} \cup -\mathcal{R} \cup \{0\})$: in the actual transactions with an opponent, you could actually accept some of them and reject some others, or just be undecided. In other words, we acknowledge that your uncertainty model may be only an *incomplete representation* of your beliefs, rather than an *exhaustive* one. This means, in particular, that your actual behaviour might even be consistent with a lower previsions that is more (but never less) precise than $\underline{P}$. There are many reasons why this may happen; see [68, Section 2.10.3] for a discussion about this point.

It is important to clearly understand the relationship of all this with time. In fact, Section 2 was entirely focused on a single point in time—which we conventionally take to be the present moment. In particular, that section assumes that you are providing your beliefs about $\Omega$ *now* and makes no claims about the dynamics of your beliefs through time. This is the case even when one considers conditional assessments, which still refer, by definition, to beliefs at present time. In this setup, there is no link, let alone a formal one, between your present and future commitments.

If we want to relate your present commitments to your future ones, we need to explicitly introduce time in the process of defining your assessments. This is what we set out to do. To this end, we consider an additional time point, besides now, that is determined by the outcome of a further experiment. The latter experiment will yield an element $B$ of a partition $\mathcal{B}$ of $\Omega$, thus informing you that the outcome $\omega \in \Omega$ of the former experiment—the one you are really after—actually belongs to $B$.

The situation then is going to be the following. At present time you define your current beliefs $\underline{P}$; from this moment to the future time point when $B$ occurs, an opponent may offer you a (finite) number of gambles and you will be committed to accept all those that belong to $\mathcal{R}$, the set associated to $\underline{P}$. On the other hand, you will also establish some other assessments in the form of a lower prevision $\underline{P}_B$: this represents your *future commitments*, which only become effective after $B$ is observed. The idea here is that from that occurrence of $B$ onwards, you will be committed to accept gambles from the set of desirable gambles $\mathcal{R}^B$ associated to $\underline{P}_B$, and no longer from $\mathcal{R}$. We consider three possible time periods when you can decide to establish your future commitments, as shown in Figure 1: now, later but before $B$, after $B$.

In practice there are a number of reasons why you might want to define $\underline{P}_B$ in different time periods. For example, you might exclude that the availability of extra time to reflect on $\underline{P}$ could lead you to modify your current conditional beliefs. In this case you might want to set your future commitments equal to your conditional beliefs, and you would do it now. Or it could be the case that at the time when you establish your present beliefs you do not even know which are the possible events to observe in the future, that is, you do not know what the partition $\mathcal{B}$ is going to be. Imagine that you come to know the form of $\mathcal{B}$ some time later and before $B$ occurs. You will probably use some of the remaining time to specifically improve on your assessments concerned with your beliefs conditional on the events of $\mathcal{B}$, and possibly commit to them (for the future) before $B$ occurs. Finally, it could well be the case that you know both $\mathcal{B}$ and $B$ only after the latter occurs. In this case you will specifically focus only on beliefs that depend on the occurrence of such a specific $B$.

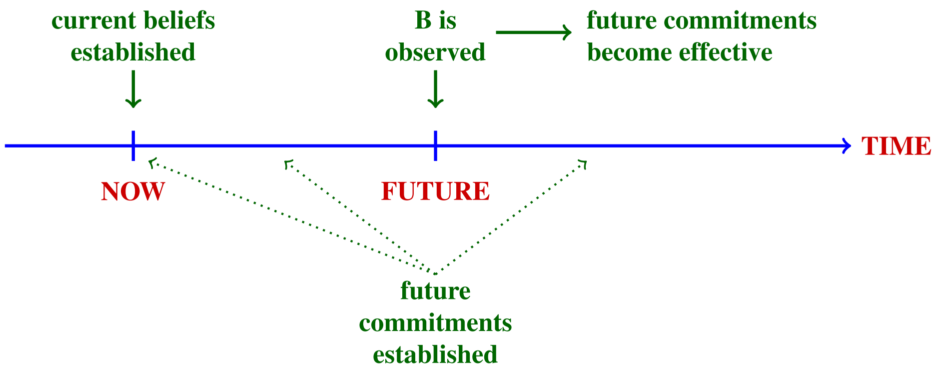


Figure 1: The temporal setup. Current beliefs are always established now. Future commitments can be established in three different periods: now; afterwards but before $B$ occurs; after $B$ occurs. Future commitments are never effective before $B$ occurs.

Needless to say, your future commitments $\underline{P}_B$ express your beliefs about $\Omega$ at the time point when you establish them (also in this case, $\mathcal{R}^B$ may be only an incomplete representation of your beliefs). Therefore, in case you establish them before $B$ occurs, they cannot in general be regarded as your beliefs *after* $B$. This is the reason why we call them future commitments rather than future beliefs. Still, they are commitments: for you are aware that in the moment when you decide to establish (i.e., declare) them, you are committing yourself to accept gambles, after $B$ occurs, from the set of desirable gambles $\mathcal{R}^B$ induced by $\underline{P}_B$. Note moreover that the beliefs represented by $\underline{P}_B$ will obviously take into account that the commitments they express will become effective after $B$ occurs, and therefore they will only focus on the case where the outcome $\omega \in \Omega$ of the original experiment belongs to $B$; in other words, they are beliefs about the set $B \subseteq \Omega$.

A few additional remarks are in order:

- In this paper we are interested in the case of two time points, as indicated above; we are not considering extensions to multiple future time points. Accordingly, we assume that the present time coincides with the beginning of the process of establishing commitments, in the sense that $\underline{P}$ represents the first commitments you have made about $\Omega$.

- Let us point out more clearly something that is already implied by the above discussion: after the present moment, when your current lower prevision $\underline{P}$ is established, you are no longer allowed to modify or drop it until event $B$ occurs. This means that the commitments it encodes actually constrain your behaviour up to the occurrence of $B$. We are aware that some may feel that this requirement (as well as others in this section) comes as too restrictive. On the other hand, we need to clearly define the problem to deal with. More flexible settings will likely be easier to address after some basic framework like the present one has been analysed.

- We are making no assumptions about the process that leads you to define the future lower prevision $\underline{P}_B$; you can define it arbitrarily. In particular, we are not assuming that it coincides with the conditional natural extension $\underline{E}(\cdot|B)$ of $\underline{P}$, that is, your present conditional beliefs. However, we shall assume that your future commitments are *known*, instead of being random variables, as is the case for the related approaches by Goldstein and van Fraassen we shall discuss in Section 7.3. Moreover, we are assuming that they are indeed commitments: once they are established, your behaviour after $B$ is going to be constrained by $\underline{P}_B$ (it is not necessarily going to be fully determined by $\underline{P}_B$, since we are not assuming that $\underline{P}_B$ is an exhaustive model). This appears to be in line with recent work by Shafer et al. [57], and less so with respect to older work by Shafer.[8]

[8] In fact, we have to say that it is not entirely clear to us yet how much Shafer was actually discussing a temporal setting rather than the updating interpretation in his early works on the subject. For instance, in the third last paragraph in [56, p. 266], Shafer seems to support the updating interpretation, while in other parts of the paper he seems to be concerned with temporal questions.

- In order to avoid confusion, let us also point out that your future commitments $\underline{P}_B$ are beliefs (at the time when you establish them) about $B$: they are *not*, and should not be interpreted as, beliefs about your future beliefs.
- On the other hand, we assume that after the definition of $\underline{P}$, *you will not receive new information relevant to* $\Omega$ *other than* $B$ (and the related partition $\mathcal{B}$); hence, we are dealing with a case of *exact information*, in the words of Shafer et al. [56].[9] This setup resembles the one commonly adopted in probability when defining the updating interpretation of conditioning: that a subject, in the process of assessing his conditional beliefs, assumes to get to know $B$ and nothing else new about $\Omega$. However, there are also differences with our setup. One that is especially clear arises if you define $\underline{P}_B$ after $B$ occurs: in that case it will not be only an assumption for you that $B$ is the only new information observed, it will arguably be a matter of fact. Another difference is indeed that your future commitments can be established later than now (check Figure 1), while conditional beliefs are always established at present time. A final difference is that in the traditional updating interpretation there is no statement claiming that your future behaviour should be constrained by your conditional beliefs: in fact, this is one of the key points of this paper, that is, distinguishing clearly your future commitments, those that will actually constrain your behaviour in the future, from your current conditional beliefs.
- The previous point assumes that your state of information stays the same from now to the occurrence of $B$. What then justifies a possible difference between $\underline{E}(\cdot|B)$ and $\underline{P}_B$ is just the availability of additional time: having more time allows you in general to rework your original assessments $\underline{E}(\cdot|B)$ by examining the available evidence more carefully so as to come up with model $\underline{P}_B$. In fact, in this paper we aim at taking into account also the situations where your initial beliefs $\underline{P}$ may have been specified only roughly, for instance for lack of time or other resources; in this case, the availability of additional time to define $\underline{P}_B$ may well change much your future commitments from your current conditional beliefs.[10] Note moreover that despite the above reworking process may be made in time in an incremental way through different stages, we only consider the final stage where you decide that the model $\underline{P}_B$ is definitive and hence declare it (i.e., establish it). For it is only then that you will commit yourself to accept gambles, after $B$ occurs, from $\mathcal{R}^B$. (We comment on this point and the previous one to some further extent in the concluding section 8.)
- Finally, we assume that you value gambles according to a linear utility scale throughout. This implies that the time when you accept some gamble does not affect its value for you. We need this assumption to be able to compare gambles accepted at different times, as in the following discussion.

Letting time enter the picture of belief assessment raises a new kind of consistency problem about your commitments that is not present otherwise: it may happen that despite both your present and future commitments are coherent when taken on their own, they may lead to some form of inconsistency when considered together. For instance, even if each of the sets of gambles $\mathcal{R}$ and $\mathcal{R}^B$, respectively induced by $\underline{P}$ and $\underline{P}_B$, are coherent, this cannot prevent you from establishing that for a certain $f \in \mathcal{L}(\Omega)$, and $\varepsilon_0 > \varepsilon_1 > 0$, the gamble $B(f-\varepsilon_0)$ is desirable now and that after $B$ occurs, the gamble $B(\varepsilon_1 - f)$ becomes desirable; these assessments (considered also that $B$ represents the event that occurs) imply that you are actually exposing yourself to the possibility of accepting the gamble $f - \varepsilon_0 + \varepsilon_1 - f = \varepsilon_1 - \varepsilon_0 < 0$, that is, to a sure loss. Note that you can undergo such a loss only by combining commitments related to different times: the reason is that, although at each time point you are coherent, at the moment we are missing some notion, or requirement, of consistency of your commitments *through* time. The rest of this paper will be devoted to introduce and discuss some notions of time consistency.

### 3.2. Basic tools

We start introducing some basic tools, in terms of the sets of desirable gambles associated to your current and future commitments, that we need for the following mathematical development:

[9]This is also related to the problem of *observability* within subjective probability. See [68, Sections 6.1 and 6.11] and [12, 21] for some additional discussion. Very briefly, the question is that conditioning is well defined under the updating interpretation, as well as in a temporal setup, if the conditioning event is the outcome of an experiment, and hence it belongs to a partition of $\Omega$. Overlooking this subtlety may lead into troubles by neglecting the role of the process by which one makes observations. All this is related to the condition of *coarsening at random* (see [8, 20, 56, 72]).

[10]One could argue then that $\underline{E}(\cdot|B)$ and $\underline{P}_B$ should coincide when they are established together. Our choice is to leave you freedom also in this case to choose $\underline{P}_B$ as you wish, so as to make a treatment uniform with all the other cases; only later we shall discuss how rationality will require in this case to set $\underline{P}_B$ equal to $\underline{E}(\cdot|B)$ (as well as to make $\underline{P}$ conglomerable; see Section 4.4 and in particular Theorem 2(c)).

- $\mathcal{R}$ is the coherent set of gambles induced by the coherent lower prevision $\underline{P}$ that represents your current beliefs by means of Eq. (4). For each $B \in \mathcal{B}$, your beliefs conditional on $B$ as induced by $\mathcal{R}$ are given by the set $\mathcal{R}^{|B}$ determined by Eq. (7).
- $\mathcal{R}^B$ is the set of desirable gambles that represents your future commitments in case $B \in \mathcal{B}$ obtains, and that is induced by the lower prevision $\underline{P}_B$ by means of Eq. (5).[11]

The specification of future commitments can be more or less informative depending on the period when they are established. If they are established after $B$ occurs, then there will be only a single set $\mathcal{R}^B$. But if you establish them before $B$ occurs, then you will have to specify a lower prevision $\underline{P}_B$ for every $B \in \mathcal{B}$, given that at that time you will not know which $B$ is going to obtain; correspondingly, there will be a set $\mathcal{R}^B$ for every $B \in \mathcal{B}$.

Let us analyse the latter case more in detail. Expressing all those sets commits you to accept any gamble in $\mathcal{R}^B$ provided that (and hence after) $B$ occurs, and this for every $B \in \mathcal{B}$. What we claim now is that this actually implies more: that you are committed to accept any gamble $f$ such that $Bf$ belongs to $\mathcal{R}^B$ for any $B \in \mathcal{B}$. In fact, when the sets $\mathcal{R}^B$, $B \in \mathcal{B}$, have been established, an opponent might offer you the following agreement: that he will give you gamble $Bf$ in case $B$ occurs, and this for all $B \in \mathcal{B}$. By the very definition of the sets $\mathcal{R}^B$, $B \in \mathcal{B}$, you will have to accept this agreement. But the agreement just says that you will accept $Bf$ *after any $B$ occurs*, so that eventually you will be rewarded with $f(\omega)$ whatever $\omega \in \Omega$ will come true. It is important to realise that this kind of acceptance of $f$ does *not* imply that you will be involved in transactions made by infinitely many gambles: you will eventually accept the single gamble $Bf$ related to the only event $B$ that will obtain.

These considerations lead us to define your future commitments in a way that everything that can happen in the future, and hence also all the agreements that you would accept, is properly represented. We can do this by a single set of gambles:

$$\mathcal{F}^{\mathcal{B}} := \{f \in \mathcal{L}(\Omega) : Bf \in \mathcal{R}^B \cup \{0\} \ \forall B \in \mathcal{B}\} \setminus \{0\} \tag{9}$$

$$= \left\{ f \in \mathcal{L}(\Omega) : f = \sum_{B \in \mathcal{B}} Bg \colon Bg \in \mathcal{R}^B \cup \{0\} \right\} \setminus \{0\}. \tag{10}$$

In Eqs. (9)–(10) we allow that for each set $B \in \mathcal{B}$ the restriction of $f$ to $B$ is equal to zero, because in this way $\mathcal{R}^B$ is correctly included in $\mathcal{F}^{\mathcal{B}}$ for all $B \in \mathcal{B}$. Saying this differently, letting each of these restrictions to possibly equal zero allows us to represent also the agreements (you would accept) stating that you will be given a gamble from a certain $\mathcal{R}^B$ in case $B$ happens, and nothing otherwise. Finally, observe that we exclude the zero gamble from $\mathcal{F}^{\mathcal{B}}$. This is harmless, just because it represents a trivial transaction, and at the same time it allows us to make $\mathcal{F}^{\mathcal{B}}$ comply with axiom D2.[12] In fact, $\mathcal{F}^{\mathcal{B}}$ satisfies a much stronger property:

**Proposition 1.** *$\mathcal{F}^{\mathcal{B}}$ is the conglomerable natural extension of $\cup_{B \in \mathcal{B}} \mathcal{R}^B$.*

*Proof.* Let us show that $\mathcal{F}^{\mathcal{B}}$ satisfies D1–D5.

D1. Consider $h \gneq 0$. Then for every $B \in \mathcal{B}$, it holds that $Bh \geq 0$, and as a consequence it belongs to $\mathcal{R}^B \cup \{0\}$. Hence, $h \in \mathcal{F}^{\mathcal{B}}$.

D2. We know that $0 \notin \mathcal{F}^{\mathcal{B}}$ by definition.

D3. Consider $h \in \mathcal{F}^{\mathcal{B}}$ and $\lambda > 0$. Then for every $B \in \mathcal{B}$ the gamble $Bh$ belongs to $\mathcal{R}^B \cup \{0\}$, whence $\lambda(Bh) = B(\lambda h)$ also belongs to $\mathcal{R}^B \cup \{0\}$ and as a consequence $\lambda h \in \mathcal{F}^{\mathcal{B}}$.

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[11] Although Eq. (5) induces a set of desirable gambles from a conditional lower prevision $\underline{P}(\cdot|B)$, nothing prevents us from applying it to $\underline{P}_B$, given that $\underline{P}_B$ formally acts like a conditional lower prevision—only its interpretation is different. Note also that to maintain a consistent notation here we denote the induced set by $\mathcal{R}^B$ rather than by $\mathcal{R}^{|B}$ as in Eq. (5).

[12] A more subtle issue is that when we create gambles in a piece-wise way along the elements of a partition as in $\mathcal{F}^{\mathcal{B}}$, we can give rise to unbounded gambles even if each set $\mathcal{R}^B$ contains only bounded ones: for instance, consider a countable partition $\mathcal{B}$ with elements $B_1, B_2, \dots, B_k, \dots$, and select the positive constant gamble $k$ in $\mathcal{R}^{B_k}$ for all $k \in \mathbb{N}$. We rule out situations of this type by requiring in (9) that $f$ belongs to $\mathcal{L}(\Omega)$, which is a set made of bounded gambles by definition. The reason is that currently the theories of coherent sets of desirable gambles, as well as of coherent lower previsions, are developed only for the case of bounded gambles (with the notable exception of [64]). Such a choice is not restrictive for the subsequent analysis since we were already assuming that beliefs be expressed only with reference to bounded gambles.

D4. Consider $h, h' \in \mathcal{F}^{\mathcal{B}}$. Then for every $B \in \mathcal{B}$ it holds that $Bh, Bh' \in \mathcal{R}^B \cup \{0\}$, whence $B(h + h') = Bh + Bh' \in \mathcal{R}^B \cup \{0\}$. This implies that $h + h' \in \mathcal{F}^{\mathcal{B}} \cup \{0\}$. To see that $h + h' \neq 0$, assume ex-absurdo that $h + h' = 0$. Since neither of these gambles is equal to zero, there must be some $B \in \mathcal{B}$ such that $Bh \neq 0$ and $Bh' = -Bh$. But then $Bh, Bh' \in \mathcal{R}^B$ and $Bh + Bh' = 0$, a contradiction with the coherence of $\mathcal{R}^B$. Hence, $h + h' \in \mathcal{F}^{\mathcal{B}}$.

D5. Consider $0 \neq h \in \mathcal{L}$ such that $Bh \in \mathcal{F}^{\mathcal{B}} \cup \{0\}$ for all $B \in \mathcal{B}$. Then it follows from Eq. (9) that $Bh \in \mathcal{R}^B \cup \{0\}$ for every $B \in \mathcal{B}$, whence $h \in \mathcal{F}^{\mathcal{B}}$.

On the other hand, any superset $\mathcal{F}$ of $\cup_{B \in \mathcal{B}} \mathcal{R}^B$ satisfying D1–D5 should include any gamble $f \neq 0$ for which $Bf \in \mathcal{R}^B \cup \{0\}$ for every $B \in \mathcal{B}$, because then $Bf \in \cup_{B \in \mathcal{B}} \mathcal{R}^B \cup \{0\}$ for every $B$ and applying D5 it follows that $f \in \mathcal{F}$. We conclude that $\mathcal{F}^{\mathcal{B}}$ is the smallest superset of $\cup_{B \in \mathcal{B}} \mathcal{R}^B$ satisfying D1–D5, i.e., its conglomerable natural extension. $\square$

This result tells us in particular that $\mathcal{F}^{\mathcal{B}}$ is conglomerable. But remember that we are still not assuming that, at a certain moment in time, an infinite sum of desirable gambles is desirable to you: the commitments expressed by the sum $\sum_{B \in \mathcal{B}} Bg$ will become effective only after a certain $B$ occurs, so that in the end you will only make the single transaction represented by $Bg$.

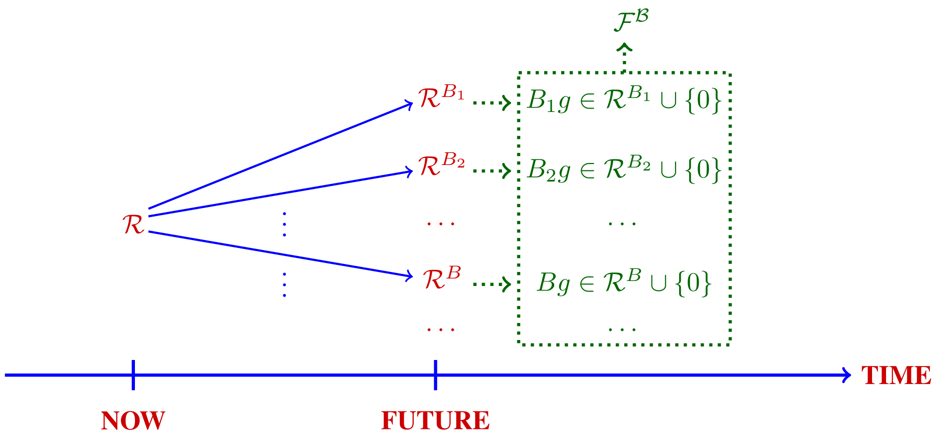


Figure 2: We can represent the establishment of commitments through time by a tree. At the root there are present beliefs $\mathcal{R}$. Later, depending on the event that occurs, a certain set $\mathcal{R}^B$ represents the commitments, such as $Bg$, that are effective since then. $\mathcal{F}^{\mathcal{B}}$ represents the totality of what can happen in the future, without assuming to know the specific event $B$ that will occur.

Another way to look at this is that the above infinite sum does not involve gambles representing commitments that you hold simultaneously, given that the different sets $\mathcal{R}^B$ are exclusive: you will never hold commitments $\mathcal{R}^{B'}$ together with $\mathcal{R}^{B''}$, for two different events $B', B''$ in $\mathcal{B}$. This can be seen also from Figure 2. Therefore $\mathcal{F}^{\mathcal{B}}$ is not a set of commitments that you will hold at some point in time; it is rather a formal tool that we use to represent everything that can happen in the future based on the commitments that you have established (separately) relative to the occurrence of different events.

The situation is different in the case of conditional beliefs. Consider a gamble $g := \sum_{B \in \mathcal{B}} Bg$ such that $Bg \in \mathcal{R}^{\rfloor B}$ for all $B \in \mathcal{B}$, where $\mathcal{R}^{\rfloor B}$ is induced from $\mathcal{R}$ by means of Eq. (7). Then for every $B \in \mathcal{B}$, you are willing to accept $Bg$ now (and *not after $B$ occurs*). But this does not mean that you are now willing to accept $\sum_{B \in \mathcal{B}} Bg$: in fact, all the gambles $Bg$ belong to $\mathcal{R}$ and hence express commitments that you hold altogether at the same point in time; and you are allowed to combine only a finite number of them through the finitary axiom D4. As a consequence, an opponent may have you at most accept a gamble like $g' := \sum_{B \in \mathcal{B}'} Bg_B$, where $\mathcal{B}'$ is a finite subset of $\mathcal{B}$. But this will not be able to represent gamble $g$ in general. As a consequence, $\mathcal{R}$ is not going to be closed with respect to D5 in general.

We see then that the mechanisms that allow us to combine gambles through sums are very different in the case where we focus on a set of beliefs maintained at the same point in time ($\mathcal{R}$) or on a set summarising sets of commitments relative to different future events that will occur ($\mathcal{F}^{\mathcal{B}}$). And it is particularly revealing in our view to see that the conglomerability of $\mathcal{F}^{\mathcal{B}}$ is a feature that arises spontaneously without being imposed on the set. Note in particular that without using infinite sums in $\mathcal{F}^{\mathcal{B}}$ we would not be able to represent all the possible future scenarios: for instance, the scenario where you will accept some $Bg \neq 0$ after $B$ occurs, and this for all $B \in \mathcal{B}$, can be represented only through the infinite sum $\sum_{B\in\mathcal{B}} Bg$ because by making it finite some elements $Bg$ (actually infinitely many of them) should equal zero. The set that arises if we use only finite sums is in fact different from $\mathcal{F}^{\mathcal{B}}$, as the following corollary shows:

**Corollary 1.** *The natural extension of* $\cup_{B\in\mathcal{B}}\mathcal{R}^B$ *is given by*

$$\mathcal{E}^{\mathcal{B}} := \left\{ f \in \mathcal{L}(\Omega) : f = h + \sum_{B\in\mathcal{B}'} Bg \colon Bg \in \mathcal{R}^B \cup \{0\}, h \geq 0, \mathcal{B}' \subseteq \mathcal{B} \text{ s.t. } |\mathcal{B}'| < \infty \right\} \setminus \{0\}. \tag{11}$$

*Proof.* It is enough to show that $\mathcal{E}^{\mathcal{B}}$ is the smallest coherent superset of $\cup_{B\in\mathcal{B}}\mathcal{R}^B$. That $\cup_{B\in\mathcal{B}}\mathcal{R}^B \subseteq \mathcal{E}^{\mathcal{B}}$ is trivial, and that any coherent superset of $\cup_{B\in\mathcal{B}}\mathcal{R}^B$ must include $\mathcal{E}^{\mathcal{B}}$ follows with a reasoning analogous to that used w.r.t. $\mathcal{F}^{\mathcal{B}}$ in Proposition 1. On the other hand, given a gamble in (11), if we rewrite

$$h + \sum_{B\in\mathcal{B}'} Bg = \sum_{B\in\mathcal{B}'} B(g+h) + \sum_{B\notin\mathcal{B}'} Bh$$

we see, through (10), that $\mathcal{E}^{\mathcal{B}} \subseteq \mathcal{F}^{\mathcal{B}}$. Then the axioms D1–D4 are trivial to verify, considered in particular that sums of gambles from $\mathcal{E}^{\mathcal{B}}$ cannot yield zero as $0 \notin \mathcal{F}^{\mathcal{B}}$. □

Observe in particular that the natural extension $\mathcal{E}^{\mathcal{B}}$ satisfies D1–D4 and not D5 in general.

On the other hand, since in Section 4 we shall focus primarily on coherent lower previsions, we shall need also a way to aggregate the several models of future commitments directly through lower previsions. In this case we have a prevision $\underline{P}_B$ on $\mathcal{L}(B)$ for all $B \in \mathcal{B}$. We summarise them by means of the functional $\underline{P}_{\mathcal{B}}$ given by

$$\underline{P}_{\mathcal{B}}(f) := \sum_{B\in\mathcal{B}} B\underline{P}_B(f_B),$$

which mathematically *acts* as a separately coherent conditional lower prevision on $\mathcal{L}(\Omega)$. However, since, strictly speaking, it is not a *conditional* lower prevision (which, by definition, represents current beliefs), we shall use for it the terminology of *separately coherent future lower prevision*.

Let us remark that although the sets $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ will be used already in Section 4, as a way to easily convey a behavioural interpretation of the results we shall pursue, it is from Section 6 onwards that these sets will actually play a primary role in the development, because we shall take them as our primitive models, not only as models derived from coherent lower previsions. This will allow us to establish results similar in spirit to the first part of the paper but in a much more general way.

## 4. Temporal consistency

Remember that we distinguish different time periods for the definition of your future commitments. In this section we start by focusing on the intermediate period, where you define them before $B$ occurs, but also after having established your current beliefs. In this situation, the relevant models in terms of lower previsions are an unconditional lower prevision $\underline{P}$ on $\mathcal{L}(\Omega)$ that represents your present beliefs, and the separately coherent future lower prevision $\underline{P}_{\mathcal{B}}$ on $\mathcal{L}(\Omega)$ that summarises your future commitments.

### *4.1. Temporal consistency*

We are ready to define our first notion of consistency across your current and future commitments. To this end, let $\mathcal{R}, \mathcal{R}^B$ $(B \in \mathcal{B})$ be the sets of gambles associated to $\underline{P}, \underline{P}_B$ by means of Eqs. (4), (5) and let $\mathcal{F}^{\mathcal{B}}$ be the conglomerable natural extension of $\cup_{B\in\mathcal{B}}\mathcal{R}^B$.

**Theorem 1.** *The following statements are equivalent:*

(a) $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ *avoids partial loss.*

(b) $\underline{P}, \underline{P}_{\mathcal{B}}$ *avoid sure loss.*

(c) $\overline{P}(f - \underline{P}_{\mathcal{B}}(f)) \geq 0$ *for every* $f \in \mathcal{L}$.

(d) $\underline{P}_{\mathcal{B}}(f) \geq 0 \Rightarrow \overline{P}(f) \geq 0$ *for every* $f \in \mathcal{L}$.

(e) $\overline{P}(f) \geq \inf \underline{P}_{\mathcal{B}}(f)$ *for every* $f \in \mathcal{L}$.

*Proof.* Let us make a circular proof.

(a) $\Rightarrow$ (b) Assume ex-absurdo that $\underline{P}, \underline{P}_{\mathcal{B}}$ do not avoid sure loss. Then from ASL we deduce that there are gambles $f, g$ such that $\sup[G(f) + G_{\mathcal{B}}(g)] < 0$, whence there is some $\delta > 0$ such that $\sup[G(f) + G_{\mathcal{B}}(g) + \delta] < 0$. Since

$$G(f) + \frac{\delta}{2} = f - (\underline{P}(f) - \frac{\delta}{2}) \in \mathcal{R} \text{ and}$$
$$G_B(g_B) + B\frac{\delta}{2} = B(g - (\underline{P}_B(g) - \frac{\delta}{2})) \in \mathcal{R}^B \ \forall B \in \mathcal{B} \Rightarrow G_{\mathcal{B}}(g) + \frac{\delta}{2} \in \mathcal{F}^{\mathcal{B}},$$

we deduce that the gamble $G(f) + G_{\mathcal{B}}(g) + \delta$ belongs to the natural extension of $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ (because the natural extension is closed w.r.t. sums of gambles). But since such a gamble is negative, we deduce that the natural extension is incoherent, a contradiction with (a). As a consequence, $\underline{P}, \underline{P}_{\mathcal{B}}$ avoid sure loss.

(b) $\Rightarrow$ (c) This follows from [68, Theorem 6.3.5].

(c) $\Rightarrow$ (d) Because if $\underline{P}_{\mathcal{B}}(f) \geq 0$ then $\overline{P}(f) \geq \overline{P}(f - \underline{P}_{\mathcal{B}}(f)) \geq 0$.

(d) $\Rightarrow$ (e) Taking into account that, since $\underline{P}$ is a coherent lower prevision, $\overline{P}(f - \inf \underline{P}_{\mathcal{B}}(f)) = \overline{P}(f) - \inf \underline{P}_{\mathcal{B}}(f)$, and $\underline{P}_{\mathcal{B}}(f - \inf \underline{P}_{\mathcal{B}}(f)) \geq 0$ because $\underline{P}_{\mathcal{B}}$ is separately coherent.

(e) $\Rightarrow$ (a) Use Remark 1, and assume there are gambles $f \in \mathcal{R}, g \in \mathcal{F}^{\mathcal{B}}$ such that $f + g \leq 0$. Note that $f \notin \mathcal{L}^+$ because in that case we would have $g \leq 0$, a contradiction with the coherence of $\mathcal{F}^{\mathcal{B}}$. Then $g \in \mathcal{F}^{\mathcal{B}}$ implies that $\underline{P}_{\mathcal{B}}(g) \geq 0$, whence, applying (e), $\overline{P}(g) \geq 0$. Using the conjugacy between upper and lower previsions, together with $f \leq -g$, we deduce that $\underline{P}(-g) \leq 0$, and the monotonicity of $\underline{P}$ implies then that $\underline{P}(f) \leq 0$. But since $f \notin \mathcal{L}^+$, we deduce from Eq. (4) that $f$ cannot belong to the set of gambles $\mathcal{R}$, a contradiction. Hence, $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss. □

This result generalises [68, Theorem 6.3.5(1) and (3)], where the implications $(b) \Rightarrow (e)$ and $(b) \Rightarrow (c)$ were established.[13] From it, we establish the following:

*Definition* 13 (**Temporal consistency**). We say that your current and future commitments $\underline{P}, \underline{P}_{\mathcal{B}}$ are *temporally consistent* when any of the equivalent conditions of Theorem 1 holds.

The rationale behind this definition should be clear: if you failed temporal consistency, an opponent could create a combination of current and future transactions that will have the overall effect of making you desire, and then accept, a gamble that is strictly smaller than 0. For example, assume that there are gambles $f$ and $g$ such that ASL fails, because there is some $\delta > 0$ such that

$$\sup\left[G(f) + G_{\mathcal{B}}(g)\right] \leq -\delta < 0.$$

Then the definition of $\underline{P}$ means that you should be disposed to buy the gamble $f$ for the price $\underline{P}(f) - \frac{\delta}{3}$, or, equivalently, that you should accept the gamble $G(f) + \frac{\delta}{3}$; on the other hand, for any $B \in \mathcal{B}$, you should be disposed to buy the gamble $g$ at the price $\underline{P}_B(g) - \frac{\delta}{3}$ after observing $B$, or equivalently, to accept the gamble $B(g - \underline{P}_B(g) + \frac{\delta}{3})$. But this means that no matter which is the $B$ you observe, you should accept the gamble $G(f) + G_{\mathcal{B}}(g) + \frac{2\delta}{3}$, which will produce a loss of at least $\frac{\delta}{3}$ *irrespective of the actual* $B$ *that will occur*. This is an inconsistency.

[13] But note that Walley was not concerned with temporal considerations as we are; his results deal with a pair of unconditional and conditional lower previsions that are both established at present time. Therefore the aims and the interpretation of the results in the two cases are very different.

*Example* 7 (Running example). Consider again the situation depicted in Example 1: take the variables $V :=$ 'virus type' (*s*easonal, *a*typical) and $T :=$ 'test result' (*p*ositive, *n*egative). Assume you model your current beliefs about the two variables by means of the coherent lower prevision $\underline{P}$ from Example 4, and that, at some later time, but before you know the results of the test, you set your future commitments equal to the separately coherent conditional lower prevision $\underline{P}(V|T)$ from Example 2. Since we have showed in Example 6 that these two lower previsions avoid sure loss, we conclude that your assessments are temporally consistent: they cannot be exploited together in order to make you subject to a sure loss. This is easy to see if we consider their associated sets of gambles by means of Eqs. (4) and (5): we have that

$$\mathcal{R} := \mathcal{L}^+ \cup \{f : f(s,n) > 0, f(s,p) > 0 \text{ and } 0.95 \min\{f(s,n), f(s,p)\} + 0.05 \min\{f(a,n), f(a,p)\} > 0\};$$

and

$$\mathcal{R}^B := \{f : \quad f(a,n) = f(s,n) = 0 \text{ and } (f(a,p) > \max\{0, -\frac{f(s,p)}{3}\} \text{ or } f(a,p) = 0 < f(s,p))\},$$
$$\mathcal{R}^{B^c} := \{f : \quad f(a,p) = f(s,p) = 0 \text{ and } (f(s,n) > \max\{0, -\frac{f(a,n)}{4}\} \text{ or } f(s,n) = 0 < f(a,n))\},$$

where we are using $B$ to denote the event $T = p$.

Then note that if the sum of a gamble $f$ in $\mathcal{R}$ with a gamble $g \in \mathcal{F}^{\mathcal{B}}$ were negative or zero, we would deduce that it must be $f(s,n) = g(s,n) = 0$, from which $f \in \mathcal{L}^+$ and therefore $g \leq 0$, a contradiction. Hence, $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss. ♦

Theorem 1 also allows us to relate the conditional beliefs we can derive from your set of current beliefs and your future commitments. Let $\mathcal{R}^{|B}$ be the set of conditional beliefs derived from $\mathcal{R}$ by means of Eq. (7), and let $\underline{E}(\cdot|B)$ be its associated lower prevision (the conditional natural extension of $\underline{P}$, from Lemma 1), given by Eq. (6). Since $\mathcal{R}^B \subseteq \mathcal{F}^{\mathcal{B}}$, then we can deduce from Theorem 1 that

$$\underline{E}(f|B) \leq \overline{P}_B(f_B) \; \forall f \in \mathcal{L}(\Omega). \tag{12}$$

Indeed, if (12) does not hold, then you are willing now to pay $\underline{E}(f|B) - \delta/2$, for any $\delta > 0$, to get $f$ under the assumption that $B$ happens, while after $B$ actually happens, you are willing to sell $f$ at price $\overline{P}_B(f_B) + \delta/2$; the result of these two transactions is $B(f - \underline{E}(f|B) + \delta/2) - Bf + \overline{P}_B(f_B) + \delta/2 = \overline{P}_B(f_B) - \underline{E}(f|B) + \delta$, which is negative provided that we choose $\delta < \underline{E}(f|B) - \overline{P}_B(f_B)$.

We can use Theorem 1 also to see, intuitively, that you do not need to modify your current beliefs in order to achieve temporal consistency: given any set of current beliefs, you can create temporally consistent future commitments. This is an important feature of temporal consistency, because it means that you can use it also after having established (and hence having 'fixed') the present beliefs. The next remark makes the point more precisely.

*Remark* 2. If your commitments become more imprecise, in the sense that $\underline{P}_1 \leq \underline{P}$ and $\underline{P}^1_{\mathcal{B}} \leq \underline{P}_{\mathcal{B}}$, then if $\underline{P}, \underline{P}_{\mathcal{B}}$ satisfy temporal consistency then so do $\underline{P}_1, \underline{P}^1_{\mathcal{B}}$. A related comment is made in [68, Proposition 2.6.3(a)]. Furthermore, temporal consistent models always exist (for instance the vacuous ones); and we can always find a future model that satisfies temporal consistency with respect to your current beliefs by making it imprecise enough: if we take the vacuous $\underline{P}_{\mathcal{B}}$, then it is temporal consistent with any initial coherent lower prevision $\underline{P}$. This shows on the one hand that temporal consistency is weaker than conglomerability: even if we start with a conglomerable, precise prevision, a conditional lower prevision that is temporally consistent with it is not necessarily precise. On the other hand, even if there are coherent lower previsions that are not conglomerable, they are always temporally consistent with respect to the vacuous $\underline{P}_{\mathcal{B}}$. We shall come back to the connection with conglomerability in Section 4.4. ♦

#### *4.1.1. Correcting temporal inconsistency*

An interesting side problem is to determine whether you can modify your (not yet established) commitments when temporal consistency is not satisfied, so as to obtain a temporal consistent model and with a correction that is as small as possible. In other words, we would like to define an analogue of the notion of natural extension for temporally consistent models, in the sense of being the closest model that satisfies temporal consistency. Since we have already remarked that a model that is included in a temporally consistent model satisfies again temporal consistency, the correction should be done by making the model more imprecise. That is, you should define the temporally consistent extension as the greatest (that is, more informative) model that is temporally consistent and is included in your assessments.

If your current and future commitments are temporally inconsistent, it follows from the definition of avoiding sure loss that there is no linear prevision $P$ satisfying $P(G(f)) \geq 0$ and $P(G_{\mathcal{B}}(f)) \geq 0$ for every gamble $f$. This means that if we consider the credal sets $\mathcal{M}_1 := \{P : P(G(f)) \geq 0 \,\forall f\}$ and $\mathcal{M}_2 := \{P : P(G_{\mathcal{B}}(f)) \geq 0 \,\forall f\}$, their intersection is empty. Equivalently, this means that the lower prevision $\underline{Q} := \max\{\underline{P}_1, \underline{P}_2\}$, where $\underline{P}_1, \underline{P}_2$ are the lower previsions associated to $\mathcal{M}_1, \mathcal{M}_2$, incurs a sure loss. Therefore, one possibility for correcting your inconsistent assessments would be to find the 'closest' lower prevision that is dominated by $\underline{Q}$ and avoids sure loss (so that it is associated to a non-empty credal set). However, such a prevision does not exist in general, as we proceed to show:

**Proposition 2.** *Let $\underline{P}$ be a coherent lower prevision incurring sure loss. Then there is no greatest lower prevision that avoids sure loss and is dominated by $\underline{P}$.*

*Proof.* First of all, we can assume without loss of generality that $\underline{P}(f) \in [\inf f, \sup f]$ for every $f$: otherwise, it suffices to consider the lower prevision $\underline{P}'$ given by

$$\underline{P}'(f) := \begin{cases} \inf f & \text{if } \underline{P}(f) < \inf f \\ \sup f & \text{if } \underline{P}(f) > \sup f \\ \underline{P}(f) & \text{otherwise.} \end{cases}$$

Then for any credal set $\mathcal{M}$ we can consider the lower previsions $\underline{P}_1 := \inf(\mathcal{M} \cup \mathcal{M}(\underline{P}))$ and $\underline{P}_2 := \inf(\mathcal{M} \cup \mathcal{M}(\underline{P}'))$, by means of which we can make a correspondence between the lower previsions that are dominated by $\underline{P}$ and avoid sure loss and those that are dominated by $\underline{P}'$ and avoid sure loss. Hence, if we show that there is not a greatest lower prevision that is dominated by $\underline{P}'$ and avoids sure loss, we shall immediately deduce that there is not a greatest lower prevision that is dominated by $\underline{P}$ and avoids sure loss either.

Assume now that there is a lower prevision $\underline{Q} \leq \underline{P}$ that avoids sure loss and such that for any other lower prevision $\underline{Q}' \leq \underline{P}$ that avoids sure loss it holds that $\underline{Q}' \leq \underline{Q}$. Since $\underline{Q}$ avoids sure loss and $\underline{P}$ does not, there must be some gamble $f$ such that $\underline{Q}(f) < \underline{P}(f)$. Let $P$ be a linear prevision satisfying $P(f) = \underline{P}(f)$ (such a prevision always exists because $\inf f \leq \underline{P}(f) \leq \sup f$), and let us define $\underline{Q}' := \min\{P, \underline{P}\}$. Then $P \in \mathcal{M}(\underline{Q}')$, so $\underline{Q}'$ avoids sure loss, and moreover $\underline{Q}' \leq \underline{P}$. However, $\underline{Q}'(f) = \underline{P}(f) > \underline{Q}(f)$, and as a consequence $\underline{Q}'$ is not dominated by $\underline{Q}$. This is a contradiction. $\square$

Hence, it is not possible to find the closest temporally consistent model to some temporally inconsistent assessments; interestingly, it may be possible to do so if you fix your current beliefs $\underline{P}$ and look for the greatest model of future commitments that is dominated by $\underline{P}_{\mathcal{B}}$ and is temporally consistent with $\underline{P}$. This is easier to establish if we work with sets of desirable gambles, as we shall see in Section 6.1.

### 4.2. *Strong temporal consistency*

Temporal consistency means that it should not be possible to combine your current and future commitments in order to make you subject to a sure loss, but it does not impose any actual restriction on how these future commitments should be defined. When we require that they are determined by your current beliefs, we obtain a strengthening of temporal consistency that we shall call strong temporal consistency:

*Definition* 14 (**Strong temporal consistency**). We say that a coherent lower prevision $\underline{P}$ and a separately coherent future lower prevision $\underline{P}_{\mathcal{B}}$ are *strongly temporally consistent* when they are temporally consistent and moreover $\underline{P}_{\mathcal{B}}$ coincides with the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$, given by Eq. (3).

Taking into account the comments about temporal consistency in Section 4.1, the behavioural interpretation of this condition is that one should not be able to exploit your current and future commitments in order to make you subject to a sure loss, when moreover your future commitments are determined by your current ones by means of natural extension. This shall be clearer when we discuss the definition of strong temporal consistency in terms of sets of gambles in Section 6.2.

We begin by noting that strong temporal consistency is related to the property of conglomerability, in the sense that if you set your future commitments equal to present conditional beliefs, then strong temporal consistency holds automatically if $\underline{P}$ is conglomerable.

**Proposition 3.** *Let $\underline{P}$ be a coherent lower prevision and let $\underline{E}(\cdot|\mathcal{B})$ denote its conditional natural extension. Then each of the following statements implies the next:*

(a) *$\underline{P}$ is conglomerable.*

(b) *$\underline{P}, \underline{E}(\cdot|\mathcal{B})$ have dominating coherent lower previsions $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$.*

(c) *$\underline{P}, \underline{E}(\cdot|\mathcal{B})$ are temporally consistent.*

*Proof.*

(a) $\Rightarrow$ (b) If $\underline{P}$ is conglomerable then it is coherent with its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$, so the thesis holds trivially.

(b) $\Rightarrow$ (c) If $\underline{Q} \geq \underline{P}$ and $\underline{Q}(\cdot|\mathcal{B}) \geq \underline{E}(\cdot|\mathcal{B})$, and $\underline{Q}, \underline{Q}(\cdot|\mathcal{B})$ are coherent, then $\underline{Q}$ is conglomerable, whence it is coherent with its conditional natural extension, which must also dominate $\underline{E}(\cdot|\mathcal{B})$ because $\underline{Q} \geq \underline{P}$. Taking into account Remark 2, we deduce that $\underline{P}, \underline{E}(\cdot|\mathcal{B})$ are temporally consistent. □

On the other hand, we may note then that if the coherent lower prevision $\underline{P}$ that represents your current beliefs does not satisfy temporal consistency with its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$, neither does any dominating coherent lower prevision $\underline{Q} \geq \underline{P}$: for if there are gambles $f, g$ such that

$$\sup[f - \underline{P}(f) + g - \underline{E}(g|\mathcal{B})] < 0,$$

then we also have

$$\sup[f - \underline{Q}(f) + g - \underline{Q}(g|\mathcal{B})] < 0,$$

taking into account that the conditional natural extension $\underline{Q}(\cdot|\mathcal{B})$ of $\underline{Q}$ dominates that of $\underline{P}$. This shows that failure of strong temporal consistency cannot be corrected by making your assessments more precise.

*Example* 8 (Running example). If in our running example we make your future commitments equal to the ones derived from $\underline{P}$ by means of natural extension, we obtain the vacuous conditional lower prevision $\underline{E}(V|T)$ from Example 5. Since its associated set of gambles is $\mathcal{F}^{|\mathcal{B}} = \mathcal{L}^{+}$, it trivially avoids partial loss with the set $\mathcal{R}$ induced by $\underline{P}$. Hence, $\underline{P}, \underline{E}(V|T)$ are strongly temporally consistent. This could be deduced immediately from Proposition 3: since the partition $\{T = p, T = n\}$ is finite, the coherent lower prevision $\underline{P}$ is trivially conglomerable, and therefore it is temporally consistent with its conditional natural extension. ♦

One particular case where $\underline{P}$ always avoids sure loss with its conditional natural extension is when you build it by means of the *marginal extension*. This is a generalisation of the law of total probability for the imprecise case that is useful in a context of hierarchical information. Consider a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}$, and let $\underline{P}$ be a coherent lower prevision defined on the set $\mathcal{K} \subseteq \mathcal{L}$ of $\mathcal{B}$-measurable gambles. Then the marginal extension of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ is the lower prevision given by $\underline{P}_1 := \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, and it can be checked [68, Theorem 6.7.2] that this lower prevision is coherent with $\underline{P}(\cdot|\mathcal{B})$.

**Proposition 4.** *Under the above conditions, the marginal extension $\underline{P}_1$ of $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ is temporally consistent with its conditional natural extension.*

*Proof.* The result follows from Proposition 3 once we show that the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}_1$ is dominated by $\underline{P}(\cdot|\mathcal{B})$. Given $B \in \mathcal{B}$, if $\underline{P}_1(B) = 0$ then $\underline{E}(\cdot|B)$ is vacuous and is trivially dominated by $\underline{P}(\cdot|B)$; and if $\underline{P}_1(B) > 0$ then $\underline{E}(\cdot|B)$ is uniquely determined from $\underline{P}_1$ using GBR, and as a consequence it must coincide with $\underline{P}(\cdot|B)$, because $\underline{P}_1, \underline{P}(\cdot|\mathcal{B})$ are coherent. □

On the other hand, the equality $\underline{E}(\cdot|\mathcal{B}) = \underline{P}_{\mathcal{B}}$ between the conditional natural extension of $\underline{P}$ and the separately coherent future lower prevision is made up of two inequalities: $\underline{E}(\cdot|\mathcal{B}) \leq \underline{P}_{\mathcal{B}}$ and $\underline{E}(\cdot|\mathcal{B}) \geq \underline{P}_{\mathcal{B}}$. The first of these inequalities means that your future commitments should take into account (i.e., be at least as precise as) the implications of the current beliefs by conditional natural extension. This suggests that strong temporal consistency can naturally be turned into a weaker consistency notion based on such an inclusion.

*Definition* 15 (**Strong backward temporal consistency**). We say that your current and future commitments are *strongly backward temporally consistent* if they are temporally consistent and $\underline{E}(\cdot|\mathcal{B}) \leq \underline{P}_{\mathcal{B}}$.

The inequality $\underline{E}(\cdot|\mathcal{B}) \leq \underline{P}_{\mathcal{B}}$ is related to a proposal that Walley did in [68, Section 6.1.2].[14] The rationale, paraphrasing his words, is that your conditional assessments $\underline{E}(\cdot|\mathcal{B})$ should be 'reliable', in the sense that when you establish them, you should inspect all the evidence carefully; if you do so, when you later come to know $B$, you might do some extra effort and make your assessments more precise, but there should be no possibility that you change your mind so as to make your assessments become more imprecise. It is interesting then to see that Walley's proposal can be thought of as a consistency requirement of current commitments onto future ones. We can see thus strong temporal consistency as a limit case of strong backwards temporal consistency: it would be the least-committal (i.e., the most imprecise) model for which strong backward temporal consistency is satisfied.

But note that strong backward temporal consistency is stronger than Walley's proposal because we are requiring in addition that $\underline{P}, \underline{P}_{\mathcal{B}}$ avoid sure loss, and this does not follow from the inequality $\underline{E}(\cdot|\mathcal{B}) \leq \underline{P}_{\mathcal{B}}$ and the fact that $\underline{P}, \underline{E}(\cdot|\mathcal{B})$ avoid sure loss: use [68, Example 6.6.10] for an example of a linear prevision $P$ whose conditional natural extension is vacuous (and which therefore trivially satisfies temporal consistency) but that is temporally inconsistent with any precise future commitments.

### 4.3. Event-wise (strong) temporal consistency

In the previous sections we have focused on a setup where future commitments are established after present beliefs and before event $B$ occurs. Now we move on to consider the simplest case to characterise, that where future commitments are established after the occurrence of $B$.

In this situation, having got to know exactly which element in the partition $\mathcal{B}$ has obtained, you will obviously focus on the lower prevision $\underline{P}_B$ associated to that $B$, which is defined on $\mathcal{L}(B)$, or, equivalently, on the subset of $\mathcal{L}(\Omega)$ given by those gambles that are zero outside $B$. Characterising consistency is then very similar to what we have already done before, with the additional requirement to focus on the only set $B$ that is available.

**Proposition 5.** *Let $\underline{P}$ be the coherent lower prevision modelling your current beliefs and let $\underline{P}_B$ be your coherent lower prevision on $\mathcal{L}(B)$, or equivalently on $\mathcal{K} := \{f \in \mathcal{L}(\Omega) : f = Bf\}$, that models your beliefs after knowing that $B$ occurs. Let $\mathcal{R}, \mathcal{R}^B$ be their associated sets of gambles by Eqs.* (4) *and* (5). *Then the following are equivalent:*

(a) *The lower prevision $\underline{P}_1$ given by*

$$\underline{P}_1(f) := \begin{cases} \max\{\underline{P}(f), \underline{P}_B(f)\} & \text{if } f \in \mathcal{K} \\ \underline{P}(f) & \text{otherwise} \end{cases}$$

*avoids sure loss.*

(b) *$\mathcal{R} \cup \mathcal{R}^B$ avoids partial loss.*

(c) *$\underline{E}(f|B) \leq \overline{P}_B(f)$ for every $f \in \mathcal{K}$, where $\underline{E}(\cdot|B)$ denotes the conditional natural extension of $\underline{P}$.*

*Proof.*

(a) $\Rightarrow$ (b) Since both $\mathcal{R}, \mathcal{R}^B$ are convex cones of gambles, we can deduce from Remark 1 that $\mathcal{R} \cup \mathcal{R}^B$ incurs partial loss if and only if there are gambles $f \in \mathcal{R}, g \in \mathcal{R}^B$ such that $f + g \leq 0$. We can assume without loss of generality that none of these gambles is positive, or we would contradict the coherence of either $\mathcal{R}$ or $\mathcal{R}^B$. As a consequence, it must be $\underline{P}(f) > 0$ and $\underline{P}_B(g) > 0$, whence

$$\sup[f - \underline{P}_1(f) + g - \underline{P}_1(g)] < \sup[f + g] \leq 0,$$

which implies that $\underline{P}_1$ incurs sure loss, a contradiction.

[14] A notion related to strong backwards temporal consistency can be found in [26]; there are, however, a few differences with our framework: the authors of [26] consider only the observation of an event, instead of a partition; they also consider a non-linear utility function, whereas we assume your utility scale is linear; and their basic model is not established in terms of lower and upper previsions.

(b) ⇒ (a) Assume ex-absurdo that there are two gambles $f, g$ such that $\sup[f - \underline{P}_1(f) + g - \underline{P}_1(g)] < 0$; then there is some $\delta > 0$ such that $f - \underline{P}_1(f) + g - \underline{P}_1(g) + \delta < 0$. Since both $\underline{P}$ and $\underline{P}_B$ are coherent on their respective domains, we can assume without loss of generality that $f \in \mathcal{K}$ and $\underline{P}_1(f) = \underline{P}_B(f)$ and that $\underline{P}_1(g) = \underline{P}(g)$. Hence, given $f_1 := f - \underline{P}_1(f) + B\frac{\delta}{2} \in \mathcal{K}$, it holds that $\underline{P}_B(f_1) = \frac{\delta}{2}$, whence $f_1 \in \mathcal{R}^B$. Similarly, given $g_1 := g - \underline{P}_1(g) + \frac{\delta}{2} = g - \underline{P}(g) + \frac{\delta}{2}$, it holds that $\underline{P}(g_1) = \frac{\delta}{2} > 0$, whence $g_1 \in \mathcal{R}$. But then $f_1 + g_1$ belongs to the natural extension of $\mathcal{R} \cup \mathcal{R}^B$ and it is smaller than or equal to 0. Hence, $\mathcal{R} \cup \mathcal{R}^B$ incurs partial loss, a contradiction.

(b) ⇒ (c) Assume ex-absurdo that there is a gamble $f \in \mathcal{K}$ such that $\underline{E}(f|B) > \overline{P}_B(f)$. Then from Lemma 1 there is some $\delta > 0$ such that $B(f - \overline{P}_B(f) - \delta) \in \mathcal{R}$; since $B(\overline{P}_B(f) - f + \frac{\delta}{2})$ belongs to $\mathcal{R}^B$, we would deduce that $-B\frac{\delta}{2}$ belongs to the natural extension of $\mathcal{R} \cup \mathcal{R}^B$, a contradiction.

(c) ⇒ (b) If $\mathcal{R} \cup \mathcal{R}^B$ incurs partial loss, then there are gambles $f \in \mathcal{R}, g \in \mathcal{R}^B$ such that $f + g \leq 0$. Since $B^c g = 0$, it follows that $B^c f \leq 0$. Thus, $Bf = f - B^c f \in \mathcal{R}$. Since $g$ cannot be positive because this would contradict the coherence of $\mathcal{R}$, we deduce that $\underline{P}_B(g) > 0$, from which we have the following contradiction:

$$0 \geq \overline{P}_B(Bf + g) \geq \overline{P}_B(Bf) + \underline{P}_B(g) \geq \underline{E}(f|B) + \underline{P}_B(g) > 0.$$

Here the first inequality follows from $f + g \leq 0$ and the monotonicity of coherent upper previsions, the second one follows from the coherence of $\overline{P}_B$ and the third one from (c). □

*Definition* 16 (**Event-wise temporal consistency**). We say that your present and future commitments are *event-wise temporally consistent* when any of the equivalent conditions in Proposition 5 holds.

Hence, by event-wise temporal consistency you establish your future commitments after observing which element $B$ of the partition happens, while making sure that your present and future commitments cannot be exploited in order to make you subject to a sure loss.[15] As a consequence of Theorem 1 and Proposition 5, if your current and future commitments are temporally consistent, then they are also event-wise temporally consistent for every $B \in \mathcal{B}$. The converse does not hold in general: if your future commitments coincide with the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$, then we always have (strong) event-wise temporal consistency for every $B \in \mathcal{B}$, but not necessarily temporal consistency. An explicit case is shown in Example 11 later on.

A consequence of the above proposition is that inequality (12), which we derived from temporal consistency, is actually equivalent to event-wise temporal consistency. This shows that violations of event-wise temporal consistency should be very rare if only present beliefs were assessed using some minimal care.

*Example* 9 (Running example). Consider again the unconditional lower prevision $\underline{P}$ that represents your current beliefs, and assume that you postpone your assessment of your future commitments until the test has been made, and that this turns out to be positive. Then temporal consistency should only be verified by means of $\underline{P}, \underline{P}(V|T = p)$, since it makes no sense anymore to take into account the assessments in $\underline{P}(V|T = n)$. Since we already showed in Example 7 that $\underline{P}, \underline{P}(V|T)$ are temporally consistent, and therefore $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss, so does its subset $\mathcal{R} \cup \mathcal{R}^B$, where $B$ denotes the event $T = p$. Hence, we also have event-wise temporal consistency. ⧫

We can extend the similarity to temporal consistency further up to strong temporal consistency, in an obvious way:

*Definition* 17 (**Event-wise strong temporal consistency**). We say that your present and future commitments are *event-wise strongly temporally consistent* if $\underline{P}_B = \underline{E}(\cdot|B)$.

The underlying idea is, as usual, that you set future commitments equal to conditional beliefs. The additional requirement that conditional beliefs and future commitments jointly avoid sure loss is automatically satisfied in the present case, taking into account Proposition 5.

[15] One might want to consider modifying the definition in the following way: given that future commitments are established after $B$ occurs, at that time you might know exactly which subset $\mathcal{R}'$ of the desirable gambles $\mathcal{R}$ associated to $\underline{P}$ have been offered to you, and hence accepted, in that first stage (see also Section 3.1); then it might make sense to consider a (weaker) definition of temporal consistency that involves (the natural extension of) $\mathcal{R}'$ rather than $\mathcal{R}$. The essence of the rationale behind the definition would not change, however, nor would the technical development change in any substantial way.

The (event-wise) consistencies we have introduced in this section are the weakest in this paper. This is due to the limited availability of information about your future commitments. On the other hand, note that the other notions, such as (strong) temporal consistency, are not only a separate replication of event-wise temporal consistency for all the possible events $B \in \mathcal{B}$: they also consider the joint effect on your current beliefs of future commitments related to different events $B$ through the inborn conglomerability of set $\mathcal{F}^{\mathcal{B}}$ induced by $\underline{P}_{\mathcal{B}}$. This is the key to the increased consistency power of those notions that can rely on $\mathcal{F}^{\mathcal{B}}$.

*4.4. Temporal coherence*

We finally focus on the case where you define your future commitments now, at the same time of your present beliefs. The peculiar feature of this case is that the assessments of present and future commitments can influence each other (this should be contrasted with strong temporal consistency, for instance, where it is only possible that future commitments are affected by present beliefs, which are established in advance and cannot be modified during the assessment of future commitments). This allows us to give much more stringent conditions than it was possible before, and actually allows us also to establish them as rationality requirements.

The first condition that becomes immediately tenable as a rationality requirement is that future commitments coincide with the conditional beliefs derived from $\underline{P}$, which, from Lemma 1, are given by

$$\underline{E}(f|B) = \sup\{\mu : B(f-\mu) \in \mathcal{R}\} = \sup\{\mu : B(f-\mu) \in \mathcal{R}^{|B}\},$$

where $\mathcal{R}$ is the set of gambles induced by $\underline{P}$ through (4) and $\mathcal{R}^{|B}$ is derived from $\mathcal{R}$ by means of (7). The lower prevision $\underline{E}(\cdot|B)$ represents your beliefs now under the assumption that $B$ occurs. Given that by assumption you are also establishing now the model $\underline{P}_B$, this should just lead you to make $\underline{P}_B$ equal to $\underline{E}(\cdot|B)$. For this reason, we stick to the equality $\underline{P}_{\mathcal{B}} = \underline{E}(\cdot|\mathcal{B})$ in this section.[16] This is also equivalent to the equality $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$, where

$$\mathcal{F}^{|\mathcal{B}} := \{f \in \mathcal{L} : Bf \in \mathcal{R}^{|B} \cup \{0\}\ \forall B \in \mathcal{B}\} \setminus \{0\}$$

denotes the conglomerable natural extension of $\cup_{B\in\mathcal{B}}\mathcal{R}^{|B}$. Note that $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$ if and only if $\mathcal{R}^{B} = \mathcal{R}^{|B}$ for all $B \in \mathcal{B}$.

**Proposition 6.** *Let $\underline{P}, \underline{P}_{\mathcal{B}}$ be a coherent lower prevision and a separately coherent future lower prevision on $\mathcal{L}$ that represent your current and future commitments, respectively. Let $\mathcal{R}$, $\mathcal{R}^B$ ($B \in \mathcal{B}$) be the sets of desirable gambles they induce by means of Eqs.* (4) *and* (5)*, and let $\mathcal{F}^{\mathcal{B}}$ the conglomerable natural extension of $\cup_{B\in\mathcal{B}}\mathcal{R}^B$. On the other hand, let $\mathcal{R}^{|B}$ be the set of conditional gambles induced by $\mathcal{R}$ by means of* (7)*, let $\mathcal{F}^{|\mathcal{B}}$ be the conglomerable natural extension of $\cup_{B\in\mathcal{B}}\mathcal{R}^{|B}$ and let $\underline{E}(\cdot|\mathcal{B})$ be the conditional lower prevision induced by $\mathcal{F}^{|\mathcal{B}}$. Then*

$$\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}} \Leftrightarrow \underline{P}_{\mathcal{B}} = \underline{E}(\cdot|\mathcal{B}).$$

*Proof.* Since $\mathcal{F}^{\mathcal{B}}$ induces $\underline{P}_{\mathcal{B}}$, the direct implication is trivial, so it suffices to prove the converse one. Fix $B \in \mathcal{B}$, and let us show that $\mathcal{R}^{|B} = \mathcal{R}^B$. Recall that

$$\mathcal{R}^B = \{f : f = Bf, f \gneq 0 \text{ or } \underline{P}_B(f_B) > 0\}$$

and

$$\mathcal{R}^{|B} = \{f : f = Bf, Bf \in \mathcal{R}\} = \{f : f = Bf, f \gneq 0 \text{ or } \underline{P}(Bf) > 0\}.$$

We skip the trivial case $f \gneq 0$. Let us show that in the remaining case it holds that $\underline{E}(f|B) > 0$ if and only if $\underline{P}(Bf) > 0$, from which the thesis follows immediately. $\underline{E}(f|B) > 0$ implies that there is $\delta > 0$ s.t. $B(f-\delta) \in \mathcal{R}$. This means that $\underline{P}(B(f-\delta)) > 0$, taking into account that we have excluded the case $Bf \gneq 0$, whence $\underline{P}(Bf) > 0$. Conversely, $\underline{P}(Bf) > 0$ implies that there is $\delta > 0$ s.t. $\underline{P}(Bf-\delta) > 0$, whence $\underline{P}(B(f-\delta)) > 0$. This means that $B(f-\delta) \in \mathcal{R}$ or, in other words, that $\underline{E}(f|B) > 0$. $\square$

[16] When discussing these questions in the more general framework of desirability in Section 6.4, we shall see that the equality of future commitments to conditional beliefs formally follows from a coherence condition.

The next condition that does appear tenable too as a rationality requirement is that also your current and future commitments cohere. If this were not the case, then some of the beliefs you are expressing at the same point in time (now) would clash with each other. As we mentioned in Section 2, $\underline{P}$ and $\underline{E}(\cdot|\mathcal{B})$ need not be coherent, because they need not satisfy CNG. The next theorem gives insights on this important question.

**Theorem 2.** *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}$ representing your current beliefs, and $\underline{E}(\cdot|\mathcal{B})$ be its conditional natural extension. Let $\mathcal{R}$ be the set of gambles associated to $\underline{P}$, and $\mathcal{F}^{|\mathcal{B}}$ the conglomerable natural extension of $\cup_{B\in\mathcal{B}}\mathcal{R}^{|B}$. Then the following are equivalent:*

(a) *$\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$ is coherent.*

(b) *$\underline{P},\underline{E}(\cdot|\mathcal{B})$ are coherent.*

(c) *$\mathcal{R}$ is conglomerable.*

*Proof.* The equivalence of (b) and (c) follows from [44, Theorem 3] and the definition of conglomerability. We proceed to prove the equivalence of (a) and (b).

(a) $\Rightarrow$ (b) Assume that $\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$ is coherent, and let us show that in that case

$$\underline{P}(f)\geq 0\;\forall f\in\mathcal{F}^{|\mathcal{B}}. \tag{13}$$

There are two possibilities: if $\mathcal{R}\subseteq\mathcal{F}^{|\mathcal{B}}$, [44, Proposition 10] implies that $\mathcal{R}=\mathcal{F}^{|\mathcal{B}}=\mathcal{L}^+$, and therefore (13) holds. On the other hand, if $\mathcal{R}\nsubseteq\mathcal{F}^{|\mathcal{B}}$ then we can consider a gamble $g\in\mathcal{R}\setminus\mathcal{F}^{|\mathcal{B}}$. This means that $\underline{P}(g)\geq 0$ and moreover that there is some $B\in\mathcal{B}$ such that $Bg\notin\mathcal{R}^{|B}\cup\{0\}$.

Assume ex-absurdo that (13) does not hold, and take $f\in\mathcal{F}^{|\mathcal{B}}$ such that $\underline{P}(f)<0$. Define $f_1:=B^cf$. Then $B'f_1\in\mathcal{R}^{|B'}\cup\{0\}$ for every $B'\in\mathcal{B}$, and moreover it cannot be $f_1=0$ or we should have $f=Bf\in\mathcal{R}$ and $\underline{P}(f)\geq 0$, a contradiction. As a consequence, $f_1\in\mathcal{F}^{|\mathcal{B}}$, and therefore $\lambda f_1\in\mathcal{F}^{|\mathcal{B}}$ for all $\lambda>0$. Moreover,

$$\underline{P}(f_1)=\underline{P}(f-Bf)\leq\underline{P}(f)+\overline{P}(-Bf)=\underline{P}(f)-\underline{P}(Bf)<0,$$

because $\underline{P}(f)<0$ and $\underline{P}(Bf)\geq 0$ taking into account that $Bf\in\mathcal{R}^{|B}\cup\{0\}$. Define $h:=\lambda f_1+g$; $h$ belongs to the natural extension of $\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$. Then it holds that

$$Bh=B(\lambda f_1+g)=Bg\notin\mathcal{R}^{|B}\cup\{0\}\Rightarrow h\notin\mathcal{F}^{|\mathcal{B}},$$

and

$$\underline{P}(h)=\underline{P}(\lambda f_1+g)\leq\underline{P}(\lambda f_1)+\overline{P}(g)=\lambda\underline{P}(f_1)+\overline{P}(g)<0\text{ for }\lambda>-\frac{\overline{P}(g)}{\underline{P}(f_1)}.$$

Hence, for $\lambda$ big enough $\underline{P}(h)<0$, whence $h$ does not belong to $\mathcal{R}$ either. As a consequence, $\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$ is different from its natural extension and therefore it is not a coherent set, a contradiction.

We conclude that Eq. (13) holds, and applying now [44, Theorem 2], we deduce that $\mathcal{F}^{|\mathcal{B}}\subseteq\mathcal{R}$. Hence, $\mathcal{R}$ is conglomerable, and since the second and the third statements are equivalent we deduce that $\underline{P},\underline{E}(\cdot|\mathcal{B})$ are coherent.

(b) $\Rightarrow$ (a) Conversely, assume ex-absurdo that $\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$ is not coherent. Then there are gambles $f\in\mathcal{R},g\in\mathcal{F}^{|\mathcal{B}}$ such that $f+g\notin\mathcal{R}\cup\mathcal{F}^{|\mathcal{B}}$; since this gamble does not belong to $\mathcal{R}$, we deduce that $\underline{P}(f+g)\leq 0$. We can assume without loss of generality that neither of these gambles is positive, or we should contradict the coherence of either $\mathcal{R}$ or $\mathcal{F}^{|\mathcal{B}}$. From Eq. (4), $f\in\mathcal{R}$ implies that $\underline{P}(f)>0$. On the other hand, $g\in\mathcal{F}^{|\mathcal{B}}$ implies that $\underline{E}(g|\mathcal{B})\geq 0$, whence $g\geq g-\underline{E}(g|\mathcal{B})$. As a consequence,

$$0\geq\underline{P}(f+g)\geq\underline{P}(f)+\underline{P}(g)\Rightarrow 0>-\underline{P}(f)\geq\underline{P}(g)\geq\underline{P}(g-\underline{E}(g|\mathcal{B})),$$

whence $\underline{P},\underline{E}(\cdot|\mathcal{B})$ do not satisfy CNG and therefore are not coherent. $\square$

In the light of Proposition 16, we see that Theorem 2 yields a truly bright outcome: that, provided that $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$, the coherence of the union of your sets of current and future commitments is equivalent to the conglomerability of $\mathcal{R}$ or, equivalently, to the conglomerability of $\underline{P}$. It is striking in particular that a very simple, natural, and especially finitary requirement such as the coherence of $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$, which is also the straightforward way to strengthen strong temporal consistency, eventually shows itself to coincide with $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$. (We remark that the use of Walley's results in the proof of this theorem is necessary for the equivalence between the second and the third items, but not for the equivalence with the first item, which is the main finding of the theorem, and which allows us to justify conglomerability in a finitary way: in our knowledge, this is achieved here for the first time.)

This gives rise to the following rationality condition:

*Definition* 18 (**Temporal coherence**). Let $\underline{P}, \underline{P}_{\mathcal{B}}$ be the coherent lower previsions representing your current and future commitments. We say that they are *temporally coherent* when $\underline{P}_{\mathcal{B}}$ coincides with the conditional natural extension of $\underline{P}$ and when moreover any of the conditions in Theorem 2 holds.

The conglomerability of an imprecise probability model does not imply that the model is equivalent to a set of conglomerable precise models; an example can be found in [68, Section 6.6.9]. This means, in particular, that even when it is rational for you to hold conglomerable beliefs, it is not necessary that your imprecise probability model be made up of countably additive linear previsions. On the other hand, when your beliefs are precise, then our results entail a tight connection of your model with disintegrability. This will be discussed in Section 4.5.

*Example* 10 (Running example). In the situation of our running example, since the partition $\{T = p, T = n\}$ is finite, the coherent lower prevision $\underline{P}$ representing your current beliefs is trivially conglomerable, and as a consequence it is coherent with its conditional natural extension $\underline{E}(V|T)$, which tells how you should assess your future commitments if you establish them at the time of your present beliefs, and hence have no time to refine the assessment $\underline{P}$. We conclude that $\underline{P}, \underline{E}(V|T)$ are temporally coherent. ⧫

### 4.5. *The precise case*

In this section we discuss the important special case where your present and future commitments are precise. Hence, we focus on the case where your present beliefs are specified via a linear prevision $P$ on $\mathcal{L}$, and we assume that also your future commitments for any $B \in \mathcal{B}$ are given by a linear prevision, which we denote by $P_B$ and that is defined on $\mathcal{L}(B)$. From these we induce the future linear prevision $P_{\mathcal{B}}$ given, for any $f \in \mathcal{L}$, by $P_{\mathcal{B}}(f) := \sum_B B P_B(f_B)$.

We start by focusing on temporal consistency:

**Corollary 2.** *Let $P$ and $P_{\mathcal{B}}$ represent your present and future commitments, respectively. Then temporal consistency holds if and only if*

$$P(f) = P(P_{\mathcal{B}}(f)) \quad \forall f \in \mathcal{L}.^{17} \tag{14}$$

*Proof.* This is an immediate consequence of Theorem 1. □

If $P(B) > 0$, then it is a consequence of coherence [68, Section 6.4.1] that the conditional natural extension on $B$ is precise and is given by Bayes' rule:

$$E(f|B) = \frac{P(Bf)}{P(B)}.$$

Moreover, we see from Eq. (12) that temporal consistency implies that

$$\underline{P}_B(f) \leq E(f|B) \leq \overline{P}_B(f)$$

and

$$\underline{E}(f|B) \leq P_B(f) \leq \overline{E}(f|B) \, \forall f \in \mathcal{L}.$$

[17] Note how this formula resembles very closely Goldstein's formalisation of his notion of temporal coherence in [21]. See Section 7.3 for details.

This means that when unconditional beliefs as well as future commitments are precise, we obtain the interesting additional result that $P_B(f) = P(f|B)$, and as a consequence

$$P(B) > 0 \Rightarrow P_B(f) = \frac{P(Bf)}{P(B)} \ \forall f \in \mathcal{L}. \tag{15}$$

Loosely speaking, we can rephrase this by saying that for a Bayesian who wants to be consistent in time, there is only one way to compute future commitments: Bayes' rule (but remember we are under the constraint of the positivity of probabilities). It is useful to notice that this result, which we have obtained from temporal consistency, actually follows from event-wise temporal consistency. This shows that (15) follows under the weakest consistency notion we have introduced in this paper, and that in the precise case our notion of temporal consistency gives rise to conditional reasoning.[18]

This is not the only thing that we can say when your current and future commitments are precise. A key observation is that in such a setting the notions of avoiding sure loss and coherence are equivalent: taking into account Theorem 2, this points to a relationship between temporal consistency and conglomerability. Such a relationship was sensed already by Walley. This is relatively clear in [68, Example 6.8.5], where Walley argues that there may be a (temporal) sure loss when conditional beliefs are used as future commitments.[19] A similar point was made also in an analogous example by Seidenfeld [52, Section 2.2], who used it to argue that Goldstein's proposal of generalising de Finetti's ideas to a temporal setting, that we shall detail in Section 7.3, was incompatible with Bayes' rule when beliefs are modelled by a finitely additive probability. Below we report Walley's example in our language for completeness:[20]

*Example* 11. Let $\mathcal{B} := \{\{n, -n\} : n \in \mathbb{N}\}$, $\Theta := \{+, -\}$, and $\Omega := \Theta \times \mathcal{B}$. $\Omega$ represents the set of non-zero integers: to see this, identify the integer $n$ with the pair $(\text{sign}(n), |n|)$. $B \in \mathcal{B}$ represents the observation of a certain absolute value, while $\Theta$ represents a hypothesis about the sign of an integer. Your current beliefs are represented by a probability $P$ defined as follows: for all $n \in \mathbb{N}$, $P(\{n\}|+) := 2^{-n}$, $P(\{-n\}|-) := 0$, and $P(\{+\}) := P(\{-\}) := 1/2$. It turns out that for all $B \in \mathcal{B}$, it holds that $P(B) > 0$ and $P(\{+\}|B) = 1$, so that also $P(\{+\}|\mathcal{B}) = 1$.

Now, consistently with (15), assume that you use Bayes' rule to define your future commitments on $\Theta$: $P_{\mathcal{B}} := E(\cdot|\mathcal{B})$. It follows that $P(\{+\}) = \frac{1}{2} \neq 1 = P(P_{\mathcal{B}}(\{+\}))$, which contradicts (14). And in fact, this would allow an opponent to buy event $\{+\}$ from you now at price $\frac{1}{2}$ and, after $B$ occurs (whatever $B$ it is), sell you $\{+\}$ at price 1, thus making a sure gain. We deduce that if you want to preserve temporal consistency as well as the possibility to use Bayes' rule to define your future commitments, it is necessary that $P$ be disintegrable. ⧫

In order to fully clarify this matter, it is useful to establish the following result, which holds irrespective of whether or not future commitments match conditional beliefs (that is, irrespective of whether or not $P_{\mathcal{B}}$ is the conditional natural extension of $P$):

**Theorem 3.** *Assume your current and future commitments are linear previsions $P, P_{\mathcal{B}}$, and let $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ be the sets of gambles they induce by means of Eqs.* (4), (5) *and* (9). *Then:*

$$\mathcal{R} \cup \mathcal{F}^{\mathcal{B}} \textit{ coherent } \Leftrightarrow \mathcal{R} \cup \mathcal{F}^{\mathcal{B}} \textit{ avoids partial loss } \Leftrightarrow P, P_{\mathcal{B}} \textit{ coherent}.$$

*Proof.* That the first condition implies the second is trivial; the second implies that $P, P_{\mathcal{B}}$ avoid sure loss by Theorem 1, and since they are linear this means that they are coherent.

Conversely, assume that $P, P_{\mathcal{B}}$ are coherent, and let us show that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ is coherent. It suffices to show that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ equals its natural extension. Since both $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are coherent, this holds if and only if $f + g \in \mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ for every $f \in \mathcal{R}, g \in \mathcal{F}^{\mathcal{B}}$. Consider such $f, g$. We can assume without loss of generality that neither of these gambles is positive, because the result holds trivially in that case. Using Eq. (4), we deduce that $P(f) > 0$. On the other hand, $g \in \mathcal{F}^{\mathcal{B}}$ implies that $P_{\mathcal{B}}(g) \geq 0$, whence

$$P(f + g) = P(f) + P(g) = P(f) + P(P_{\mathcal{B}}(g)) > 0;$$

[18]Other justifications of Bayesian updating as a temporal rule can be found in [26, 45, 57, 70]. See also Section 7.

[19]But Walley was not fully explicit in claiming that conditional beliefs would be taken as the definition of future commitments (more generally speaking, he seldom talks of future commitments in his book). Overlooking this subtlety can make Walley's argumentation easily misunderstood: in fact, if one takes those as conditional (non-future) commitments, then there is no loss, as instead the example is supposed to show.

[20]This example is a specific instance of the question discussed right after Definition 13.

the second equality follows from Corollary 2, taking into account that the coherence of $P, P_{\mathcal{B}}$ implies that they are temporally consistent; for the inequality use that the coherence of $P$ implies that $P(P_{\mathcal{B}}(g)) \geq 0$. As a consequence, $f + g$ has a positive prevision, and therefore it belongs to $\mathcal{R}$. We conclude that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ is coherent. $\square$

We can finally analyse in more detail the implications of the precise setting. We focus on the special case where $P(B) > 0$ for all $B \in \mathcal{B}$. Then Eq. (15) and Theorem 3 imply that:

**Corollary 3.** *If $P(B) > 0$ for all $B \in \mathcal{B}$ and $P^{\mathcal{B}}$ is derived from $P$ by means of Bayes' rule, then the following conditions are equivalent:*

(a) *$P, P^{\mathcal{B}}$ are temporally consistent.*

(b) *$P, P^{\mathcal{B}}$ are strongly temporally consistent.*

(c) *$P, P^{\mathcal{B}}$ are temporally coherent.*

(d) *$P$ is disintegrable.*

The above equivalence does not hold for event-wise temporal consistency, because it considers only the occurrence of a single event $B$. On the other hand, one can argue that event-wise temporal consistency is not the notion to apply in the present case, and more generally that it does not make sense to declare future commitments after present beliefs: in fact, since there is no choice other than using Bayes' rule to define future commitments, why should Bayesians want to postpone this task? They know from the very beginning that by setting future commitments different from conditional beliefs, they would expose themselves to incur a loss (remember also that we are assuming that you receive no new information about $\Omega$ until $B$ obtains). Being Bayesian together with conceding—right now—that future commitments might differ from conditional beliefs appears irrational. Stated differently, temporal coherence appears to be a rationality requirement in the present case.[21]

*Example* 12 (Running example). Assume that in our running example we have precise information, stating that the prevalence of the seasonal and the atypical virus is 95% and 5%, respectively, and that the probability that the test is positive is of 60% and 90% in each of these cases. This corresponds to the assessments

$$P(V = s) = 0.95, P(V = a) = 0.05, P(T = p|V = s) = 0.6, P(T = p|V = a) = 0.9$$

from which, applying the law of total probability, we obtain the joint model

$$P(s, p) = 0.57, P(s, n) = 0.38, P(a, p) = 0.045, P(a, n) = 0.005.$$

What our results tell us is that, in order to achieve temporal coherence (or temporal consistency, or strong temporal consistency) your future model $P(V|T)$ should be determined by Bayes' rule, so that:

$$P(V = a|T = p) = 0.073 \text{ and } P(V = a|T = n) = 0.013,$$

and since there is no freedom in how to establish your future commitments in this precise case, it makes no sense to postpone this task until the test has been performed. ⧫

*Remark* 3. Let us consider now what happens in the more general situation where it may be that $P(B) = 0$ for some $B \in \mathcal{B}$. In this case, the natural extension $\underline{E}(\cdot|B)$ of $P$ conditional on $B$ is vacuous, and therefore differs from $P_B$ (which we require to be precise). The connection between the different consistency notions is weaker then. Moreover, event-wise temporal consistency, as well as declaring future commitments after present beliefs, cannot be ruled out: present beliefs do not constrain future commitments when $P(B) = 0$. (See Proposition 19 in Section 6.5 for additional insights on this case.) ⧫

[21] However, note that if one decided to use the modification of event-wise temporal consistency sketched in note 14, it might happen that the set of actually made (current) transactions $\mathcal{R}'$ is strictly smaller than $\mathcal{R}$ and hence that its natural extension may not be a precise probabilistic model. In that case, Bayes' rule would not necessarily be the unique consistent rule to use even though all probabilities were positive.

Our discussion so far does not exhaust all the possibilities to express precise assessments. For instance, when $\Omega$ is continuous, it is common to hold precise beliefs in the form of a linear prevision $P$ that assigns zero probability to all the subsets of $\Omega$ while having precise, and hence non-vacuous, conditional beliefs $P(\cdot|\mathcal{B})$. In this case, the model of present (unconditional and conditional) beliefs is not only made of $P$, simply because $P(\cdot|\mathcal{B})$ cannot be derived from it. In terms of desirable gambles, this means that your conditional beliefs cannot be derived from the set of gambles $\mathcal{R}$ associated to $P$ by means of (4). However, it is still possible to deal with questions of temporal consistency under these types of precise assessments by considering other sets of desirable gambles different from $\mathcal{R}$; how this can be done is the purpose of the next sections (see in particular Section 5.2).

## 5. Coherent sets of gambles

So far, we have assumed that your assessments are modelled by means of (unconditional and conditional) lower previsions. These lower previsions encode your commitments to accept certain gambles. In our previous sections, we considered the sets $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ determined by $\underline{P}, \underline{P}_{\mathcal{B}}$ using Eqs. (4), (5) and (9), and showed that the different consistency notions for $\underline{P}, \underline{P}_{\mathcal{B}}$ can be given a behavioural interpretation in terms of these sets.

However, as we shall argue in this section, in certain situations there are other sets of desirable gambles that are also related to $\underline{P}, \underline{P}_{\mathcal{B}}$ and that may be more informative than the ones considered in Section 4. Because of this, it may be useful to model your assessments using directly the sets of commitments you are willing to accept. In order to make this clear, we shall introduce next a number of aspects of the theory of sets of desirable gambles, and detail their connection with the models of lower previsions.

### 5.1. Almost- and strict desirability

In the treatment done so far, we have focused on the sets of gambles induced by coherent lower previsions. One of the most important things to realise now is that coherent lower previsions induce, through (4), only a special case of coherent sets of desirable gambles, which can be characterised as follows:

*Definition* 19 (**Strictly desirable gambles**). $\mathcal{R}$ is called a set of *strictly desirable gambles* when it is coherent and moreover satisfies the following condition:

D0. For every $f \in \mathcal{R} \setminus \mathcal{L}^+$ there is some $\delta > 0$ such that $f - \delta \in \mathcal{R}$.

D0 is a condition of openness: a set of strictly desirable gambles is a convex cone that, excluding the region $\mathcal{L}^+ \subseteq \mathcal{R}$ from consideration, coincides with its interior. In the following we shall say that the set is open, thus neglecting the case of $\mathcal{L}^+$, with an abuse of terminology. We shall adopt the notation $\underline{\mathcal{R}}$ for a set of strictly desirable gambles in case we need to distinguish it from different types of sets.

Given a set $\mathcal{R}$ of strictly desirable gambles, we can induce a coherent lower prevision from it in a way similar to what we have already done in (6) for the conditional case:

$$\underline{P}(f) \coloneqq \sup\{\mu : f - \mu \in \mathcal{R}\}. \tag{16}$$

It can be checked that Eqs. (4) and (16) commute; as a consequence, there is a one-to-one correspondence between coherent lower previsions and sets of strictly desirable gambles; something similar applies to the conditional case (Eqs. (5) and (6)). This correspondence extends also towards the notion of conglomerability we have given in Definitions 5 and 10: a coherent lower prevision $\underline{P}$ is conglomerable if and only if its associated set of strictly desirable gambles $\mathcal{R}$ is conglomerable.

At the other extreme of the coherent open sets of desirable gambles, there are the closed sets of desirable gambles. These are cones whose interior is a set of strictly desirable gambles, and which include the border of the cone. These closed cones are called sets of almost-desirable gambles, and are characterised as follows:

*Definition* 20 (**Almost-desirable gambles**). $\mathcal{R}$ is called *almost-desirable* when it satisfies axiom

D0′. $f + \varepsilon \in \mathcal{R}\ \forall \varepsilon > 0 \Rightarrow f \in \mathcal{R}$,

the following modified versions of axioms D1 and D2:

D1′. $\inf f > 0 \Rightarrow f \in \mathcal{R}$,

D2′. $\sup f < 0 \Rightarrow f \notin \mathcal{R}$,

as well as axioms D3 and D4.

D0′ is a closure condition, which means that the uniform limit of a decreasing sequence of gambles in $\mathcal{R}$ also belongs to $\mathcal{R}$; axioms D1′ and D2′ guarantee that $\mathcal{R}$ includes the gambles that are strictly positive and excludes those that are strictly negative, when these gambles are at the same time bounded away from zero. Note that a set of almost-desirable gambles is not coherent because axioms D0′–D1′ imply that $0 \in \mathcal{R}$, thus violating D2.

A coherent lower prevision $\underline{P}$ on $\mathcal{L}$ induces a set of almost-desirable gambles by

$$\mathcal{R} := \{f \in \mathcal{L} : \underline{P}(f) \geq 0\}. \tag{17}$$

If we denote by $\underline{\mathcal{R}}$ the strictly desirable set induced by $\underline{P}$ through (4), we obtain that $\mathcal{R}$ corresponds to the closure of $\underline{\mathcal{R}}$ in the topology of uniform convergence [42, Proposition 4], so any almost-desirable gamble can be seen as the uniform limit of a sequence of strictly desirable gambles. For this reason, we shall also denote by $\overline{\mathcal{R}}$ a set of almost-desirable gambles.

In this paper almost-desirability is useful to provide the definition of avoiding sure loss in the case of gambles:

*Definition* 21 (**Avoiding sure loss for gambles**). We say that a set of gambles $\mathcal{R}$ *avoids sure loss* when it is included in a set of almost-desirable gambles.

Since any set of almost-desirable gambles that includes $\mathcal{R}$ must also include its natural extension $\mathcal{E}_{\mathcal{R}}$ because of axioms D0′, D1′, D3 and D4, we deduce that $\mathcal{R}$ avoids sure loss if and only if $\mathcal{E}_{\mathcal{R}}$ is included in a set of almost-desirable gambles. Moreover,

$$\mathcal{R} \text{ avoids sure loss} \Leftrightarrow \overline{\mathcal{E}}_{\mathcal{R}} \text{ is a set of almost-desirable gambles},$$

where $\overline{\mathcal{E}}_{\mathcal{R}}$ denotes the closure of $\mathcal{E}_{\mathcal{R}}$ in the topology of uniform convergence. To see the direct implication, note that any set of almost-desirable gambles that includes $\mathcal{R}$ must include $\overline{\mathcal{E}}_{\mathcal{R}}$ because of the axioms of almost-desirability, and moreover $\overline{\mathcal{E}}_{\mathcal{R}}$ is a set of almost-desirable gambles if and only if it satisfies D2′, so if $\overline{\mathcal{E}}_{\mathcal{R}}$ is not a set of almost-desirable gambles none of its supersets can be.

Moreover, it can be checked that if $\overline{\mathcal{E}}_{\mathcal{R}}$ violates D2′ then also $\mathcal{E}_{\mathcal{R}}$ does it, and from this we deduce the existence of a positive linear combination $f$ of gambles in $\mathcal{R}$ such that $\sup f < 0$. This allows us to get some intuition about this definition: we say that $\mathcal{R}$ avoids sure loss when it is not possible that a positive linear combination of gambles in $\mathcal{R}$ results in a gamble that produces a loss of at least some $\varepsilon > 0$, no matter the outcome of the experiment.

A set of gambles that avoids partial loss avoids in particular sure loss, because the closure of any coherent set of desirable gambles is a set of almost-desirable gambles; however, the converse does not hold in general, since a set of almost-desirable gambles, such as $\{f \geq 0\}$, may incur a partial loss. Note moreover than a coherent set of desirable gambles avoids in particular partial and sure loss.

We conclude by noting that, in a similar way to coherent lower previsions, a coherent set of desirable gambles $\mathcal{R}$ is also associated to a credal set:

$$\mathcal{M}(\mathcal{R}) := \{P : P(f) \geq 0 \, \forall f \in \mathcal{R}\},$$

and it can be checked that this credal set coincides with $\mathcal{M}(\underline{P})$, where $\underline{P}$ is the coherent lower prevision induced by $\mathcal{R}$ by Eq. (16).

### 5.2. *Introducing general desirability*

From now on we consider the general case of coherent sets of desirable gambles, i.e., those sets that satisfy axioms D1–D4 without being necessarily strictly desirable. In the present section we introduce such a general case and discuss why it is important to focus on it.

We start by noticing that, as opposed to the case of strict desirability, the correspondence between coherent lower previsions and coherent sets of gambles is one-to-many: in fact, any coherent set of desirable gambles $\mathcal{R}$ inducing $\underline{P}$ by means of Eq. (16) satisfies

$$\underline{\mathcal{R}} \subseteq \mathcal{R} \subsetneq \overline{\mathcal{R}}, \tag{18}$$

where $\underline{\mathcal{R}}$ is the set of strictly desirable gambles that induces $\underline{P}$ and $\overline{\mathcal{R}}$ is the topological closure of $\mathcal{R}$ (as well as of $\underline{\mathcal{R}}$). In fact, $\overline{\mathcal{R}}$ is also the set of almost-desirable gambles induced by $\underline{P}$ by means of Eq. (17). In other words, all the infinitely many coherent sets of gambles $\mathcal{R}$ that satisfy the inclusion (18) are going to induce $\underline{P}$. Such an inclusion highlights once more that the differences between all these sets lies only in their topological border, given that $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ share the same interior.

Remember that in the case of strict desirability a gamble $f \notin \mathcal{L}^+$ belongs to $\underline{\mathcal{R}}$ if and only if $\underline{P}(f) > 0$. For a general coherent set of desirable gambles $\mathcal{R}$, it may hold instead that both $f \in \mathcal{R}$ and $\underline{P}(f) = 0$. This has important consequences for the conditional case. To see this, note that we can obtain a separately coherent conditional lower prevision from a coherent set of desirable gambles in the following way:

$$\underline{P}(f|B) \coloneqq \sup\{\mu : B(f - \mu) \in \mathcal{R}\}, \tag{19}$$

which generalises and subsumes both (6) and (16). Now, whenever $\underline{P}(B) = 0$ for an event $B \in \mathcal{B}$, we must have also $\underline{P}(Bg) \leq 0$ for any gamble $g \in \mathcal{L}$ (cf. [44, Lemma 1]), so that there is no gamble $g$ such that $Bg \in \underline{\mathcal{R}}$; by applying (19) to $\underline{\mathcal{R}}$, we see then that the supremum $\mu$ such that $B(f - \mu) \in \underline{\mathcal{R}}$ is $\mu = \inf_B f$. This means that if we apply Eq. (19) to a set of strictly desirable gambles, the lower prevision of any gamble $f$ conditional on $B$, with $\underline{P}(B) = 0$, is necessarily vacuous. Note how this is consistent with (3) (in fact, the lower prevision conditional on $B$ obtained through $\underline{\mathcal{R}}$ is just the conditional natural extension of $\underline{P}$, as we can see from Lemma 1), and that this means in particular that a set of strictly desirable gambles is completely uninformative about any conditional inference when $\underline{P}(B) = 0$.

This need not be the case for general desirability, as an immediate consequence of the property that allows both $\underline{P}(Bf) = 0$ and $Bf \in \mathcal{R}$ to hold. The implications of such a property are very broad, perhaps broader that one might expect at first: in fact, it has been shown in [44, Theorem 25(i)] (and stated also in [68, Appendix F4]) that whenever $\underline{P}$ is (jointly) coherent with a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$, then there is a coherent set $\mathcal{R}$ that induces them both. Joint coherence is particularly easy to characterise when we deal with a finite possibility space $\Omega$ and all the conditioning events have positive upper probability: in that case, a coherent lower prevision $\underline{P}$ and a separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ are jointly coherent if and only if $\underline{P}(\cdot|B) \in [\underline{E}(\cdot|B), \underline{R}(\cdot|B))]$ for all $B \in \mathcal{B}$, where $\underline{E}(\cdot|B)$ is the conditional natural extension given by Eq. (3) and

$$\underline{R}(f|B) \coloneqq \sup\{\mu : \underline{P}(B(f - \mu)) \geq 0\}$$

is called the *regular extension* (see [40, Section 4.3]).[22] This implies that whenever $\underline{E}(\cdot|\mathcal{B}) \neq \underline{R}(\cdot|\mathcal{B})$,[23] there are infinitely many separately coherent conditional lower previsions $\underline{P}(\cdot|\mathcal{B})$ that are jointly coherent with $\underline{P}$. For each of those, we can find a coherent set of desirable gambles $\mathcal{R}$ that induces both $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$. This shows that even if the difference between desirability and strict desirability is only in the topological border of the involved sets, the border actually makes all the difference when it comes to making inferences in the conditional case with $\underline{P}(B) = 0$.

Moreover, note that the case $\underline{P}(B) = 0$ is particularly important in applications. For instance, consider the case where $\Omega$ is the bi-dimensional set of real numbers $\mathbb{R}^2$. In a case like this, it is common practice in precise probability to express uncertainty through a density function that assigns zero probability to every pair $(\omega_1, \omega_2) \in \Omega$; at the same time, the inferences conditional on the observation of $\omega_2 \in \mathbb{R}$ are obtained through the conditional density, and hence are not vacuous. This prevents the conditional and unconditional models from being represented through a coherent set of strictly desirable gambles, because this would be incompatible with the non-vacuity of the conditional inferences. The models can instead be induced by a single coherent set of desirable gambles, because the pair $\underline{P}, \underline{P}(\cdot|\mathcal{B})$, corresponding to the unconditional and the conditional density, are jointly coherent (see [68, Section 7.7.2]).

Even though these situations cannot be represented using a single set of strictly desirable gambles, or, which is equivalent, using a single coherent lower prevision, one could still try to use a *collection* of coherent lower previsions to represent your current beliefs, such as a jointly coherent pair $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ for your present unconditional and conditional beliefs, respectively—where $\underline{P}(\cdot|\mathcal{B})$ would not need to coincide with the conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ of $\underline{P}$. But this would technically complicate much the analysis we are pursuing in this paper compared to using desirability, taking

[22] Conditioning a set of mass functions by regular extension corresponds to applying Bayes' rule to each mass function that assigns positive probability to the conditioning event $B$. If there is no such mass function, then the regular extension yields the set of all the mass functions, i.e., the vacuous model.

[23] This may happen only when $\overline{P}(B) > \underline{P}(B) = 0$; when $\underline{P}(B) > 0$ both are equal to the unique value for which GBR is satisfied.

into account that we should also need to use a further separately coherent lower prevision for your future commitments. Perhaps more important, it has been shown that even collections of separately coherent conditional lower previsions are not as expressive as coherent sets of desirable gambles; this is the case also when we consider finite spaces of possibilities, as shown by the following example based on [42, Example 10].

*Example* 13. Two people express their beliefs about a fair coin using coherent sets of desirable gambles. The possibility space $\Omega := \{h, t\}$, represents the two possible outcomes of tossing the coin, i.e., heads and tails. For the first person, the desirable gambles $f$ are characterised by $f(h) + f(t) > 0$; for the second person, a gamble $f$ is desirable if either $f(h) + f(t) > 0$ or $f(h) = -f(t) < 0$. Call $\mathcal{R}_1$ and $\mathcal{R}_2$ the set of desirable gambles for the first and the second person, respectively. It can be verified that both sets are coherent. Moreover, they originate the same conditional and unconditional lower previsions. In the unconditional case we obtain $\underline{P}(f) = \frac{f(h)+f(t)}{2}$; this corresponds, correctly, to assigning $0.5$ probability to both heads and tails. In the conditional case, we again correctly obtain that each person would assign probability 1 to either heads or tails assuming that one of them indeed occurs: $\underline{P}(f|\{h\}) = f(h), \underline{P}(f|\{t\}) = f(t)$. This exhausts the conditional and unconditional lower previsions that we can obtain from $\mathcal{R}_1$ and $\mathcal{R}_2$, given that $\Omega$ has only two elements. It follows that $\mathcal{R}_1$ and $\mathcal{R}_2$ are indistinguishable as far as probabilistic statements are concerned. But now consider the gamble $f := (-1, 1)$, which yields a loss of 1 unit of utility if the coin lands heads and a gain of 1 unit otherwise: whereas $f$ is not desirable for the first person, it is actually so for the second. This distinction of the two persons' behaviour cannot be achieved through probabilities—and in fact gamble $f$ lies in the border of each of the two sets. This shows, in addition, that the relationship between collections of separately coherent conditional lower previsions and coherent sets of desirable gambles is also one-to-many. ⧫

In summary, the focus on general desirability allows us be truly general and at the same time it does not overcomplicate the technical development. On the other hand, we can find yet another reason to focus on desirability, as opposed to coherent lower previsions, in that it allows us to naturally regard coherent sets of desirable gambles as a logic[24] (this is helpful, among other things, to discuss the relationship of the present work with the field of belief revision, see Section 7.5).

### 5.3. Basic consistency notions for general desirability

Assume that you check $\mathcal{L}$ for desirability: this means that at some point you will isolate the subset $\mathcal{R}$ of gambles in $\mathcal{L}$ that you desire. Set $\mathcal{R}$ is a belief model that can be regarded as a generalisation of a set of sentences in propositional logic, where $\mathcal{L}$ has the role of the language. Notice how in propositional logic the sentences represent what is certain to you, while $\mathcal{R}$ only represents what you deem desirable; this change of perspective is the passage from which uncertainty comes into play.

The desirability analog of the deductive closure operator in logic is the mechanism that allows us to obtain the gambles in $\mathcal{L}$ whose desirability is implied by those in $\mathcal{R}$. To see how this mechanism works, consider first that since gambles express rewards in units of a linear utility, then any positive linear combination of a finite number of desirable gambles is desirable too. Let us call this the 'posi' of a set:

$$\mathrm{posi}(\mathcal{R}) := \left\{ \sum_{j=1}^{r} \lambda_j f_j \colon f_j \in \mathcal{R}, \lambda_j > 0, r \geq 1 \right\};$$

$\mathrm{posi}(\mathcal{R})$ is the smallest convex cone[25] that includes $\mathcal{R}$. Moreover, any gamble in $\mathcal{L}^+$ is desirable as well, given that it may increase the utility without ever decreasing it (whence $\mathcal{L}^+$ plays the role of the tautologies in logic). In other words, the mechanism we are after simply works as follows:

$$\mathcal{E}_{\mathcal{R}} := \mathrm{posi}(\mathcal{R} \cup \mathcal{L}^+),$$

[24] Joining probability and logic is the focus of some recent work by Howson [28, 29], who, interestingly, discusses also the question of conglomerability (but not desirability).

[25] A set $\mathcal{R}$ is a convex cone if $\mathrm{posi}(\mathcal{R}) = \mathcal{R}$.

where $\mathcal{E}_{\mathcal{R}}$ is then the analog of the deductive closure in propositional logic (and in fact $\mathcal{E}_{\mathcal{R}}$ can be shown to satisfy Tarski's axioms[26] for the *finitary consequence operator*;[27] for a description of these axioms, see [62, Chapter 5§1], axioms 2–4).

The reason why we use $\mathcal{E}_{\mathcal{R}}$ is first of all to check that $\mathcal{R}$ is a rational set of assessments, which means that it does not lead to the zero gamble being desirable:

*Definition* 22 (**Avoiding partial loss for gambles**). We say that $\mathcal{R}$ avoids partial loss if $0 \notin \mathcal{E}_{\mathcal{R}}$.

The name of this condition is due to the fact that once a set avoids partial loss, then no gamble $g \leq 0$ can belong to it [42, Corollary 2]. This notion of avoiding partial loss is equivalent to the existence of a coherent superset of $\mathcal{R}$ (as stated in Definition 8), and in fact in that case $\mathcal{E}_{\mathcal{R}}$ is nothing else but the natural extension of $\mathcal{R}$ in Definition 9. In propositional logic the analog of avoiding partial loss is the notion of a consistent set of sentences; and when in propositional logic we say that a set of sentences is consistent and logically closed, or a theory, in desirability we say that set $\mathcal{R}$ is coherent: this is equivalent to having $0 \notin \mathcal{R} = \mathcal{E}_{\mathcal{R}}$.

In logic, a special role is taken by complete theories: in a *complete theory*, for every sentence in the language it holds that either the sentence or its negation is in the theory. In desirability, the situation is very much alike: we say that a coherent set of gambles $\mathcal{R}$ is *complete*, or *maximal*, if for every non-zero gamble $f \in \mathcal{L}$, either $f \in \mathcal{R}$ or $-f \in \mathcal{R}$. Geometrically, a maximal set corresponds to a cone degenerated into a hyperplane. Maximal sets are tightly related to precise probability, because deriving a lower prevision from a maximal set by means of Eq. (16) yields a linear prevision; more generally speaking, linear previsions are in one-to-one correspondence with the interiors of maximal sets.

As a side note, let us point out that the relationship between desirability and logic that we have only sketched here, is discussed in much greater detail in [6, Section 5] (which is partly based on a former work in a similar spirit [45]). One interesting point made in that paper, among others, is that propositional logic can formally be embedded in the logic originated by coherent lower previsions, and that linear previsions correspond to complete logical theories. This shows in a definite sense how imprecision in probability, as well as in desirability, is what allows us to move away from complete theories into the much more expressive and used field of general logical theories.

### 5.4. Advanced consistency notions for general desirability

In this section we consider the notion of coherence *relative to a subset* $\mathcal{Q}$ of $\mathcal{L}$, which generalises the previous notion of coherence. The reason why this generalisation is introduced, is that it is not always realistic to expect that you can check all the gambles in $\mathcal{L}$ for desirability; you will often focus on a subset $\mathcal{Q}$ of them, and identify in $\mathcal{R}$ the subset of gambles in $\mathcal{Q}$ that you find desirable. To define coherence in this setting, we first extend $\mathcal{R}$ into $\operatorname{posi}(\mathcal{R} \cup \mathcal{L}^+) =: \mathcal{E}_{\mathcal{R}}$ and check that $0 \notin \mathcal{E}_{\mathcal{R}}$, because otherwise $\mathcal{R}$ would have no coherent extension to $\mathcal{L}$. In case $0 \notin \mathcal{E}_{\mathcal{R}}$, we proceed to define coherence in a very natural way: we say that you are coherent if the restriction of the natural extension $\mathcal{E}_{\mathcal{R}}$ to $\mathcal{Q}$ recreates $\mathcal{R}$. This means that not only you rationally defined $\mathcal{R}$, but that you were also fully aware of the desirability implications of your assessments within the set $\mathcal{Q}$ that you examined. This is made precise below:

*Definition* 23 (**Coherence relative to a subset of $\mathcal{L}$**). Say that $\mathcal{R}$ is *coherent relative to* $\mathcal{Q}$ if $\mathcal{R}$ avoids partial loss and $\mathcal{Q} \cap \mathcal{E}_{\mathcal{R}} \subseteq \mathcal{R}$ (and hence $\mathcal{Q} \cap \mathcal{E}_{\mathcal{R}} = \mathcal{R}$). In case $\mathcal{Q}$ coincides with $\mathcal{L}$ then we simply say that $\mathcal{R}$ is *coherent*.

This definition is indeed equivalent to axioms D1–D4 when $\mathcal{Q} = \mathcal{L}$ [42, Proposition 2]. It can be understood as a more primitive definition of coherence, which shows perhaps more intuitively what is the rationale behind those axioms. Part of the beauty of this definition is in that it joins simplicity with generality: in fact, from this single definition one can derive not only all the theory of coherent sets of desirable gambles but also all the theory of coherent lower previsions (except for the part that requires conglomerability in addition, as in the case of Walley's), as well as de Finetti's theory as a special case—by imposing the extra axiom of completeness.

Definition 23 has been developed for the case where you examine a single set $\mathcal{Q} \subseteq \mathcal{L}$ of gambles and isolate out of it the set $\mathcal{R}$ of gambles that are desirable to you. However, for the aims of this paper, we need to consider also a slightly more general setup. The motivation is that we need to deal with a pair of sets of desirable gambles: one for your present

[26] Yet, note that Tarski's axiom 1 restricts the treatment to sets that are at most countable. This restriction does not apply to $\mathcal{L}$ nor to the logical formalism of coherent sets of desirable gambles.

[27] This is actually a more formal rewording of the claim, we repeat in a few places throughout the paper, that our setup is finitary.

beliefs and one for your future commitments. These two sets are the result of two different assessment procedures, and we might want the two models to cohere with each other in some sense.

To address this problem it is convenient for the moment to represent the situation abstractly, without reference to the temporal setting of this paper. In this representation, you first examine set $\mathcal{Q}_1 \subseteq \mathcal{L}$ and declare that the coherent subset $\mathcal{R}_1 \subseteq \mathcal{Q}_1$ is desirable; then you examine set $\mathcal{Q}_2 \subseteq \mathcal{L}$ and declare that the coherent subset $\mathcal{R}_2 \subseteq \mathcal{Q}_2$ is desirable. In other words, such a situation is characterised by a collection $\{(\mathcal{R}_1, \mathcal{Q}_1), (\mathcal{R}_2, \mathcal{Q}_2)\}$ of assessed models, and the problem now is how to define the overall coherence of the collection. A basic requirement is that each model $(\mathcal{R}_i, \mathcal{Q}_i)$, $i = 1, 2$, is coherent according to Definition 23. To define then the actual coherence of the collection, we can proceed in two ways. One possibility is to replay the ideas at the basis of the traditional notion of coherence straightforwardly:

*Definition* 24 (**Coherence of a collection**). Let $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$, and $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$. Say that the collection is *coherent* if $\mathcal{R}$ is coherent relative to $\mathcal{Q}$.

However, there is another avenue that we can take. The idea is that the logical implications of the sets $\mathcal{R}_1, \mathcal{R}_2$, should not force any gamble in $\mathcal{Q}_1 \setminus \mathcal{R}_1$, nor any gamble in $\mathcal{Q}_2 \setminus \mathcal{R}_2$, to become desirable:

*Definition* 25 (**Strong coherence of a collection**). Let $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{E}_\mathcal{R} := \mathrm{posi}(\mathcal{R} \cup \mathcal{L}^+)$. Say that the collection is *strongly coherent* if $\mathcal{R}$ avoids partial loss and $\mathcal{E}_\mathcal{R} \cap \mathcal{Q}_i \subseteq \mathcal{R}_i$ for $i = 1, 2$.

In other words, what we require here is that the logical implications of the collection, expressed by $\mathcal{E}_\mathcal{R}$, cohere with each model $(\mathcal{Q}_i, \mathcal{R}_i)$, $i = 1, 2$, separately. The next proposition shows that strong coherence is indeed stronger than coherence. It is related to ideas underlying [42, Theorem 11, points (1) and (3)].

**Proposition 7.** *Let $\mathcal{R}_i$ be a set of desirable gambles coherent relative to $\mathcal{Q}_i$, for $i = 1, 2$. Let $\mathcal{E}_\mathcal{R}$ be the natural extension of $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$, and let $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$. Then the following are equivalent:*

(a) *The collection is coherent and $\mathcal{R} \cap \mathcal{Q}_i \subseteq \mathcal{R}_i$ for $i = 1, 2$.*

(b) *The collection is strongly coherent.*

*Proof.*

(a) ⇒ (b) Since $\mathcal{R}$ is coherent relative to $\mathcal{Q}$, then it avoids partial loss. Moreover, $\mathcal{E}_\mathcal{R} \cap \mathcal{Q} \subseteq \mathcal{R}$ implies that $\mathcal{E}_\mathcal{R} \cap \mathcal{Q} \cap \mathcal{Q}_i \subseteq \mathcal{R} \cap \mathcal{Q}_i$ or, in other words, that $\mathcal{E}_\mathcal{R} \cap \mathcal{Q}_i \subseteq \mathcal{R} \cap \mathcal{Q}_i \subseteq \mathcal{R}_i$, applying the assumption.

(b) ⇒ (a) By hypothesis, $\mathcal{R}$ avoids partial loss. In addition, we know that $\mathcal{E}_\mathcal{R} \cap \mathcal{Q}_i \subseteq \mathcal{R}_i$ for $i = 1, 2$. By taking the union on each side of the inclusion for $i = 1, 2$, we see that $\mathcal{E}_\mathcal{R} \cap \mathcal{Q} \subseteq \mathcal{R}$. □

A case of special interest for this paper is that where $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{L}$. In this case we see immediately that strong coherence amounts to having $\mathcal{R}_1 = \mathcal{R}_2$, taking into account that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be coherent sets.[28] This equality can be represented through the pair of inclusions $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\mathcal{R}_1 \supseteq \mathcal{R}_2$. If we look at the definition of strong coherence, we see that the former inclusion states that $\mathcal{R}_2$ must not be inconsistent with the logical implications of $\mathcal{R}_1$, and the latter, that $\mathcal{R}_1$ must not be inconsistent with the logical implications of $\mathcal{R}_2$. In other words, strong coherence can be regarded as a bidirectional requirement of coherence. We formalise the unidirectional requirement as follows:

*Definition* 26 (**One-way strong coherence**). Let $\mathcal{R}_1, \mathcal{R}_2$ be two coherent sets of desirable gambles. We say that $\mathcal{R}_2$ *strongly coheres* with $\mathcal{R}_1$ when $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

One-way strong coherence implies the coherence of the collection $\{(\mathcal{R}_1, \mathcal{L}), (\mathcal{R}_2, \mathcal{L})\}$, since that is equivalent to the coherence of $\mathcal{R}_1 \cup \mathcal{R}_2$.

The unidirectional requirement of strong coherence is an important notion for this paper. The underlying reason, which will become clear in the following sections, is that our temporal setup creates additional constraints with respect to the abstract representation above. These constraints are related to the existence of a temporal order of the models for which we might consider a requirement of strong coherence (such as your present and future commitments); and this order may be compatible only with a unidirectional consistency notion, as it is the case, for instance, of strong temporal consistency in Section 6.2. In these cases, one-way strong coherence is the strongest consistency requirement that is possible to consider. On the other hand, there are cases where the question of the order is less constraining, such as in the case of strong temporal coherence in Section 6.4, and then it is possible to apply also bidirectional strong coherence.

[28] It can be checked that this holds as soon as $\mathcal{Q}_1 = \mathcal{Q}_2$; note, however, that this does not mean that the sets $\mathcal{Q}_1, \mathcal{Q}_2$ uniquely determine $\mathcal{R}_1, \mathcal{R}_2$.

## 6. Temporal consistency notions for coherent sets of gambles

Assume now that you assess your current and future commitments directly in terms of sets of desirable gambles. Let us consider thus that your current beliefs are a coherent set $\mathcal{R}$ of desirable gambles. Your future commitments, that become effective after a certain $B \in \mathcal{B}$ occurs, are instead represented by means of a coherent set $\mathcal{R}_B$ of desirable gambles with respect to $\mathcal{L}(B)$, or, equivalently, by means of the set $\mathcal{R}^B \subseteq \mathcal{L}$ it determines through $\mathcal{R}^B := \{Bf : f \in \mathcal{R}_B\}$. Note that $\mathcal{R}^B$ is coherent relative to the set $\{f \in \mathcal{L}(\Omega) : f = Bf\}$. In case you establish your future commitments for all $B \in \mathcal{B}$, then we combine all the sets $\mathcal{R}^B$ into their conglomerable natural extension $\mathcal{F}^{\mathcal{B}}$, as in Proposition 1.

In this section, we shall investigate how the different consistency notions from Section 4 can be established when your assessments are modelled by means of the above sets.

### 6.1. Temporal consistency

In this first case the relevant sets are $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$. The following definition is a straightforward extension to sets of desirable gambles of what we have already studied in Section 4:

*Definition* 27 (**Temporal consistency for gambles**). We say that your current and future commitments are *temporally consistent* if $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss.

Once again, the rationale behind this definition is that if you failed temporal consistency, an opponent could create a combination of current and future transactions that will have the overall effect of making you desire, and then accept, a gamble $g \leq 0$. For example, assume that $f - \varepsilon$ belongs to $\mathcal{R}$ for some $\varepsilon > 0$, and that at the same time $-Bf$ belongs to $\mathcal{R}^B$ for all $B \in \mathcal{B}$. Then an opponent might decide now to sell you $f$ at price $\varepsilon$ immediately, and to ask you for $Bf$ after $B$ happens. This will earn him a gain of $\varepsilon$ *irrespective of the actual $B$ that will occur*: that is, he will make you incur a sure loss (in time). As a side remark, observe that when $\Omega$ is infinite, this specific example is prevented from happening under Definition 27 just because that definition is based on $\mathcal{F}^{\mathcal{B}}$: we could not achieve this by using finite combinations of elements from sets $\mathcal{R}^B$, $B \in \mathcal{B}$.

Let us give some insight about the definition we have just introduced.

**Proposition 8.** *Let $\mathcal{R}, \mathcal{R}_1$ be coherent sets of desirable gambles. Then*

(a) *$\mathcal{R} \cup \mathcal{R}_1$ avoids partial loss if and only if $f \in \mathcal{R}_1 \Rightarrow -f \notin \mathcal{R}$.*

(b) *As a consequence, the following conditions are equivalent:*

  (b.1) *Your current and future commitments are temporally consistent.*

  (b.2) *The following implication holds:*

$$f \in \mathcal{F}^{\mathcal{B}} \Rightarrow -f \notin \mathcal{R}. \tag{20}$$

  (b.3) *$\mathrm{posi}(\mathcal{R} \cup \mathcal{F}^{\mathcal{B}})$ is coherent.*

*Proof.*

(a) Let $\mathcal{R}' := \mathcal{R} \cup \mathcal{R}_1$, and call $\mathcal{E}_{\mathcal{R}'}$ its natural extension. Given that both $\mathcal{R}$ and $\mathcal{R}_1$ are coherent, we obtain that

$$\begin{aligned}\mathcal{E}_{\mathcal{R}'} = \mathrm{posi}(\mathcal{R}') &= \left\{\sum_{i=1}^{n} \lambda_i f_i \colon n \geq 1, \lambda_i > 0, f_i \in \mathcal{R}'\right\} \\ &= \{f + g \colon f \in \mathcal{R} \cup \{0\}, g \in \mathcal{R}_1 \cup \{0\}, f \neq 0 \text{ or } g \neq 0\}.\end{aligned}$$

Now, $\mathcal{R}'$ avoids partial loss if and only if $0 \notin \mathcal{E}_{\mathcal{R}'}$. Note that this last condition holds if and only if $f \in \mathcal{R}_1 \Rightarrow -f \notin \mathcal{R}$. The direct implication is trivial. For the converse, $0 \in \mathcal{E}_{\mathcal{R}'}$ requires that there are $f \in \mathcal{R}, g \in \mathcal{R}_1$ so that $f = -g$, because $0 \notin \mathcal{R}'$ since both $\mathcal{R}$ and $\mathcal{R}_1$ are coherent.

(b) This follows from (a) and [42, Proposition 3(d)]. □

This proposition provides an interpretation of temporal consistency: for this condition to hold, if you commit yourself to accept some gamble in the future, you should not accept the opposite gamble now. This is related to the ideas of Goldstein and van Fraassen we shall discuss in Section 7.3.[29]

*Remark* 4. Similarly to Remark 2, temporal consistency holds automatically when you make your sets of temporally consistent commitments more imprecise, in the sense that if you consider two sets of current beliefs $\mathcal{R}_1 \subseteq \mathcal{R}$ and two sets of future commitments $\mathcal{F}_1^{\mathcal{B}} \subseteq \mathcal{F}^{\mathcal{B}}$ and $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent, then trivially also $\mathcal{R}_1$ and $\mathcal{F}_1^{\mathcal{B}}$ are temporally consistent. The interpretation here is that, since temporal consistency means that the convex cone generated by your current and future commitments does not produce partial losses, if you become more cautious (that is, more imprecise) in your assessments and remove some gambles from your sets of current and future commitments you will obtain a smaller convex cone, which as a consequence will not produce partial losses either. ⧫

The set $\mathcal{R}$ of current beliefs induces a coherent lower prevision $\underline{P}$ by means of Eq. (19); similarly, for every $B \in \mathcal{B}$ the set $\mathcal{R}^B$ induces a lower prevision $\underline{P}_B$ by means of Eq. (19); as a consequence, the set $\mathcal{F}^{\mathcal{B}}$ induces a separately coherent future lower prevision $\underline{P}_{\mathcal{B}}$. The sets of strictly desirable gambles associated to these two lower previsions are included in $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$, respectively. Then it is easy to deduce from Theorem 1 the following:

**Theorem 4.** *Let us denote by $\underline{P}, \underline{P}_{\mathcal{B}}$ the lower previsions induced by $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ respectively. Then the following conditions are equivalent:*

(a) $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ *avoids sure loss.*

(b) $\mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathcal{F}^{\mathcal{B}}) \neq \emptyset$.

(c) $\underline{P}, \underline{P}_{\mathcal{B}}$ *are temporally consistent.*

*As a consequence, if $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent, so are the lower previsions $\underline{P}, \underline{P}_{\mathcal{B}}$ they induce.*

*Proof.* Let us make a circular proof.

(a) ⇒ (b) Assume that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids sure loss. This means that it is included in some set $\mathcal{D}$ of almost-desirable gambles, and as a consequence the lower prevision $\underline{Q}$ given by $\underline{Q}(f) := \sup\{\mu : f - \mu \in \mathcal{D}\}$ is coherent [68, Theorem 3.8.1]. Moreover, $\mathcal{D} = \{f : \underline{Q}(f) \geq 0\}$ whence, given $P \in \mathcal{M}(\underline{Q})$, it holds that $P(f) \geq 0$ for every $f \in \mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$, i.e., $P \in \mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathcal{F}^{\mathcal{B}})$.

(b) ⇒ (c) Consider $P \in \mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathcal{F}^{\mathcal{B}})$, and assume ex-absurdo that $\underline{P}, \underline{P}_{\mathcal{B}}$ do not avoid sure loss. Then there are gambles $f, g$ such that $\sup[G(f)+G_{\mathcal{B}}(g)] < 0$, whence there is some $\delta > 0$ such that $\sup[G(f)+G_{\mathcal{B}}(g)+\delta] < 0$. Since

$$G(f) + \frac{\delta}{2} = f - \left(\underline{P}(f) - \frac{\delta}{2}\right) \in \underline{\mathcal{R}} \subseteq \mathcal{R} \text{ and}$$

$$G_B(g) + B\frac{\delta}{2} = B\left(g - (\underline{P}_B(g) - \frac{\delta}{2})\right) \in \underline{\mathcal{R}}^B \subseteq \mathcal{R}^B \; \forall B \in \mathcal{B} \Rightarrow G_{\mathcal{B}}(g) + \frac{\delta}{2} \in \mathcal{F}^{\mathcal{B}},$$

we deduce that $P(G(f) + G_{\mathcal{B}}(g) + \delta) \geq 0$. But on the other hand the coherence of $P$ implies that $P(G(f) + G_{\mathcal{B}}(g) + \delta) \leq \sup[G(f) + G_{\mathcal{B}}(g) + \delta] < 0$, a contradiction. Hence, $\underline{P}, \underline{P}_{\mathcal{B}}$ avoid sure loss.

(c) ⇒ (a) If $\underline{P}, \underline{P}_{\mathcal{B}}$ are temporally consistent then it follows from Theorem 1 that given their associated sets of strictly desirable gambles $\mathcal{R}_1, \mathcal{R}_1^B, B \in \mathcal{B}$, the union $\mathcal{R}_1 \cup \mathcal{F}_1^{\mathcal{B}}$ avoids partial loss. As a consequence, $\mathcal{R}_1 \cup \mathcal{F}_1^{\mathcal{B}}$ avoids sure loss, which means that it is included in a set of almost desirable gambles $\mathcal{D}$. But this set $\mathcal{D}$ must include the union $\overline{\mathcal{R}}_1 \cup \overline{\mathcal{F}}_1^{\mathcal{B}}$ of the closures of $\mathcal{R}_1, \mathcal{F}_1^{\mathcal{B}}$, and as a consequence it also includes $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$. It follows that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids sure loss.

[29] Interestingly, the first point in the proposition relates also to propositional logic, where it is said that a set of sentences is consistent when it is not the case that a proposition and its negation both belong to that set.

For the second part it suffices to use that $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ must include the sets of strictly desirable gambles induced by $\underline{P}, \underline{P}_{\mathcal{B}}$, and as a consequence if $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent so are these sets of strictly desirable gambles. The result follows then from Theorem 1. $\square$

Let us show that the implication in the second part of this theorem is not an equivalence:

*Example* 14. Let $\Omega := \{1,2,3,4\}, B := \{1,2\}$ and $\mathcal{B} := \{B, B^c\}$. Let your current beliefs be given by

$$\mathcal{R} := \{f \in \mathcal{L}(\Omega) : (f(1)+f(2)+f(3)+f(4) > 0) \text{ or } (f(1)+f(2)+f(3)+f(4) = 0, f(1) > 0)\}$$

and let your future commitments be given by

$$\mathcal{R}^B := \{f \in \mathcal{L}(\Omega) : f = Bf, f(1)+f(2) > 0 \text{ or } f(1) = -f(2) < 0\},$$
$$\mathcal{R}^{B^c} := \{f \in \mathcal{L}(\Omega) : f = B^c f, f(3)+f(4) > 0 \text{ or } f(3) = -f(4) < 0\}.$$

It can be checked that all the above sets of (present and future) commitments are coherent. Then since the partition $\mathcal{B}$ is finite we have $\mathcal{F}^{\mathcal{B}} = \mathcal{E}^{\mathcal{B}}$ and this set is equal to

$$\{f \in \mathcal{L}(\Omega) : (f(1)+f(2) > 0 \text{ or } f(1) = -f(2) \leq 0) \text{ and } (f(3)+f(4) > 0 \text{ or } f(3) = -f(4) \leq 0)\} \setminus \{0\}.$$

Moreover, $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ does not avoid partial loss, because the gamble $(1,-1,1,-1)$ belongs to $\mathcal{R}$ and its opposite $(-1,1,-1,1)$ belongs to $\mathcal{F}^{\mathcal{B}}$, meaning that their sum 0 belongs to $\operatorname{posi}(\mathcal{R} \cup \mathcal{F}^{\mathcal{B}})$.

On the other hand, $\mathcal{R}$ induces the linear prevision $P$ given by

$$P(f) := \frac{f(1)+f(2)+f(3)+f(4)}{4} \quad \forall f \in \mathcal{L},$$

and $\mathcal{F}^{\mathcal{B}}$ induces the conditional linear prevision $P_{\mathcal{B}}$ given by

$$P_B(f) := \frac{f(1)+f(2)}{2}, \; P_{B^c}(f) := \frac{f(3)+f(4)}{2};$$

$P, P_{\mathcal{B}}$ satisfy GBR because $P_{\mathcal{B}}$ is derived from $P$ by means of Bayes' rule, and, taking into account that $\mathcal{B}$ is finite, also CNG. Hence, $P, P_{\mathcal{B}}$ are coherent, and as a consequence also temporally consistent. ⧫

In particular, we deduce from this example and Theorem 1 that we may have temporally inconsistent $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ while the lower previsions $\underline{P}, \underline{P}_{\mathcal{B}}$ they induce satisfy any of the equivalent conditions from that result, and in particular where their associated sets of strictly desirable gambles are temporally consistent.

When we consider the sets of strictly desirable gambles induced by your current and future lower previsions, then Theorem 1 gives us also the opportunity to discuss an interesting side point concerned with the definition of $\mathcal{F}^{\mathcal{B}}$. Remember that we have allowed each $Bf$ in Eq. (9) to possibly equal zero (as we have argued right after that equation). Now we can show that, as long as we focus on coherent lower previsions (and only then), it is immaterial whether we allow each $Bf$ to equal zero or not:

**Proposition 9.** *Let $\mathcal{R}, \mathcal{R}^B$ be the sets of strictly desirable gambles induced by $\underline{P}, \underline{P}_B, B \in \mathcal{B}$, and let us define $\mathcal{F} := \{f : Bf \in \mathcal{R}^B \; \forall B \in \mathcal{B}\} \subseteq \mathcal{F}^{\mathcal{B}}$. Then $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss $\Leftrightarrow$ $\mathcal{R} \cup \mathcal{F}$ avoids partial loss.*

*Proof.* The direct implication follows trivially from the inclusion $\mathcal{F} \subseteq \mathcal{F}^{\mathcal{B}}$. To see the converse, note that the conditional lower previsions induced by $\mathcal{F}$ and $\mathcal{F}^{\mathcal{B}}$ coincide, because of the one-to-one correspondence between coherent lower previsions and sets of strictly desirable gambles in Eqs. (16) and (4). Let us prove that if $\mathcal{R} \cup \mathcal{F}$ avoids partial loss then the lower previsions $\underline{P}, \underline{P}_{\mathcal{B}}$ they induce avoid sure loss; the result will follow then from Theorem 1. Assume ex-absurdo that $\underline{P}, \underline{P}_{\mathcal{B}}$ incur a sure loss. Then there are gambles $f, g$ such that $\sup[G(f) + G_{\mathcal{B}}(g)] < 0$, whence there is some $\delta > 0$ such that $G(f) + G_{\mathcal{B}}(g) + \delta < 0$. Since

$$G(f) + \frac{\delta}{2} = f - \left(\underline{P}(f) - \frac{\delta}{2}\right) \in \mathcal{R} \text{ and}$$
$$G_B(g) + B\frac{\delta}{2} = B\left(g - (\underline{P}_B(g) - \frac{\delta}{2})\right) \in \mathcal{R}^B \; \forall B \in \mathcal{B} \Rightarrow G_{\mathcal{B}}(g) + \frac{\delta}{2} \in \mathcal{F},$$

we deduce that the gamble $G(f) + G_{\mathcal{B}}(g) + \delta < 0$ belongs to $\text{posi}(\mathcal{R} \cup \mathcal{F})$, and as a consequence this set incurs a partial loss. This is a contradiction. As a consequence, $\underline{P}, \underline{P}_{\mathcal{B}}$ avoid sure loss and therefore $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss. □

On the other hand, we can also show that the equivalence established in Proposition 8 is only an implication when the sets of gambles are not strictly desirable:

*Example* 15. Let us consider the space and sets of Example 14, but considering instead $\mathcal{R}^{B^c} := \{f \in \mathcal{L}(\Omega) : f = B^c f, f(3) + f(4) > 0\}$. Then $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ incurs partial loss, because $f := (1, -1, 0, 0) \in \mathcal{R}$ and $-f \in \mathcal{F}^{\mathcal{B}}$. However, for every $g \in \mathcal{F}$ it holds that $g(1) + g(2) + g(3) + g(4) \geq g(3) + g(4) > 0$, whence $\mathcal{F} \subseteq \mathcal{R}$ and as a consequence $\mathcal{R} \cup \mathcal{F}$ is coherent and in particular avoids partial loss. ♦

We turn now to the problem of correcting temporally inconsistent assessments. As we discussed in Section 4.1.1, when your current and future commitments are temporally inconsistent it may be interesting to determine the closest model that satisfies temporal consistency.

Taking into account the negative result we obtained for the case of lower previsions (see Proposition 2), we shall take here a different route: we shall start from temporally inconsistent $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$, but shall study instead which is the greatest coherent subset of $\mathcal{F}^{\mathcal{B}}$ that is temporally consistent with $\mathcal{R}$. Taking into account Proposition 7, a set is temporally consistent with $\mathcal{R}$ if and only if it is included in $(-\mathcal{R})^c$. Hence, we shall look for the greatest coherent subset of $\mathcal{R}_1 := (-\mathcal{R})^c \cap \mathcal{F}^{\mathcal{B}}$, where 'greatest' means that every coherent subset of $\mathcal{R}_1$ is included in it.

**Proposition 10.** (a) *The greatest coherent subset of $\mathcal{R}_1$, if it exists, is always equal to $\mathcal{R}_1$.*

(b) *If $\mathcal{R}$ is a maximal set of gambles, then $\mathcal{R}_1$ is coherent.*

*Proof.* (a) It suffices to show that for every gamble $f$ in $\mathcal{R}_1$ there is a coherent subset of $\mathcal{R}_1$ that includes $f$. To see this, note that the natural extension of $\{f\}$ is given by

$$\mathcal{E}_{\{f\}} := \{g = \lambda f + h : \lambda \geq 0, h \geq 0\} \setminus \{0\}.$$

This is included in $\mathcal{F}^{\mathcal{B}}$ because the latter is a coherent set of gambles that includes $f$. To see that it is also included in $(-\mathcal{R})^c$, assume that there is some $\lambda \geq 0$ and some $h \geq 0$ such that $\lambda f + h$ is non-zero and does not belong to $(-\mathcal{R})^c$. This means that $\lambda f + h \in -\mathcal{R}$, or, equivalently, that $-\lambda f - h \in \mathcal{R}$. Since this is a coherent set, we deduce that also $-\lambda f \in \mathcal{R}$, and consequently so does $-f$. But we are assuming that $f \in \mathcal{R}_1 \subseteq (-\mathcal{R})^c$, whence $-f \notin \mathcal{R}$, a contradiction.

(b) If $\mathcal{R}$ is a maximal set of gambles,

$$(-\mathcal{R})^c = \{f : f \notin -\mathcal{R}\} = \{f : -f \notin \mathcal{R}\} = \mathcal{R} \cup \{0\},$$

whence $\mathcal{R}_1 = \mathcal{R} \cap \mathcal{F}^{\mathcal{B}}$ is coherent because it is the intersection of two coherent sets. □

Statement (b) of this proposition cannot be extended to the case where $\mathcal{R}$ is not maximal:

*Example* 16. Consider $\Omega := \{1, 2, 3, 4\}$, and let your current beliefs be given by

$$\mathcal{R} := \mathcal{L}^+ \cup \{f : \min\{f(1), f(3)\} > 0\}.$$

Let your future commitments be the same as in Example 14. Then $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$ are not temporally consistent, because for instance

$$f := (1, -2, 1, -2) \in \mathcal{R} \text{ and } -f := (-1, 2, -1, 2) \in \mathcal{F}^{\mathcal{B}};$$

to see that in this case $\mathcal{R}_1 = (-\mathcal{R})^c \cap \mathcal{F}^{\mathcal{B}}$ is not coherent, consider the gambles $h_1 := (1, 0, -2, 3)$ and $h_2 := (-2, 3, 1, 0)$. Then $h_1, h_2 \in \mathcal{F}^{\mathcal{B}}$, whence $h_1 + h_2 \in \mathcal{F}^{\mathcal{B}}$. On the other hand, $h_1 \in (-\mathcal{R})^c$ because $-h_1 = (-1, 0, 2, -3) \notin \mathcal{R}$, and $h_2 \in (-\mathcal{R})^c$ because $-h_2 = (2, -3, -1, 0) \notin \mathcal{R}$. However, $h_1 + h_2 = (-1, 3, -1, 3) \notin (-\mathcal{R})^c$ because $-h_1 - h_2 \in \mathcal{R}$. Hence, $h_1, h_2 \in \mathcal{R}_1$ but $h_1 + h_2 \notin \mathcal{R}_1$, and therefore this set is not coherent. ♦

This means that, although there may not be an optimal way of correcting inconsistent assessments when you modify both your sets of commitments at the same time (see Proposition 2), such an optimal correction may be possible when you fix your set of current beliefs and modify only the set of future commitments, i.e., when you look for the greatest set $\mathcal{F}_1^{\mathcal{B}}$ of future commitments such that $\mathcal{R} \cup \mathcal{F}_1^{\mathcal{B}}$ avoids partial loss. This second scenario is actually something that well fits the current setup where future commitments are defined after current beliefs.

In a similar vein as Proposition 9, we can in addition establish the following:

**Corollary 4.** *Assume your current beliefs are determined by a linear prevision $P$, and that they are temporally inconsistent with the separately coherent future lower prevision $\underline{P}_{\mathcal{B}}$. Then if $P(B) > 0$ for every $B \in \mathcal{B}$ there is a greatest lower prevision $\underline{P}_1^{\mathcal{B}} \leq \underline{P}_{\mathcal{B}}$ that is temporally consistent with respect to $P$.*

*Proof.* If $P(B) > 0$ for every $B$, then it follows from Corollary 3 that the only linear conditional prevision that satisfies temporal consistency with $P$ is the $E(\cdot|\mathcal{B})$ determined from $P$ by means of Bayes' rule. Then Proposition 5 implies that any $\underline{P}'_{\mathcal{B}}$ that is temporally consistent with $P$ must be dominated by $E(\cdot|\mathcal{B})$. Hence, the greatest $\underline{P}_1^{\mathcal{B}}$ that is dominated by $\underline{P}_{\mathcal{B}}$ and satisfies temporal consistency with $P$ is simply $\underline{P}_1^{\mathcal{B}} := \min\{\underline{P}_{\mathcal{B}}, E(\cdot|\mathcal{B})\}$ (taking into account also Remark 2). □

Hence, an optimal correction is always possible when your current beliefs are precise and all the conditioning events have positive probability. More generally, it is not clear whether we can do an optimal correction whenever your current beliefs are precise, because the implication between the temporal consistency of gambles and that of the previsions they induce is not an equivalence, as we showed in Example 14.

### *6.2. Strong temporal consistency*

If you express your assessments by means of sets of desirable gambles, the idea of strong temporal consistency means that (i) $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss; and (ii) the future commitments $\mathcal{F}^{\mathcal{B}}$ strongly cohere (cf. Definition 26) with the conditional information embedded in $\mathcal{R}$, and which we can represent by the conglomerable natural extension of $\cup_{B \in \mathcal{B}} \mathcal{R}^{|B}$:

$$\mathcal{F}^{|\mathcal{B}} := \{f \in \mathcal{L} : Bf \in \mathcal{R}^{|B} \cup \{0\} \ \forall B \in \mathcal{B}\} \setminus \{0\},$$

where the sets $\mathcal{R}^{|B}$ are derived from $\mathcal{R}$ by means of Eq. (7).

*Definition* 28 (**Strong temporal consistency for gambles**). Your current and future commitments are said to be *strongly temporally consistent* if they are temporally consistent and $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$.

Remember that at the time when $\mathcal{F}^{\mathcal{B}}$ is established, in the present setting there is no possibility to revise $\mathcal{R}$, given that it represents commitments that are effective at that time. Therefore it is only possible for you to make future commitments not inconsistent with present beliefs (and not vice versa). We express this using one-way strong coherence through $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{F}^{\mathcal{B}}$. The equality that we have in Definition 28 follows then by focusing on the least-committal future model that strongly coheres with $\mathcal{F}^{|\mathcal{B}}$. If we instead do not focus on such a least-committal model, we obtain a weaker consistency condition:

*Definition* 29 (**Strong backward temporal consistency for gambles**). We say that your current and future commitments are *strongly backward temporally consistent* if they are temporally consistent and $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{F}^{\mathcal{B}}$.

The latter could in particular be a more realistic consistency condition than the former as it seems unreasonable to expect that future commitments always match conditional beliefs: why should this be the case given that you have additional time to refine your conditional beliefs into future commitments? It may instead be more reasonable to expect that in a number of cases future commitments are more precise than conditional beliefs, as in Definition 29, just because of the extra time you can devote to assess them.

*Remark* 5. One might wonder whether the rationale behind Definitions 28–29 should rather be applied to $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$, leading to $\mathcal{R} = \mathcal{F}^{\mathcal{B}}$ (or $\mathcal{R} \subseteq \mathcal{F}^{\mathcal{B}}$). This approach is not viable in the temporal setup we are dealing with again for the reason that present beliefs cannot be modified at the time of establishing $\mathcal{F}^{\mathcal{B}}$: in fact, considered the special structure of $\mathcal{F}^{\mathcal{B}}$, the only possibility to have $\mathcal{R} \subseteq \mathcal{F}^{\mathcal{B}}$ would be that $\mathcal{R}$ is also defined similarly through sums of gambles defined piece-wise on different elements of the partition $\mathcal{B}$. But this would mean that $\mathcal{R}$ is built using only conditional information so that for every $f \in \mathcal{R}$ and every $B \in \mathcal{B}$ it holds that $Bf \in \mathcal{R} \cup \{0\}$, and as a consequence $\mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}}$.

This is clearly not going to be the case in general, and therefore this approach is not applicable. Notice, however, that in case $\mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}}$, then Definition 29 represents indeed a strong coherence requirement between $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$, so that the fact that the previous definitions focus on $\mathcal{F}^{|\mathcal{B}}$ is not restrictive. ♦

In the remainder of this section, we explore some of the implications of strong temporal consistency. Let us start by giving an equivalent formulation of this notion:

**Proposition 11.** *Assume that $\mathcal{R}^B = \mathcal{R}^{|B}$ for all $B \in \mathcal{B}$, and consider $f \in \mathcal{L}$. Then* (20) *can be rewritten equivalently as follows:*

$$Bf \in \mathcal{R} \cup \{0\}\ \forall B \in \mathcal{B} \Rightarrow -f \notin \mathcal{R}. \tag{21}$$

*If in addition $\mathcal{R}$ is maximal, then strong temporal consistency is equivalent to the conglomerability of $\mathcal{R}$.*

*Proof.* Let us show that (20) implies (21). We skip the trivial case $f = 0$. For a $B \in \mathcal{B}$ s.t. $Bf \in \mathcal{R}$, we have that $Bf \in \mathcal{R}^{|B} = \mathcal{R}^B$. Applying the definition of $\mathcal{F}^{\mathcal{B}}$ in (9), we obtain that $f = \sum_{B\in\mathcal{B}\colon Bf\neq 0} Bf \in \mathcal{F}^{\mathcal{B}}$ and hence $-f \notin \mathcal{R}$, using (20). We consider now the converse implication. Take $f \in \mathcal{F}^{\mathcal{B}}$. Then $f = \sum_{B\in\mathcal{B}} Bf$, with $Bf \in \mathcal{R}^B \cup \{0\} = \mathcal{R}^{|B} \cup \{0\}$, whence $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$. Applying (21), we see that $-f \notin \mathcal{R}$.

For the second part, note that Eq. (21) together with the maximality of $\mathcal{R}$ imply that $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$, i.e., that $\mathcal{R}$ is conglomerable; and conversely, if $\mathcal{R}$ is conglomerable, then $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$ implies that $f \in \mathcal{R} \cup \{0\}$, whence by coherence $-f \notin \mathcal{R}$ and Eq. (21) holds. □

Expression (21) allows us also to clarify an important point:

**Proposition 12.** *Your set $\mathcal{R}$ of current beliefs fails* (21) *if and only if every coherent set $\mathcal{R}' \supseteq \mathcal{R}$ fails* (21).

*Proof.* For the direct implication, consider $-f \in \mathcal{R}$ s.t. $Bf \in \mathcal{R} \cup \{0\}\ \forall B \in \mathcal{B}$. Assume ex-absurdo that there is a coherent set $\mathcal{R}' \supseteq \mathcal{R}$ that satisfies (21). Then $Bf \in \mathcal{R}' \cup \{0\}\ \forall B \in \mathcal{B}$, so that $-f \notin \mathcal{R}' \supseteq \mathcal{R}$. This is a contradiction. The converse implication is trivial. □

In other words, the only possible coherent extension of $\mathcal{R}$ that satisfies (21) is $\mathcal{R}$ itself. This means that if $\mathcal{R}$ fails (21), there is no such an extension; and on the other hand, if it satisfies (21), we do not need to compute the extension, as we have it already. This should be compared with the conglomerable natural extension of $\mathcal{R}$, which may be different from $\mathcal{R}$, and can in particular be difficult to compute, as illustrated in [44]. The situation is analogous if we work with lower previsions rather than sets of desirable gambles. This property can in fact be regarded as a particular case of Remark 4, when you assume that your future commitments coincide with your conditional beliefs.

### 6.2.1. MET-beliefs

We next consider the important special case in which your current beliefs have been constructed by the *marginal extension theorem* (see [41, 68], and especially [44, Proposition 29]). We call them *MET-beliefs*. This corresponds to a situation of hierarchical information, where in addition to the conditional information on each element of the partition we have unconditional information expressed in terms of a set of desirable gambles that are constant on the different elements of the partition (recall that such gambles are called $\mathcal{B}$-measurable). In order to aggregate these pieces of information into a joint model, we consider two avenues.

Let $\mathcal{R}_0$ be a set coherent relative to the set of $\mathcal{B}$-measurable gambles (and hence it is a set of $\mathcal{B}$-measurable gambles itself) and for every $B \in \mathcal{B}$ let $\mathcal{R}_{|B}$ be a coherent set of desirable gambles with respect to $\mathcal{L}(B)$. Then the conglomerable natural extension of $\mathcal{R}_0 \cup \bigcup_{B\in\mathcal{B}} \mathcal{R}^{|B}$ is given by

$$\mathcal{R} := \left\{ h + \sum_{B\in\mathcal{B}} Bg_B \colon h \in \mathcal{R}_0 \cup \{0\}, g_B \in \mathcal{R}_{|B} \cup \{0\} \right\} \setminus \{0\}. \tag{22}$$

This set can be expressed equivalently as

$$\mathcal{R} = \left\{ h + g \colon h \in \mathcal{R}_0 \cup \{0\}, g \in \mathcal{F}^{|\mathcal{B}} \cup \{0\} \right\} \setminus \{0\}.$$

Moreover, it holds that $\mathcal{R}^{|B}$ is the conditional set of gambles derived from $\mathcal{R}$ by Eq. (7): consider a gamble $f \in \mathcal{L}$ such that $f = Bf \in \mathcal{R}$. Then we can write $Bf = h_1 + h_2$, for some $h_1 \in \mathcal{R}_0 \cup \{0\}, h_2 \in \mathcal{F}^{|\mathcal{B}} \cup \{0\}$, $h_1 + h_2 \neq 0$. We skip the trivial case made of $Bh_1 = 0$. Since the gamble $h_1$ is $\mathcal{B}$-measurable, $Bh_1$ is constant on some real number. If this constant is negative, then there must be some $B' \neq B$ such that $B'h_1 \gneq 0$, or we should contradict the coherence of $\mathcal{R}_0$. But in that case since $B'(h_1 + h_2) = 0$ we should have $B'h_2 \lneq 0$, a contradiction with the definition of $\mathcal{F}^{|\mathcal{B}}$. As a consequence, we must have $Bh_1 \geq 0$. Whence, if $Bh_2 \in \mathcal{R}^{|B}$, then $f \in \mathcal{R}^{|B}$ because $f = Bf \geq Bh_2 \in \mathcal{R}^{|B}$; otherwise, if $Bh_2 = 0$, then $f$ is equal to the positive constant $Bh_1$, so that $f \in \mathcal{R}^{|B}$.

In the second case, we consider the natural extension of $\mathcal{R}_0 \cup \bigcup_{B\in\mathcal{B}} \mathcal{R}^{|B}$. In that case, your beliefs are not necessarily[30] conglomerable as we consider finite sums only. By considering the natural extension $\mathcal{E}^{|\mathcal{B}}$ of $\cup_{B\in\mathcal{B}}\mathcal{R}^{|B}$ in (11), then the natural extension of $\mathcal{R}_0 \cup \bigcup_{B\in\mathcal{B}} \mathcal{R}^{|B}$ is given by:

$$\mathcal{R} \coloneqq \Big\{h + g\colon h \in \mathcal{R}_0 \cup \{0\}, g \in \mathcal{E}^{|\mathcal{B}} \cup \{0\}\Big\} \setminus \{0\}. \tag{23}$$

Note that $\mathcal{E}^{|\mathcal{B}}$ is a subset of $\mathcal{F}^{|\mathcal{B}}$, and as a consequence this set is included in that defined by (22).

Reasoning as before, we deduce that this set also induces the conditional assessments $\mathcal{R}^{|B}, B \in \mathcal{B}$: it is trivial that $\mathcal{R}^{|B} \subseteq \{f \in \mathcal{L}\colon f = Bf \in \mathcal{R}\}$; to see the converse inclusion, note that the set of conditional assessments induced by (23) must be included in that induced by (22), which we have showed to be given by $\mathcal{R}^{|B}, B \in \mathcal{B}$.

In summary, when your current beliefs are constructed by means of the marginal extension theorem, we can also take into account conglomerability (and then we end up with the set $\mathcal{R}$ in Eq. (22)) or not (and then we end up with the set $\mathcal{R}$ in (23)). Let us show that $\mathcal{R}$ satisfies strong temporal consistency in each of these two cases.

**Proposition 13.** *Let $\mathcal{R}$ be defined either by* (22) *or by* (23)*, and consider $f \in \mathcal{L}$. Then*

$$Bf \notin \mathcal{R}\ \forall B \in \mathcal{B} \Rightarrow f \notin \mathcal{R}.$$

*As a consequence, in either case $\mathcal{R}$ satisfies strong temporal consistency with its conditional beliefs.*

*Proof.* We focus on the case that $\mathcal{R}$ is defined by (22); the remaining case is analogous. Assume ex-absurdo that there is some $f \in \mathcal{R}$ such that $Bf \notin \mathcal{R}$ for every $B \in \mathcal{B}$. Then, $f = h + \sum_{B\in\mathcal{B}} Bg_B$ for some $h \in \mathcal{R}_0 \cup \{0\}$ and $g_B \in \mathcal{R}_{|B} \cup \{0\}$, $B \in \mathcal{B}$. Consider any element $B \in \mathcal{B}$; we see that $Bf = Bh_B + Bg_B$, where $h_B$ is a constant and $g_B \in \mathcal{R}_{|B} \cup \{0\}$.

Now, for any $B \in \mathcal{B}$ it cannot be $h_B > 0$, because since $Bg_B \in \mathcal{R} \cup \{0\}$ we should deduce that $Bf = Bh_B + Bg_B \in \mathcal{R}$, a contradiction. As a consequence, it must be $h \leq 0$. On the one hand, it cannot be $h \lneq 0$ or we contradict $h \in \mathcal{R}_0$. On the other hand, $h = 0$ means that $f = \sum_{B : Bf \neq 0} Bg_B$, whence $Bf = Bg_B$ for every $B$, and since $Bf \notin \mathcal{R}$ by hypothesis we deduce that $Bg_B = 0$ for every $B$. This means that $f = 0 \notin \mathcal{R}$.

Let us move now to the second part. Take $f \in \mathcal{L}$ such that $Bf \in \mathcal{R} \cup \{0\}$ for all $B \in \mathcal{B}$. Then $B(-f) \notin \mathcal{R}$ for all $B \in \mathcal{B}$. The thesis follows applying the first part of the result and Proposition 10. $\square$

This means that in the case of MET-beliefs, strong temporal consistency is automatically secured if future commitments equal conditional beliefs, irrespective of whether current beliefs are conglomerable or not. This last case seems to be particularly interesting. To see this more clearly, recall that we have showed that the set $\mathcal{R}$ induces the conditional assessments $\mathcal{R}^{|B}$ for $B \in \mathcal{B}$ irrespective of whether it is defined by Eqs. (22) or (23). If $\mathcal{R}$ is defined by Eq. (22) we deduce that $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$, and as a consequence $\mathcal{R}$ is conglomerable, which implies in particular that it satisfies (21). When $\mathcal{R}$ is defined by (23), since it determines the same conditional assessments, we deduce that it also satisfies (21): if it did not, we should have some gamble $f \in \mathcal{F}^{|\mathcal{B}}$ such that $-f \in \mathcal{R}$, and this would contradict the conglomerability of the coherent set of gambles defined by (22).

Note also that this does not contradict Proposition 10, because if $\mathcal{R}$ is defined by (23) and it is not conglomerable, then it cannot be maximal, because it is strictly included in the conglomerable coherent set defined by (22).

Proposition 12 gives us also an opportunity to simplify the initial notion of (i.e., non-strong) temporal consistency in case of MET-beliefs:

[30] They are conglomerable if $\mathcal{B}$ is finite, as in this case the sets in (22) and (23) coincide.

**Proposition 14.** *If your current beliefs are given by* (22)*, then temporal consistency holds if and only if*

$$f \in \mathcal{F}^{\mathcal{B}} \Rightarrow -f \notin \mathcal{F}^{|\mathcal{B}}. \tag{24}$$

*In case they are given by* (23) *and temporal consistency holds, then condition* (24) *holds.*

*Proof.* Let us show that under (22) temporal consistency is implied by (24). Take $f \in \mathcal{F}^{\mathcal{B}}$. Then $f = \sum_{B\in\mathcal{B}:\, Bf\neq 0} Bf$, and for every $B$ such that $Bf \neq 0$ it holds that $Bf \in \mathcal{R}^{B} \subseteq \mathcal{F}^{\mathcal{B}}$. We have by (24) that $B(-f) \notin \mathcal{F}^{|\mathcal{B}}$, whence $-Bf \notin \mathcal{R}^{|B}$ and as a consequence $B(-f) \notin \mathcal{R}$ for all $B$. Proposition 12 then establishes the necessity of (20). Now, observing that the set in (23) is included in the set in (22), we obtain that necessity holds also under (23). Finally, that temporal consistency implies (24) is trivial under (22) because in that case $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$. □

The characterisation of temporal consistency established in Proposition 13 does not hold when your current beliefs are not constructed by marginal extension:

*Example* 17. Consider $\Omega := \{1,2,3,4\}, B := \{1,2\}, \mathcal{B} := \{B, B^c\}$ and let us consider the linear previsions $P_1, P_2$ on $\mathcal{L}$ determined by

$$P_1(\{1\}) := 0.75, P_1(\{3\}) := 0.25,\ P_2(\{2\}) := 0.25, P_2(\{4\}) := 0.75.$$

Let $\underline{P} := \min\{P_1, P_2\}$ and define your future commitments for any gamble $f$ by

$$P_B(f) := 0.5P_1(f|B) + 0.5P_2(f|B) = 0.5f(1) + 0.5f(2),$$
$$P_{B^c}(f) := 0.5P_1(f|B^c) + 0.5P_2(f|B^c) = 0.5f(3) + 0.5f(4).$$

Let $\mathcal{R}$ be the set of strictly desirable gambles associated to $\underline{P}$, and $\mathcal{R}_B, \mathcal{R}_{B^c}$ the sets of strictly desirable gambles associated to $P_B, P_{B^c}$. Note that by definition $\mathcal{F}^{|\mathcal{B}}$ is included in $\mathcal{R}$, because the partition $\mathcal{B}$ is finite, so in this case temporal consistency implies Eq. (24). Let us show that they are not equivalent.

Given the gamble $g := 2\mathbb{I}_{\{2,3\}} - \mathbb{I}_{\{1,4\}}$ it holds that

$$P_B(g) = -0.5 + 2\cdot 0.5 = 0.5 > 0,$$
$$P_{B^c}(g) = 0.5\cdot 2 - 0.5 = 0.5 > 0,$$

whence $g \in \mathcal{F}^{\mathcal{B}}$; however, $P_1(-g) = P_2(-g) = -2\cdot 0.25 + 1\cdot 0.75 = 0.25$, whence

$$\underline{P}(-g) = \min\{0.25, 0.25\} = 0.25 > 0 \Rightarrow -g \in \mathcal{R}.$$

Thus, $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$ are not temporally consistent. To see that Eq. (24) holds, note that if a gamble $f$ belongs to $\mathcal{F}^{\mathcal{B}}$ then either $f_B$ or $f_{B^c}$ (or both) is non-zero. Assume for instance that $f_B$ is non-zero; the remaining cases are established similarly. It holds that $f_B \in \mathcal{R}_B$, whence:

- either $f_B \gneq 0$, and then $-f_B \lneq 0 \notin \mathcal{R}_{|B}$;
- or $f_B \not\geq 0$, and then it must be $P_B(f) = 0.5f(1) + 0.5f(2) > 0$; if $-f_B \in \mathcal{R}_{|B}$, then $\underline{P}(-Bf_B) = \min\{-0.75f(1), -0.25f(2)\} \geq 0$, which means that $\max\{f(1), f(2)\} \leq 0$ and which contradicts $f(1) + f(2) > 0$. We conclude that in this case $-f_B \notin \mathcal{R}_{|B}$ either.

Thus, when $\mathcal{R}$ is not established by marginal extension Eq. (24) is not equivalent to temporal consistency. ♦

It is possible to simplify further the expression of temporal consistency under MET-beliefs:

**Proposition 15.** *Eq.* (24) *holds if and only if* $\mathcal{R}_{|B} \cup \mathcal{R}_B$ *avoids partial loss for all* $B \in \mathcal{B}$.

*Proof.* Let us focus on the direct implication. Take $B \in \mathcal{B}$. Since both $\mathcal{R}_{|B}$ and $\mathcal{R}_B$ are coherent with respect to $\mathcal{L}(B)$, we can apply Proposition 7 to conclude that $\mathcal{R}_{|B} \cup \mathcal{R}_B$ avoids partial loss if and only if $g_B \in \mathcal{R}_B$ implies $-g_B \notin \mathcal{R}_{|B}$. Then consider $g_B \in \mathcal{R}_B$, so that $Bg_B \in \mathcal{F}^{\mathcal{B}}$. By (24), $B(-g_B) \notin \mathcal{F}^{|\mathcal{B}}$, whence $-g_B \notin \mathcal{R}_{|B}$. Consider the converse implication. Take $f \in \mathcal{F}^{\mathcal{B}}$, so that $f = \sum_{B\in\mathcal{B}:\, Bg_B\neq 0} Bg_B$, with $g_B \in \mathcal{R}_B$ for all $B$. The assumption implies that for all $B$, $-g_B \notin \mathcal{R}_{|B}$. This implies that $\sum_{B\in\mathcal{B}:\, Bg_B\neq 0} B(-g_B) = -f \notin \mathcal{F}^{|\mathcal{B}}$. □

### 6.3. *Event-wise temporal consistency*

If your assessments are expressed by means of coherent sets of gambles, we can also consider a notion of event-wise temporal consistency, whose rationale is the same as that behind temporal consistency:

*Definition* 30 (**Event-wise temporal consistency for gambles**). We say that your present and future commitments are *event-wise temporally consistent* if $\mathcal{R} \cup \mathcal{R}^B$ avoids partial loss.

As discussed in Section 4.3, this notion is interesting when your future commitments are established *after* you have observed the element $B$ of the partition $\mathcal{B}$. It follows immediately from Proposition 5 that if $\mathcal{R} \cup \mathcal{R}^B$ avoids sure loss then the lower previsions they induce satisfy the corresponding notion of event-wise temporal consistency we have introduced in Section 4.3. A reformulation of event-wise temporal consistency follows immediately in a way similar to the more general case of temporal consistency:

**Proposition 16.** *The following statements are equivalent:*

(a) *Your present and future commitments satisfy event-wise temporal consistency.*

(b) $\mathcal{R}^{\rceil B} \cup \mathcal{R}^B$ *avoid partial loss.*

(c) *Define* $\mathcal{R}^{B'} := \{f \in \mathcal{L}^+ : f = B'f\}$ *for all* $B' \neq B, B' \in \mathcal{B}$*, and let* $\mathcal{F}^{\mathcal{B}}$ *be given by* (9)*. Then* $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ *avoid partial loss.*

*Proof.* Since $\mathcal{R}^{\rceil B} \subseteq \mathcal{R}$ and $\mathcal{R}^B \subseteq \mathcal{F}^{\mathcal{B}}$, we deduce that (a) implies (b) and that (c) implies (a). Let us show then that (b) implies (c). Let $f \in \mathcal{F}^{\mathcal{B}}$. By definition of $\mathcal{F}^{\mathcal{B}}$, it holds that $Bf \in \mathcal{R}^B \cup \{0\}, B^c f \geq 0$. Assume ex-absurdo that $-f = -Bf - B^c f \in \mathcal{R}$. In this case $B^c f \neq 0$, because otherwise $-Bf = -f \in \mathcal{R}^{\rceil B}$, contradicting our assumptions. As a consequence, $B^c f \gneq 0$ so that $\mathcal{R}$ includes the gamble $(-Bf - B^c f) + B^c f = -Bf \in \mathcal{R}^{\rceil B}$, again contradicting the assumptions. $\square$

This proposition makes it clear that event-wise temporal consistency relates only to your beliefs conditional on $B$, as far as your current beliefs are concerned. In other words, we can think of event-wise temporal consistency as a weakening of temporal consistency that focuses only on the subset $\mathcal{R}^{\rceil B}$ of $\mathcal{R}$. In a similar way, one can also extend the notion of event-wise strong temporal consistency to sets of desirable gambles.

### 6.4. *Strong temporal coherence*

If your current and future commitments are established at the same time, then these assessments can affect each other. Moreover, just because the commitments are established at the same time, it would be irrational in particular that conditional and future commitments do not cohere with each other. Using strong coherence in Definition 25, we deduce that $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{\rceil \mathcal{B}}$ is a rationality requirement in the present setting.

For similar reasons, it would be also irrational that the full set of present beliefs $\mathcal{R}$ does not cohere with future commitments. To this end, the strongest coherence condition that we can apply is one-way strong coherence in the form $\mathcal{F}^{\rceil \mathcal{B}} \subseteq \mathcal{R}$. This inequality makes sense just because the two sets are established at the same time, so that future commitments can actually affect present beliefs. It is instead not reasonable to impose in addition the opposite inclusion unless it is already the case that you specify $\mathcal{R}$ only through conditional information (see Remark 5).

We can also see with a simple example why the condition $\mathcal{F}^{\rceil \mathcal{B}} \subseteq \mathcal{R}$ does intuitively make sense. Consider this case: assume that there is $g := \sum_{B \in \mathcal{B}} Bg$, with $Bg \in \mathcal{R}^{\rceil B} \cup \{0\}$ for all $B \in \mathcal{B}$, such that $g \in \mathcal{F}^{\rceil \mathcal{B}} \setminus \mathcal{R}$. As we have discussed in Section 3.2, $g$ represents an agreement about the future that you would accept now, and it implies that you will be rewarded by $g(\omega)$, whatever $\omega \in \Omega$ comes true. But when you know that accepting the agreement has the same implications for you of accepting $g$, then this should make you see that $g \notin \mathcal{R}$ is not compatible with your other assessments. This should lead you to resolve the incoherence, either by making $g$ belong to $\mathcal{R}$ or by removing $g$ from $\mathcal{F}^{\rceil \mathcal{B}}$. In other words, you should agree that it is rational for you that $\mathcal{F}^{\rceil \mathcal{B}} \subseteq \mathcal{R}$.

We are thus led to the following:

*Definition* 31 (**Strong temporal coherence**). We say that your current and future commitments are *strongly temporally coherent* if $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{\rceil \mathcal{B}} \subseteq \mathcal{R}$.

Strong temporal coherence is a strengthening of strong temporal consistency: it is enough to observe that under strong temporal coherence $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}} = \mathcal{R}$ is coherent, and hence it avoids partial loss. More importantly, strong temporal coherence is equivalent to the conglomerability of $\mathcal{R}$, provided that $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$.

As we have already said, it appears to be the first time that conglomerability is obtained out of finitary considerations as in this case. Walley, for instance, has repeatedly argued in favor of conglomerability [68, Section 6.8.4], but the support that he gave to it appears eventually to go back to the idea that one should allow countably many transactions to be made (see, for instance, the penultimate paragraph in page 320 of Walley's book [68], or note 13 to Section 6.9 in the same book; other argumentations in [68, Section 6.8.4] are based on the use of the so-called 'conglomerative principle', which lies at the basis of his notion of coherence, and which involves the sum of infinitely many gambles). In contrast, our setup is finitary in that respect: all the coherence conditions we deal with are eventually based on the natural extension, which involves only finitely many transactions; and moreover, even though $\mathcal{F}^{\mathcal{B}}$ is defined through sums that may be infinite, the transactions related to elements of $\mathcal{F}^{\mathcal{B}}$ always involve finitely many gambles. And yet, as we have shown, a coherence condition together with the availability of future commitments, can make it.

We should be careful in establishing the scope of our claim that conglomerability should be taken as a rationality requirement. The notion of conglomerability is important and it has been the subject of much controversy since its early definition by de Finetti [9]: the controversy concerns precisely whether or not it is rational to impose conglomerability. Our standpoint is the following: conglomerability should be regarded as a rationality requirement when present beliefs and future commitments are established at the same point in time, for the reasons discussed above. We do not claim that conglomerability should be imposed more generally than this. In particular, we do not see reasons to impose conglomerability in the setups where your future commitments are established after your present beliefs.[31] For similar reasons, we do not support requiring conglomerability in probabilistic models whenever these are not used to constrain future behaviour: for example, when a model is used only in the unconditional case, or when it is used in the conditional case under the contingent interpretation.

Despite our claim lives within the special frame indicated above, it should not be overlooked that its scope remains quite wide. For example, it is a common statistical practice to interpret conditional beliefs as future commitments, as if it were a sort of 'default' case; and in that case, our claim applies. This implies that by taking such a practice seriously, in statistics one should always work with conglomerable models. This is straightforward to do when the partition $\mathcal{B}$ is finite (and this is obviously the case for finite $\Omega$), because conglomerability holds automatically in that case, as a consequence of axiom D4, or by the super-additivity of coherent lower previsions. In other words, in this case, we have the remarkable result that strong temporal coherence is equivalent to $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$, that is, the equality of future commitments with conditional beliefs. On the other hand, statistics is very often concerned with infinite partitions and in this case working with conglomerable models can be mathematically very involved. This has recently become even clearer [44]; more work is needed to address this problem. Alternatively, one might want to question the default case mentioned above. Some discussion in this sense can be found for instance in [68, Section 6.11.1] (see also Section 8). This could lead one, in some statistical problems, to prefer temporal consistency (or even event-wise temporal consistency) to strong temporal coherence.

Strong temporal coherence implies that $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ is coherent. This gives us some motivation to study the next coherence condition:

*Definition* 32 (**Temporal coherence for gambles**). We say that your current and future commitments are *temporally coherent* if $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$ and $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ is coherent.

Temporal coherence is different from strong temporal consistency because it may happen that $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ avoids partial loss but it is not coherent, as shown in [44, Example 7]. On the other hand, temporal coherence is also different from strong temporal coherence because we may have $\mathcal{R} \subsetneq \mathcal{F}^{|\mathcal{B}}$, as in [44, Example 2]; however, when $\mathcal{R}$ is a set of strictly desirable gambles, both conditions are equivalent, as we can see from Theorem 2.

In the case of lower prevision, temporal coherence and the corresponding notion for lower previsions coincide: remember, from Proposition 16, that $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$ means that your future commitments must be specified by $\underline{E}(\cdot|\mathcal{B})$, the conditional natural extension of $\underline{P}$; this, together with Theorem 2, shows that if you express your current and future

[31] However, in some of those cases it may be rational to impose disintegrability when working with precise models, see Section 4.5 (in particular the discussion after Theorem 3).

commitments by means of lower previsions, they are temporally coherent if and only if the sets of strictly desirable gambles they induce are.

More generally speaking, we can show that temporal coherence leads to some surprising facts. To this end, observe that temporal coherence holds trivially when either $\mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}}$ or $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$. It is useful to understand whether there are possibilities other than these in order to comply with temporal coherence. The situation is established in the following lemma:

**Lemma 2.** *Let $\mathcal{R}$ be a coherent set of desirable gambles and let $\mathcal{F}^{|\mathcal{B}}$ be the set of conditional gambles it induces. Then*

$$\mathcal{R}, \mathcal{F}^{|\mathcal{B}} \text{ temporally coherent } \Rightarrow \mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}} \text{ or } \mathcal{F}^{|\mathcal{B}} \subseteq \overline{\mathcal{R}}.$$

*Proof.* Assume ex-absurdo that $\mathcal{R}, \mathcal{F}^{|\mathcal{B}}$ are temporally coherent and that $\mathcal{R} \nsubseteq \mathcal{F}^{|\mathcal{B}}$ and $\mathcal{F}^{|\mathcal{B}} \nsubseteq \overline{\mathcal{R}}$. Then there is a gamble $f \in \mathcal{F}^{|\mathcal{B}} \setminus \overline{\mathcal{R}}$, whence $\underline{P}(f) < 0$. Take on the other hand $g \in \mathcal{R} \setminus \mathcal{F}^{|\mathcal{B}}$. Then since $g \in \mathcal{R}$ we deduce that $\underline{P}(g) \geq 0$, and since it does not belong to $\mathcal{F}^{|\mathcal{B}}$ there is some $B \in \mathcal{B}$ such that $Bg \notin \mathcal{R}^{|B} \cup \{0\}$.

Let us consider the gamble $f_1 := B^c f$. Then $B' f_1 \in \mathcal{R}^{|B'} \cup \{0\}$ for every $B' \in \mathcal{B}$, and moreover it cannot be $f_1 = 0$ or we should have $f = Bf \in \mathcal{R}$, a contradiction. As a consequence, $f_1 \in \mathcal{F}^{|\mathcal{B}}$, and therefore $\lambda f_1 \in \mathcal{F}^{|\mathcal{B}}$ for all $\lambda > 0$. Moreover,

$$\underline{P}(f_1) = \underline{P}(f - Bf) \leq \underline{P}(f) + \overline{P}(-Bf) = \underline{P}(f) - \underline{P}(Bf) < 0,$$

because $\underline{P}(f) < 0$ and $\underline{P}(Bf) \geq 0$ taking into account that $Bf \in \mathcal{R}^{|B} \cup \{0\}$. Define $h := \lambda f_1 + g \in \operatorname{posi}(\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}})$. Then it holds that

$$Bh = B(\lambda f_1 + g) = Bg \notin \mathcal{R}^{|B} \cup \{0\} \Rightarrow h \notin \mathcal{F}^{|\mathcal{B}},$$

and

$$\underline{P}(h) = \underline{P}(\lambda f_1 + g) \leq \underline{P}(\lambda f_1) + \overline{P}(g) = \lambda \underline{P}(f_1) + \overline{P}(g) < 0 \text{ for } \lambda > -\frac{\overline{P}(g)}{\underline{P}(f_1)}.$$

Hence, for $\lambda$ big enough $\underline{P}(h) < 0$, whence $h$ does not belong to $\mathcal{R}$ either. As a consequence, $\operatorname{posi}(\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}) \neq \mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ and therefore $\mathcal{R}, \mathcal{F}^{|\mathcal{B}}$ are not temporally coherent, a contradiction. $\square$

This result holds for any coherent set of desirable gambles $\mathcal{R}$, not necessarily strictly desirable ones. Remarkably, when $\mathcal{R}$ is a coherent set of strictly desirable gambles the inclusion $\mathcal{F}^{\mathcal{B}} \subseteq \overline{\mathcal{R}}$ is equivalent to the inclusion $\mathcal{F}^{\mathcal{B}} \subseteq \mathcal{R}$, i.e., to the conglomerability of $\mathcal{R}$ [44, Theorem 2]. We see then that in the case of strict desirability the notion of temporal coherence coincides with strong temporal coherence, taking into account that in such a special case $\mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}}$ holds trivially only when $\mathcal{R} = \mathcal{L}^+$ so that the first inclusion is sufficient to describe all the cases. The fact that those two notions coincide in the case of strict desirability was indeed already proved in Theorem 2.

We shall next argue that temporal coherence is not a suitable notion for the more general framework where you define your commitments through sets of (not necessarily strictly) desirable gambles. This is the case because strong temporal coherence is indeed stronger than temporal coherence when $\mathcal{R}$ is not a set of strictly desirable gambles; temporal coherence is rather an intermediate notion between the conglomerability of $\mathcal{R}$ and that of $\underline{\mathcal{R}}$ (the latter being the set of strictly desirable gambles associated to $\mathcal{R}$).

**Theorem 5.** *Let $\mathcal{R}$ be a coherent set of gambles, and let $\mathcal{F}^{|\mathcal{B}}$ be its associated conditional set of gambles. Then*

$$\mathcal{R} \text{ conglomerable } \Rightarrow \mathcal{R} \cup \mathcal{F}^{|\mathcal{B}} \text{ coherent } \Rightarrow \underline{\mathcal{R}} \text{ conglomerable}.$$

*Proof.* To see the first implication, note that if $\mathcal{R}$ is conglomerable then $\mathcal{F}^{|\mathcal{B}}$ is included in $\mathcal{R}$, whence $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}} = \mathcal{R}$ coherent.

To see the second implication, assume that $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ is coherent. Then, taking into account Lemma 2, we have two possibilities:

- the first is that $\mathcal{R} \subseteq \mathcal{F}^{|\mathcal{B}}$; then for every $f \in \mathcal{R}$ and every $B \in \mathcal{B}$, it holds that $Bf \in \mathcal{R} \cup \{0\}$. Let $\underline{P}$ be the coherent lower prevision induced by $\mathcal{R}$. Then for every $k > 0$ it holds that

$$\underline{P}(\mathbb{I}_B - k\mathbb{I}_{B^c}) \geq \underline{P}(B) + \underline{P}(-k\mathbb{I}_{B^c}) = \underline{P}(B) - k\overline{P}(B^c),$$

  and this is positive for $k$ small enough provided that $\underline{P}(B) > 0$. But in that case we should have a gamble $f := \mathbb{I}_B - k\mathbb{I}_{B^c}$ in $\mathcal{R}$ such that $B' f \notin \mathcal{R} \cup \{0\}$ for any $B' \neq B$, a contradiction. Hence, it must be $\underline{P}(B) = 0$ for every $B$, whence $\underline{P}$ is trivially conglomerable and as a consequence $\underline{\mathcal{R}}$ is conglomerable by [44, Theorem 3].

- The second possibility is that $\mathcal{F}^{|\mathcal{B}} \subseteq \overline{\mathcal{R}}$; in that case applying [44, Theorem 2] we deduce that $\underline{\mathcal{R}}$ is conglomerable. $\square$

The converse implications in the above result do not hold in general:

*Example* 18. To see that temporal coherence does not imply conglomerability, note that it may happen that $\mathcal{R}$ is a proper subset of $\mathcal{F}^{|\mathcal{B}}$; in that case, temporal coherence would hold trivially because $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$ is coherent, but this set would not be included in $\mathcal{R}$ and therefore the latter would not be conglomerable. An example of such a situation can be found in [44, Example 2].

To see that the conglomerability of $\underline{\mathcal{R}}$ does not imply temporal coherence, consider the set $\mathbb{N}$ of positive natural numbers, $\Omega := \mathbb{N}, B_n := \{2n-1, 2n\}, \mathcal{B} := \{B_n : n \in \mathbb{N}\}$ and $P$ a finitely additive probability satisfying $P(\{n\}) = 0$ for every $n$, $P(\mathbb{I}_{odd}) = 0.25$. Consider $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$, for $\mathcal{R}_1 := \{f : P(f) > 0\}$ and $\mathcal{R}_2 := \{f : P(f) = 0, \exists n_f \in \mathbb{N} \text{ s.t. } f\mathbb{I}_{n_f \rightarrow} \geq 0, \sum_n f(n) > 0\}$, where $n_f \rightarrow$ denotes the set $\{n \in \mathbb{N} : n \geq n_f\}$.

Then $\mathcal{R}$ includes any gamble $f \in \mathcal{L}^+$: it follows from the coherence of $P$ that $P(f) \geq 0\ \forall f \in \mathcal{L}^+$. If $P(f) > 0$, then $f \in \mathcal{R}_1$; if $P(f) = 0$, then $f \in \mathcal{R}_2$, taking $n_f = 1$ and using that $f \neq 0$. Applying (18), it follows that $\mathcal{R}$ lies between the set of strictly desirable gambles and that of almost-desirable gambles associated to $P$.

Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are cones that do not include the zero gamble, so to see that the zero gamble does not belong to $\operatorname{posi}(\mathcal{R})$ it suffices to see that it cannot be expressed as the sum of a gamble of $\mathcal{R}_1$ with a gamble from $\mathcal{R}_2$. But if we take $f \in \mathcal{R}_1, g \in \mathcal{R}_2$, we have that $P(f+g) = P(f) + P(g) > 0$, so it cannot be $f + g = 0$. Hence, $\operatorname{posi}(\mathcal{R})$ is a coherent set of gambles inducing $P$.

Since $P(B_n) = 0$ for every $n$, we deduce that $P$ is trivially conglomerable and as a consequence $\underline{\mathcal{R}}$ is conglomerable. However, given the gamble $f := 2\mathbb{I}_{odd} - \mathbb{I}_{even}$, it holds that $B_n f \in \mathcal{R}$ for every $n \in \mathbb{N}$, whence $f \in \mathcal{F}^{|\mathcal{B}}$. However, $P(f) = -0.25 < 0$, whence $-f \in \mathcal{R}$. Hence, $0 \in \operatorname{posi}(\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}})$, whence $\mathcal{R} \cup \mathcal{F}^{|\mathcal{B}}$ incurs partial loss and as a consequence it is not coherent. ⧫

In other words, in the general case, temporal coherence no longer implies that $\mathcal{F}^{|\mathcal{B}} \subseteq \mathcal{R}$. This means that it is no longer suited to prevent you from incurring the (second) kind of irrationality across your current and future commitments that we have illustrated at the beginning of this section. In the case of general desirability it is then necessary to impose the notion of (one-way) strong temporal coherence.

A geometrical interpretation of temporal coherence can help us to better see the specific features, and weaknesses for the case of general desirability, of such a notion. In particular, Corollary 6 in Appendix A implies that for temporal coherence it is necessary that

$$\mathcal{M}(\mathcal{R}) \cup \mathcal{M}(\mathcal{F}^{|\mathcal{B}}) \text{ is convex.} \tag{25}$$

It follows from this that the lower prevision corresponding to the credal set $\mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathcal{F}^{|\mathcal{B}})$ must agree with $\max\{\underline{P}, \underline{P}(\cdot|\mathcal{B})\}$ (see Proposition 20 in Appendix A). This condition is sufficient only when both $\mathcal{R}$ and $\mathcal{F}^{|\mathcal{B}}$ are sets of strictly desirable gambles; however, $\mathcal{F}^{|\mathcal{B}}$ will never be a set of strictly desirable gambles as soon as we have an infinite number of sets $\{B_n\}_n$ in $\mathcal{B}$ such that $\mathcal{R}_{B_n}$ is different from $\mathcal{L}^+(B_n)$, even if all these are sets of strictly desirable gambles: for, in that case, we could select a gamble $f_{B_n} \in \mathcal{R}_{B_n} \setminus \mathcal{L}^+(B_n)$ such that $\underline{P}(B_n f_{B_n}) \in (0, \frac{1}{n})$ for every $n$, and then their sum $f$ would belong to $\mathcal{F}^{\mathcal{B}}$ even if $f - \varepsilon$ does not belong to $\mathcal{F}^{\mathcal{B}}$ for any $\varepsilon > 0$.

That condition (25) is a very peculiar one to comply with, can also be seen if we exclude the cases where one of the two sets is included in the other, which we have identified at the beginning of Section 4.4 as the two possible one-way strong coherence conditions applied to $\mathcal{R}$ and $\mathcal{F}^{\mathcal{B}}$. The remaining case is characterised by two convex sets that partially overlap[32] and whose union is naturally convex too. This situation is quite constraining, as it can easily be verified with simple examples in the three-dimensional space. As a side remark, note how the difference in the respective necessary conditions makes it clear that temporal coherence is much more of a stringent condition than strong temporal consistency.

Moreover, it is difficult to find a 'temporal' meaning to condition (25) in the case just discussed. In our view, this appears to be saying that temporal coherence is meaningful when it coincides with a strong coherence condition, and that it is hardly so otherwise. In summary, we regard strong temporal coherence as the essential notion of coherence for the case where you establish your present and future commitments at the same time.

[32]They overlap because temporal coherence implies temporal consistency, and for the latter to hold it is necessary that $\mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\mathcal{F}^{|\mathcal{B}}) \neq \emptyset$ (see Theorem 4(b)).

### 6.5. The precise case

The above notions can also be applied to sets of gambles inducing precise conditional and unconditional previsions. When these sets of gambles are strictly desirable, we can establish a tight relationship with disintegrability:

**Proposition 17.** *Assume that $\mathcal{R}$ is a set of strictly desirable gambles that induces the linear prevision $P$. Assume, in addition, that your future commitments $\mathcal{F}^{\mathcal{B}}$ induce a linear future prevision $P_{\mathcal{B}}$. Then:*

$$\mathcal{R}, \mathcal{F}^{\mathcal{B}} \text{ temporally consistent } \Rightarrow P \text{ disintegrable } \Rightarrow \mathcal{R}, \mathcal{F}^{|\mathcal{B}} \text{ temporally consistent.}$$

*As a consequence, if in addition $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{|\mathcal{B}}$ then $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent if and only if $P$ is disintegrable.*

*Proof.* The first implication follows from Theorem 3, taking into account that if $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent so are $\mathcal{R}, \mathcal{F}_1^{\mathcal{B}}$, where $\mathcal{F}_1^{\mathcal{B}}$ is the conglomerable natural extension of the strictly desirable gambles associated to $P_{\mathcal{B}}$. The second follows from Theorem 2, taking into account that the disintegrability of $P$ implies its conglomerability, and from [44, Theorem 3]. The second statement is an immediate consequence of the first. □

More in general, we can apply the notions of temporal consistency and coherence to sets of gambles that are not necessarily strictly desirable. In that case, we can establish the following:

**Proposition 18.** *Let $\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ be coherent sets of desirable gambles such that they induce linear unconditional and future previsions $P, P_{\mathcal{B}}$. Then:*

$$\mathcal{R} \cup \mathcal{F}^{\mathcal{B}} \text{ coherent } \Rightarrow \mathcal{R} \cup \mathcal{F}^{\mathcal{B}} \text{ avoids partial loss } \Rightarrow P, P_{\mathcal{B}} \text{ coherent.}$$

*Proof.* That the first condition implies the second is trivial; the second implies that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids sure loss, whence by Theorem 4 $P, P_{\mathcal{B}}$ avoid sure loss, and since they are linear this means that they are coherent. □

However, the above implications are not equivalences: on the one hand, the sets of current and future commitments in Example 14 induce coherent linear previsions but their union incurs partial loss. To see that the avoiding partial loss of $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ does not imply its coherence, consider the following example:

*Example* 19. Consider $\Omega \coloneqq \{1, 2, 3, 4\}, B \coloneqq \{1, 2\}, \mathcal{B} \coloneqq \{B, B^c\}$, and the set $\mathcal{R}$ of current beliefs given by

$$\{f \in \mathcal{L}(\Omega) : f(1)+f(2)+f(3)+f(4) > 0\} \cup \{f \in \mathcal{L}(\Omega) : f(1)+f(2)+f(3)+f(4) = 0, \min\{f(1), f(3)\} > 0\}$$

and let your future commitments be given by

$$\begin{aligned} \mathcal{R}^B &: = \{f \in \mathcal{L}(\Omega) : f = Bf, f(1) + f(2) > 0 \text{ or } f(1) = -f(2) < 0\}, \\ \mathcal{R}^{B^c} &: = \{f \in \mathcal{L}(\Omega) : f = B^c f, f(3) + f(4) > 0\}. \end{aligned}$$

Then $\mathcal{R}$ induces the linear prevision associated to the uniform distribution on $\Omega$, and $\mathcal{F}^{\mathcal{B}}$ induces the conditional prevision $P_{\mathcal{B}}$ derived from $\mathcal{R}$ by means of Bayes' rule. To see that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ avoids partial loss, note that

$$\mathcal{R} \cup \mathcal{F}^{\mathcal{B}} \subseteq \{f \in \mathcal{L}(\Omega) : f(1)+f(2)+f(3)+f(4) > 0\} \cup \{f \in \mathcal{L}(\Omega) : f(1)+f(2)+f(3)+f(4) = 0, f(3) \geq 0\} \setminus \{0\},$$

and that the set on the right-hand side is coherent. To see on the other hand that $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$ is not coherent, note that given $f \coloneqq (1, -3, 1, 1) \in \mathcal{R}$ and $g \coloneqq (-1, 1, 0, 0) \in \mathcal{F}^{\mathcal{B}}$, their sum is $f + g = (0, -2, 1, 1)$, which does not belong to $\mathcal{R} \cup \mathcal{F}^{\mathcal{B}}$. ⧫

A straightforward deduction could then be that the precise case differs from the case of desirable gambles to that of previsions, which was discussed in Section 4.5. This is particularly evident from the fact that the implications in Proposition 18 were equivalences in Theorem 3. But this conclusion is based on the precision (or linearity) of the previsions that can be derived from a set of desirable gambles. This is not the precision of the set itself: the latter is rather the completeness, or maximality, of the set. By assuming that the involved sets are maximal, and considering the setting of event-wise temporal consistency, we obtain a very interesting outcome:

**Proposition 19.** *Let your current and future commitments $\mathcal{R}, \mathcal{R}^B$ be event-wise temporally consistent. Assume in addition that $\mathcal{R}$ and $\mathcal{R}_B := \{f_B \in \mathcal{L}(B) : Bf_B \in \mathcal{R}^B\}$ are both maximal. Then it holds that $\mathcal{R}^B = \mathcal{R}^{\rfloor B}$.*[33]

*Proof.* First, note that if $\mathcal{R}$ is maximal, then so is $\mathcal{R}_{\rfloor B}$: indeed, if there is $f_B \neq 0$ such that $f_B \notin \mathcal{R}_{\rfloor B}$, then $Bf_B \notin \mathcal{R}$; the maximality of $\mathcal{R}$ implies that $-Bf_B \in \mathcal{R}$ and hence $-f_B \in \mathcal{R}_{\rfloor B}$.

Now, assume ex-absurdo that $\mathcal{R}^B \neq \mathcal{R}^{\rfloor B}$, and in particular that there is a non-zero gamble $f = Bf$ such that $f \in \mathcal{R}^B$ and $f \notin \mathcal{R}^{\rfloor B}$. Since $\mathcal{R}_{\rfloor B}$ is maximal, $-f \in \mathcal{R}^{\rfloor B} \subseteq \mathcal{R}$ and this contradicts that $\mathcal{R} \cup \mathcal{R}^B$ avoids partial loss. It follows that $\mathcal{R}^B \subset \mathcal{R}^{\rfloor B}$. Then there is a non-zero gamble $f = Bf$ such that $f \in \mathcal{R}^{\rfloor B} \subseteq \mathcal{R}$ and $f \notin \mathcal{R}^B$. But since $\mathcal{R}_B$ is maximal, $-f \in \mathcal{R}^B$ and we contradict again that $\mathcal{R} \cup \mathcal{R}^B$ avoids partial loss. We deduce that under event-wise temporal consistency, it holds that $\mathcal{R}^B = \mathcal{R}^{\rfloor B}$. □

This tells us that, once we use the proper condition of precision for desirable gambles, future commitments should be equal to conditional beliefs even under the weakest consistency notion of this paper. This leads us to replay the reasoning in Section 4.5: since conditioning is the only (event-wise) temporally consistent rule in the case of maximal sets, why should one postpone the task of assessing future commitments? Being maximal (i.e., Bayesian) together with conceding—right now—that future commitments might differ from conditional beliefs appears irrational. As a consequence, we obtain that strong temporal coherence becomes a rationality requirement whenever we stick to using maximal sets. This means that in the case of maximal sets it holds that: (i) conditioning is the only rule to compute future commitments; (ii) and that $\mathcal{R}$ must be conglomerable. Moreover, taking into account Proposition 10, we obtain a result similar to Corollary 3—yet without assuming that probabilities be positive:

**Corollary 5.** *Let $\mathcal{R}$ be a maximal set of desirable gambles representing your current beliefs, and let $\mathcal{F}^{\mathcal{B}} = \mathcal{F}^{\rfloor \mathcal{B}}$. Then the following conditions are equivalent:*

(a) *$\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are temporally consistent.*

(b) *$\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are strongly temporally consistent.*

(c) *$\mathcal{R}, \mathcal{F}^{\mathcal{B}}$ are strongly temporally coherent.*

(d) *$\mathcal{R}$ is conglomerable.*

## 7. Relationship with other approaches

In this section, we consider a number of other approaches that relate probability and time, and discuss their connection with the work we have carried out in this paper.

### 7.1. Dynamic coherence

This paper is related to the work on *dynamic coherence* carried out by Skyrms [59, 60, 61] and Armendt [2, 3], among others, and with a strong influence from Ramsey's work [48] (see also [17, 31] for other references on this topic). These authors justify the rationality of a temporal updating principle by means of the impossibility of building a book against it, and in this way they support Bayes' rule of conditioning, as well as Jeffrey's and van Fraassen's rules we shall discuss later on. They distinguish between two concepts of coherence: the *static* or *synchronic* one, where your present and future commitments are all established at the same time, and where coherence can be justified using de Finetti's or Ramsey's arguments; and the *dynamic* or *diachronic* one, where your present and future commitments are established at different times. In this case, the use of a book argument (i.e., of a finite combination of acceptable gambles that yields a sure loss) has been criticised by Maher [39] and Levi [37, 38], among others (see [3] for a response to some of these criticisms). A diachronic book argument was first proposed by David Lewis in order to justify Bayesian updating, as reported by Teller [63, note 1 to Section 1.3].

[33] Strictly speaking, the notion of maximality does not hold for the sets $\mathcal{R}^{\rfloor B}$ and $\mathcal{R}^B$, because we can add gambles to them keeping their coherence; it applies instead to their counterparts $\mathcal{R}_{\rfloor B}$ and $\mathcal{R}_B$; it is this simple technical reason that makes us introduce the latter sets in the theorem.

That dynamic coherence is closely related in spirit to the present work is clear if we reconsider the ideas behind temporal consistency and coherence: in the first, we have considered that your future commitments are established later than present beliefs, and as a consequence we are considering a diachronic argument, in the above language; in the second, instead, we assume that your present and future commitments are established together, and hence we are in the synchronic framework.

On the other hand, our work is more general than dynamic coherence in three ways: first, we are allowing for an imprecise model of your assessments, which allows us to apply our results in cases of uncertain or indeterminate information; second, in our formalisation we are allowing for infinite partitions of the set of outcomes, and as a consequence we have related our approach to the notions of conglomerability and disintegrability; and finally, we have moved away from the language of previsions (or probabilities) into the richer language of sets of desirable gambles. This has implications, for instance, for the type of losses we need to consider, which are partial rather than sure as in the traditional book approach. Moreover, it has implications in that we are not led to conclude in general that rationality requires computing future commitments by conditioning (we comment on this to a wider extent in Section 8).

### 7.2. *Probability kinematics*

An interesting and useful approach to compute future commitments was provided by Richard Jeffrey in his celebrated theory of *probability kinematics* [30, 32] (see also [13]). Jeffrey's rule is defined in the following way.

Consider a countable partition $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$ of the set of outcomes $\Omega$ of the experiment, each of them with positive probability, and let $P_0$ be your current probability model. Suppose you are interested in an event $A \subseteq \Omega$, and that after defining $P_0$, you receive some evidence that leads you to revise your probabilities $P_0(B_n)$ ($n \in \mathbb{N}$) into new (or posterior) probabilities $P_1(B_n)$, without changing your conditional probabilities: that is, $P_0(A|B_n) = P_1(A|B_n)$ for all $n$. In this case your probabilities are said to satisfy probability kinematics, and your posterior probability for $A$ should be defined as

$$P_1(A) := \sum_n P_0(A|B_n)P_1(B_n).$$

Jeffrey's rule is useful when the new evidence represents an uncertain observation about $\mathcal{B}$, rather than coinciding with an event of such a partition. If in particular your unconditional commitments do not change in time, i.e., if $P_1(B_n) = P_0(B_n)$ for all $n$, we recover the law of total probability, and therefore this can be regarded as generalising Bayesian updating (and in particular the idea of using Bayes' rule to compute future commitments). Remark also on the assumption of positivity of unconditional probabilities, which is formally related to our comments in Section 4.5.

Jeffrey's approach is dual to ours, in the following sense: while we allow future commitments to differ from the conditional ones and the unconditional beliefs are determined only at the present time, Jeffrey requires that the conditional probabilities should not change in time, but allows the unconditional ones to be modified in the light of new evidence. Still, in [2] and [59], it has been argued in favor of probability kinematics by means of dynamic coherence.

### 7.3. *The temporal sure thing and reflection principles*

Two very related approaches to the work carried out in this paper are due to Goldstein [21, 22, 23] and van Fraassen [66, 67].

Goldstein [21, 23] required your current and future beliefs to satisfy the *temporal sure thing principle*:

> "Suppose that you have a sure preference for $A$ over $B$ at (future) time t. Then you should not have a strict preference for $B$ over $A$ now."

In our language, the temporal sure preference principle states that if you knew that $f$ is desirable to you in the future, then you should not desire $-f$ now, which is in clear connection with Eq. (20) in Proposition 7 (it is not exactly the same because the underlying notion we are concerned with is avoiding partial loss, which is stronger than the book-equivalent idea of avoiding sure loss).

Now, if we denote by $P_0$ your current probability model and by $P_1(\cdot|\mathcal{B})$ your future probability model conditional on the observation of some evidence, the temporal sure thing principle implies [21, Section 3] that your current model should satisfy

$$P_0(f) = P_0(P_1(f|\mathcal{B})) \tag{26}$$

for every gamble $f$.

Around the time of Goldstein's initial proposal, van Fraassen [66, page 244] established his *reflection principle*:

"My current probability for an event A, conditional that at some later time I assign A probability $r$, should also be equal to $r$",

which can be expressed mathematically also as

$$P_o(A|P_1(A) = r) = r. \tag{27}$$

Van Fraassen also allows for vague assessments in [67], determining then a *general reflection principle*:

"My current opinion about event E must lie in the range spanned by the possible opinions I may come to have about E at some later time t, as far as my present opinion is concerned."

This general principle yields Eq. (27) in the particular case of precise probabilities.

Both these approaches are related, like ours, to the impossibility of building a book against you using your current and future commitments.[34] However, their focus is on the current beliefs, and future beliefs are treated as some uncertain quantity. This is very clear by the way van Fraassen ends his principle: 'as far as my *present* opinion is concerned' (for similar reasons, van Fraassen says in [66] that we may speak of a Dutch *strategy* instead of a Dutch book). In other words, all these principles determine how your current beliefs should be related to the future ones, *if you knew them*. On the contrary, our focus is on a time where your current and future commitments have already been established, so the latter do not act as uncertain objects for us. Hence, even if their formula of time consistency Eq. (26) looks basically identical to our Eq. (14), the two of them are saying very different things. In particular, since Goldstein and van Fraassen regard future commitments as uncertain quantities, claims about them are made through expectations. Stated differently, Eq. (26) is actually concerned with a consistency property of current beliefs alone, that cannot prevent you from incurring a temporal sure loss. In contrast, endorsing Eq. (14) does prevent you from incurring a sure loss. For this to be possible, however, future commitments must be known, as we indeed assume in this paper.

Another difference between our work and Goldstein's is related to conglomerability. In fact, Goldstein maintains that his approach does not imply the conglomerability of present beliefs and in fact he supports finitely additive models. This claim has originated some controversy. We can find Walley, for instance, deducing that Eq. (26) does lead to the conglomerability of present beliefs (see [68, note 11 to Section 6.5]); and other researchers who have criticised Goldstein's temporal sure preference principle as incompatible with finite additivity, that is, non-conglomerable models (see [33, Section 2.3]; Goldstein's reply is in [24], and renewed criticism is in [34]). On the other hand, our notion of strong temporal coherence leads to conglomerability; more generally speaking, in the case of precise probability the relationship of our approach to disintegrability is very tight even under much weaker notions than strong temporal coherence.

### *7.4. The work of Shafer, Gillet and Scherl*

In an interesting paper [57], Shafer, Gillet and Scherl use a dynamic approach to justify Walley's updating rule as a temporal rule.[35] This work has some points in common with the work we have carried out here, such as, for instance, the distinction of different time periods for the establishment of future commitments. On the other hand, their approach is based on Shafer and Vovk's [58] two-player reinterpretation of de Finetti's and Walley's subjective approach to probability.

The idea is to consider the process of assessing subjective probabilities as a game between two players: *House*, which determines the probabilities of certain events, and *Gambler*, the one that determines the stakes at which he is disposed to bet on the different events. A third player, called *Reality*, can be used to determine what are the events that actually happen in the end. In our language you play the role of House while Gambler is an opponent of yours. Shafer, Gillet and Scherl use two different approaches to establish the consistency of your assessments: the first relates to the

[34] See also [22, 67] for other justifications of these principles.

[35] Shafer, Gillet and Scherl appear to regard Walley's updating interpretation of conditioning as defining a temporal setting—one that prescribes computing future beliefs out of present ones by conditioning—, and then try to justify it accordingly (this was already pointed out in [7, footnote 10 in p. 1408]). In our view, Walley's updating interpretation of conditioning is instead only concerned with your *current* beliefs under the assumption that event $B$ occurs. The justification of this interpretation of updating is already in Walley's theory, and it follows in particular from the axioms D1–D4 of desirability through his 'updating principle'.

common idea of the book, that is, to avoiding losses; the second is more peculiar to Shafer and Vovk's game-theoretic probability and is called Cournot's principle. According to the latter, and loosely speaking, the probabilities can be seen as consistent when it is not possible for Gambler to exploit them in order to become infinitely rich without risking bankruptcy. It is a principle that relates to a long-run interpretation of probability and that allows the authors to draw stronger conclusions than those they achieve by the book-based consistency.

One of the aims in [57] is to see how probabilities should be temporally updated. The authors consider both the precise (in Section 1) and the imprecise (in Section 2) cases, and also distinguish the case where you have *exact* information [56], which means that all you know in the future is the event $B \in \mathcal{B}$ that is observed (and that coincides with our case in this paper), or when you can have additional information besides $B$. They show that under some conditions, which roughly coincide with our strong temporal coherence setup, updating beliefs by means of Walley's GBR is consistent in their sense; then they try to make the case also for information that is not exact (Section 2.4) and conclude that a particular case of temporal consistency must be satisfied.

With respect to that work, there are a few novelties in our paper:

- while in [57] it is assumed that updating is made by means of GBR and afterwards it is justified that this is consistent under some conditions, in our paper we make no assumptions about how the future commitments are established; in particular, we show that there are several other possibilities satisfying temporal consistency, and that even when you establish your future commitments in advance, as in the case of temporal coherence, there are other possible rules compatible with our consistency notions (such as the regular extension).

- Our treatment allows for infinite partitions of future events, whereas only finite partitions are considered in [57]. This is what has allowed us to deal with the notion of conglomerability.

- Moreover, in [57] it is assumed that all the conditional events have positive lower probability, which is not the case in this paper.

*Remark* 6. It is also interesting to consider a more subtle difference between the two approaches. In [57, Section 1.6] the authors claim that Bayes' rule cannot be derived in the precise case when future commitments are established after current beliefs. This seems to follow from the impossibility of Gambler to know the future prices at the time when present commitments are effective: in fact, to make you undergo a loss, Gambler must design two bets, one for present and one for future commitments, that act jointly to that end. On the other hand, in the analogous case in our paper (temporal, or event-wise temporal, consistency), we instead do derive Bayes' rule—provided that probabilities are positive. This is the case because we do not stress as much as in [57] the operational nature of the game: for us it is enough that an inconsistency is possible in order to exclude the corresponding rule; we do not enforce that there should be an actual protocol by which an opponent could exploit it. Notice, however, that also Shafer, Gillet and Scherl eventually rule out those inconsistencies (in Section 1.7) and derive Bayes' rule, by invoking Cournot's principle. ⧫

### *7.5. Belief revision*

It is also interesting to comment on the connection between our approach and the work on *belief revision* developed among others by Gärdenfors in [18] (see also [1]). Roughly speaking, belief revision refers to the general process by which you modify your belief model to keep it up to date with the information you access (in particular, belief revision has been given a temporal interpretation by means of *temporal logic* [5] or *dynamic doxastic logic* [51, 65]). For more information on this topic, we recommend Peppas' gentle introduction to the field [47] and also the web site `http://www.beliefrevision.org/`.

In Gärdenfors' view, three basic procedures are relevant when changes in a belief model are concerned: *expansion*, where you add a new element to your belief set that is consistent with it; *contraction*, where you remove one element from the belief set; and *revision*, where you add a new element to your belief set that is inconsistent with the latter. In any of these cases, the modifications in your belief set should be made so that it remains logically consistent: this means, as a consequence, that when belief revision is performed, some elements of the belief set must be removed in order to maintain consistency. Moreover, the underlying rationale is that the removal of some elements should be made so as to make a minimal change to the preexisting belief set. This gives rise to a number of axioms, first introduced by Alchourrón, Gärdenfors and Makinson in [1], producing the so-called *AGM* models.

The AGM axioms have been originally put forward in a logical context, where knowledge is represented by sentences from a certain language $\mathcal{L}$. There have been attempts to extend them to probability (even in the seminal book by Gärdenfors), but many of these attempts seem to have faced the limitations of not being able to fully deal with probability as a logic. This can be overcome by representing probability through desirable gambles, which, as we have argued already, has a very direct connection with logic. In particular, de Cooman shows in [6] how some of the AGM axioms can be extended very naturally to an imprecise probability scenario by using desirability. Some earlier work on the same line of research was made by Moral and Wilson [45] (other work connecting imprecise probability and belief revision has been carried out, among others, by Dubois and Prade in the framework of possibility measures [15, 16]).

Regarding the work in this paper, at the moment we do not see a direct relationship with belief revision.[36] Although we have considered two sets of assessments established at (possibly) different points in time, the problem we have studied is that of characterising consistency across the two sets, and this is not a problem of belief revision. In particular, for the most part we are not interested nor give guidelines as to how to modify your set of current beliefs in the light of the event $B$: we just concentrate on characterising all the possible temporally consistent, or coherent, changes; as a consequence, we are certainly not focused in particular on *minimal changes*. This point is probably what marks the most significant difference from belief revision (at least from the one originally formulated by Gärdenfors). As we understand it, the rationale of doing a minimal change originates from the underlying idea that the original belief set is correct, or stable, before accessing the new information; therefore it should be preserved as much as possible while incorporating new information. Our setup is instead conceived to accommodate also situations where your current beliefs may have been roughly specified, for instance for lack of time, and thus can be (and it is actually desirable that they are) subject even to big changes if there is additional time to take the available evidence more carefully into consideration. More generally speaking, even the founding idea of ‘rational change’ in belief revision does not seem to directly apply to our setup: in our setup every change may be reasonable in some situation,[37] sometimes even the changes that are not temporally consistent, simply because current beliefs may have been inaccurately specified.

Note, moreover, that even if present and future commitments satisfy the appropriate consistency notion for the situation (temporal consistency, temporal coherence or event-wise temporal consistency, depending on when the future commitments are established), we are not concerned with the task of enlarging the set of beliefs so as to accommodate both: the idea is rather to drop the current beliefs in favor of the future commitments.

The only place where we appear to be closer to a belief revision problem is when we briefly study how to modify your assessments when temporal consistency is violated (Sections 4.1.1 and 6.1), and that could be seen as a procedure of belief contraction. However, our approach is slightly different from belief contraction, for a number of reasons: instead of modifying the union of the set of current and future commitments in order to remove the inconsistent gambles, we see that it is more productive to investigate how the set of future commitments can be contracted in order to obtain consistency with the set of current beliefs (the reason why this second problem is interesting in our approach is that in the case of temporal consistency, current beliefs cannot be modified after they are established and before event $B$ occurs). We show that there is not in general an optimal way of doing so, which in our context means that there is not a greatest subset of the future commitments satisfying temporal consistency. This links to the well-known fact within the belief revision theory that contraction is not always possible under the assumption of a minimal change, for which a number of solutions have been proposed (see [47] for information and references). However, and perhaps surprisingly, we do show that in the case of maximal (i.e., precise) desirability assessments, there may exist an optimal correction (see in particular Proposition 9 and Corollary 4).

Another special trait of our work is the structure of the set $\mathcal{F}^{\mathcal{B}}$ of future commitments. We recall that this is not a set of commitments that you hold at some point in time; it is rather a summary of different sets of commitments $\mathcal{R}^{B}$ that become effective depending on the element $B \in \mathcal{B}$ that comes true (see Section 3.1). This is another difference with respect to belief revision, where the set that is produced after the changes is a new set of beliefs. And yet $\mathcal{F}^{\mathcal{B}}$ is really a fundamental concept in order to define appropriate consistency notions across present and future commitments: it is the very structure of $\mathcal{F}^{\mathcal{B}}$ that allows us to properly define the losses (to avoid) whenever the partition $\mathcal{B}$ is infinite;

[36] A distinction was made by Katsuno and Mendelzon [35] between belief revision and belief *updating*: the latter applies when you have additional evidence that transforms your set of possible worlds, and in that case traditional techniques of belief revision are no longer applicable. The approach we follow in this paper cannot be easily linked to Katsuno and Mendelzon's work either: see [57, Section 4.3] for a related discussion in the context of Shafer et al.'s approach, which is very similar in spirit to ours.

[37] Provided that we are not focusing on strong temporal coherence or on precise beliefs, as rationality imposes very tight constraints in these cases.

it is moreover the structure of $\mathcal{F}^{\mathcal{B}}$ that has allowed us to find a finitary justification for the notion of conglomerability.

On the other hand, and despite the differences that we currently see between belief revision and our work, we find that many aspects of their relationship are still unclear; moreover, we are aware that we may well have missed contributions in the vast literature of belief revision that might have changed our mind about some of the aspects we have been commenting on. For these reasons, we think it would be useful to study more deeply the interplay of the ideas in this work and in the literature of belief revision.

## 8. Concluding remarks and future outlooks

In this section we would like to discuss what we think we have understood after the analysis we have carried out in this paper. We recall that our focus has been on developing mathematical tools to characterise whether or not your probabilistic assessments are consistent in time. We have restricted the attention in particular to the simplest situation made of two time points: now, and a subsequent point that depends on the observation of an event $B$ in a partition $\mathcal{B}$ of the possibility space $\Omega$. We have represented your current beliefs by a set of desirable gambles $\mathcal{R}$ (or by the special case made of a coherent lower prevision $\underline{P}$). We have assumed, in addition, that after $B$ occurs, you will hold new commitments, thus dropping $\mathcal{R}$, and that these 'future commitments' are known. Moreover, we have found it useful to consider three time periods when you might establish your future commitments: now, later but before $B$ occurs, after $B$ occurs.

The case where you establish your future commitments now, together with your current beliefs, is the one closest in spirit to the traditional probabilistic, and statistical, setup. We have argued that in this case rationality should lead you to apply a *strong temporal coherence* condition. From this, we have deduced that your model of present beliefs should be conglomerable. This result is meaningful because it provides for the first time, in our knowledge, a justification of conglomerability obtained through considerations of temporal coherence, where the coherence notion is—in a definite sense—finitary. This has been surprising to ourselves in the first place, because conglomerability is a non-finitary concept. Most importantly, such a feature of our approach seems to provide new elements to settle the 83-years long, and still on-going, controversy about whether or not conglomerability should be imposed on probabilistic models.

Another important question affected by strong temporal coherence is that of the choice of the 'rule to update beliefs' under imprecise probability. Here we should clearly distinguish two situations. If by updating beliefs we mean the traditional updating interpretation of conditioning—the one that is not really concerned with future commitments—, then the question was already thoroughly analysed and discussed by Walley: the only updating rule supported by (non-temporal) coherence arguments is conditioning.[38] This means using the generalised Bayes rule in case the conditioning event has positive lower probability (and hence Bayes' rule in the precise case). When this is not the case, the choice is wider but it can still be formulated as conditioning a coherent set of desirable gambles (see Definition 12). If, on the other hand, by updating beliefs we mean a temporal setting that involves future commitments established at present time, then we can use the analysis in this paper to deduce that also in this case there is only one choice: again, that of conditioning.[39]

Let us remark once again that these results are obtained in the specific case where you establish your future commitments now, together with present beliefs, and that you are in fact committed to them, in the sense that they will constrain your future behaviour. When could this be the case in practice? Most probably this would happen when you create your present uncertainty model carefully, that is, doing your best effort to examine the available evidence and formalise your current beliefs. In this situation, you would probably exclude that the availability of extra time to reason could lead you to change significantly your uncertainty model in absence of new information (remember that in this paper we assume that the only new information you will receive about $\Omega$ is $B$). As a consequence, you would commit yourself, already at present time, to have your future behaviour constrained by conditional beliefs. This will give you the opportunity to strengthen your model of present beliefs through the implications of conglomerability.

[38] Some caution, or additional considerations, should be used in case you were strongly focused on the Gamma-maximin criterion as a way to solve decision-theoretic problems (see, e.g., [4, 25], and especially the criticism to Gamma-maximin in [53]).

[39] These outcomes should not be over-interpreted: for the conditional model constrains rather than determines your behaviour in general; you are always allowed to accept gambles to which you did not commit in advance (see the discussion in Section 3.1). On the other hand, these constraints can be very weak on some occasions, especially when generalised Bayes rule is used to compute conditional inferences from credal sets. In these cases it could be useful to rely on uncertainty models more informative than sets of probabilities, such as coherent sets of desirable gambles and the related conditioning (see Section 5.2). This would be also a way not to compromise the possibility to achieve strong temporal coherence.

This situation is relatively close to the traditional view of some fields of research. For instance, in the case of knowledge-based systems, you ideally do your best effort to model domain knowledge by a, possibly imprecise, probabilistic model; once the system is built and successfully tested, then it supports decisions by making inferences through conditioning. Another example can be statistics: a model is carefully built via (again, possibly imprecise) prior and likelihood, and future actions are chosen to be constrained by the posterior inferences. It is for these reasons that we think that strong temporal coherence has something to say in particular for the traditional probabilistic and statistical setup: it appears that there you should use conglomerable models and make inferences by conditioning them.

On the other hand, it will not always be possible for you to create a model with full care. This will be the case also in the previous fields of research, whose description above has been partly idealised, and will be definitely so if we focus on reasoning and decisions in daily life: for it is relatively uncommon that at any time you have an accurate model of the evidence around you that is stable and not subject to revision in absence of information. Usually, the process by which you form beliefs is very dynamic: you start with a rough model of the evidence, whose accuracy is constrained by time limitations or by lack of other resources; then the availability of extra time usually helps you to rework your model and make it more stable. This process is reactivated whenever you can and think it is worth (and of course also when you access new information). Note how this setup seems to be more in the scope of artificial intelligence than the previous one. The scenario in which you establish your future commitments at present time appears to be just too narrow for this case: why should you commit yourself now to constrain your future behaviour by conditional beliefs when you may well doubt that they actually reflect a careful analysis of the evidence at hand? Most probably, you will instead establish your future commitments at some later time (note that nothing prevents you from realising later that conditional beliefs were instead accurate enough and can be taken as future commitments). This is precisely where the more flexible framework of temporal consistency enters the picture.

In particular, our results on temporal consistency (by this we also mean, for short, event-wise temporal consistency in what follows) can be used as a guidance in the process of belief assessment so as to maintain consistency between present and future commitments: this will prevent an opponent from making you incur a (sure or partial) loss. Nevertheless, in our view this cannot be given the status of a rationality requirement, in the sense that temporal consistency cannot be imposed in general on probabilistic models: for it is always possible that your original model $\mathcal{R}$ was too inaccurately specified, so that you might want to reconsider part of the assessments of $\mathcal{R}$ in the passage to future commitments, even though this will create an inconsistency between the two models.[40]

Stated differently, even though we think that it should be desirable for you to be self-consistent in time, we find it unreasonable to impose it on you in general. In fact, the lesson we draw from the analysis in this paper is another: the crucial point is not that you should force yourself later to define future commitments that are consistent with your present beliefs; it is rather on adopting a procedure to assess beliefs that gives you some minimal quality guaratees throughout. Remember, in fact, that temporal consistency is a relatively weak notion (see, e.g., Theorem 4(b)): that your future commitments conflict with your present beliefs implies that somewhere in the process of assessing your beliefs there has been a very serious flaw. This means that if you implement that process using some minimal care, you will automatically minimise the possibility to be temporally inconsistent. Quality can be achieved by relying on the tools that we have developed to check temporal consistency: for they make you aware when you contradict the beliefs you stated first (but still allowing you to do so), so that any change of beliefs is well-reflected on. Moreover, you achieve quality by tuning the strength of your judgments relative to the evidence at hand as well as to the depth by which you have analysed it. The good news is that this is not too difficult to do in an imprecise probability setting: it is enough to make your assessments the more imprecise the weaker your knowledge (it goes without saying that you should also aim at making them the more precise the stronger your knowledge, if you want your models to be useful).

The situation could instead be quite difficult if you wanted to stick to precise probability: for there are states of weak knowledge that cannot be represented by precise models (ignorance is one case, and there are many others that are less extreme). In these cases, the formalism you have chosen will de facto oblige you to make stronger judgments than those you actually support; this, together with the fact that temporal consistency becomes a very rigid notion in the precise case (most often[41] allowing only for conditioning in order to create future commitments), will make it quite likely for you to incur a temporal loss. In this case you would behave like a person that is used to make bold

[40] This will be even more the case in a setup where it is allowed, unlike in this paper, to receive information about $\Omega$ besides $B$, and which might even not be representable as a subset of $\Omega$.

[41] Remember the possible exceptions discussed in Remark 3, note 21 and Remark 6.

claims even when the evidence does not support them, and that is obliged to retract them at some later time. The central question here, and somewhat loosely speaking, is that precise probability is too narrow of a framework to allow strong temporal coherence and temporal consistency to be distinguished: these two notions nearly collapse into a single one.

A summary of the discussion up to this point is that our research in this paper indicates that there are two behavioural probabilistic theories of uncertainty that should be considered when the focus in on temporal considerations. Referring to the most general mathematical model in this paper, one is that of coherent sets of desirable gambles with an additional axiom that accounts for the conglomerability of the model (that is, the theory based on axioms D1–D5;[42] see [43] for the case of lower previsions); this should be used in the case of strong temporal coherence. In the remaining cases, the theory should be the one based on axioms D1–D4 and complemented by considerations of temporal consistency. Both theories offer opportunities and challenges. In the first case, a definite challenge is to make the theory of practical use in general, because the conglomerability axiom is non-finitary, and this means, in a logical language, that the deductive closure (which in that case is the conglomerable natural extension) cannot be computed in any finitary way.[43] In the second case, the theory is mathematically much easier to deal with and, on the other hand, it is largely there to be developed with regard to considerations of temporal consistency: for example, a vast number of possibilities open up here to define temporally consistent updating rules.

We would like to conclude this paper by signaling some of the most prominent open problems stemming from our work; an important one, in our view, is the extension of the notions of (strong) temporal consistency and strong temporal coherence to several steps in the future, which would allow us to link our work to stochastic processes. We think that if we can represent these steps by means of hierarchical information, the treatment should be similar to that of the marginal extension theorem we have considered in Section 6.2.1, using the general version of this result established in [41]. In this sense, it would be interesting to investigate the connections with the notion of *cut-conglomerability* considered by de Cooman and Hermans in [7].

On the other hand, we may also consider the case where information cannot be represented in a hierarchical way, for instance when we consider several different partitions, not necessarily nested, at the same point in time. We believe that in that case temporal consistency and strong temporal coherence will probably be related to the notions of weak and strong coherence by Walley [68, Chapter 7]. This would probably entail the generalisation of the results in Appendix A to several sets of desirable gambles, which leads us to believe that the notions will become much more stringent.

Another interesting open problem would be to investigate in more detail the relationships of our work with the approaches summarised in Section 7. In particular, it would be useful to detail the relationships between our approach and belief revision, studying how the set of current beliefs should be contracted or revised taking into account the (possibly inconsistent) information included in the set of future commitments. We think this could be particularly useful in the case you access information that is not exact. In fact, let us recall that in this work we have restricted the attention to the case of exact information, where $B$ is the only new information you access about the possibility space $\Omega$. We have done so in the attempt to focus on a clearly defined setting, isolating the core of the temporal questions from other types of difficult problems, and because the case of exact information is particularly important in traditional probability. Now that the basic temporal questions have been analysed, it would be possible to try a generalisation to the case of inexact information. We regard this as a very important research avenue for the future. Part of the results in this paper will probably continue to hold in such a generalised setup; the challenge will be to merge them with a model of the process by which information is accessed.

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[42] We remark once again that axiom D5 is defined relative to the fixed partition $\mathcal{B}$ and bears no implications on full conglomerability.

[43] But remember that for finite spaces of possibilities D5 boils down to a finitary axiom that follows automatically from D4.

## Appendix A. On the coherence of the union of two sets of desirable gambles

In this technical appendix, we provide some insight about the coherence of the union of two coherent sets of desirable gambles. This is used in Section 6.4 to discuss the inadequacy of temporal coherence in the case of sets of desirable gambles. Nevertheless, the results we provide here have some interest on their own, and this is particularly the case of the summary made at the end of this appendix in Corollary 6.

We start with a simple observation, which is also related to Remark 1.

**Lemma 3.** *Given two coherent sets of desirable gambles $\mathcal{R}_1, \mathcal{R}_2$, their union $\mathcal{R}_1 \cup \mathcal{R}_2$ is coherent if and only if*

$$f \in \mathcal{R}_1, g \in \mathcal{R}_2 \Rightarrow f + g \in \mathcal{R}_1 \cup \mathcal{R}_2. \tag{A.1}$$

*Proof.* It is trivial that $\mathcal{R}_1 \cup \mathcal{R}_2$ satisfies D1–D3, given that both $\mathcal{R}_1, \mathcal{R}_2$ are coherent. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is coherent if and only if it satisfies D4. But D4 holds trivially in case $f$ and $g$ are taken from the same set; whence D4 is equivalent to (A.1). ☐

On the other hand, note that given coherent sets $\mathcal{R}_1, \mathcal{R}_2$, it holds that

$$\mathcal{M}(\mathrm{posi}(\mathcal{R}_1 \cup \mathcal{R}_2)) = \mathcal{M}(\mathcal{R}_1 \cup \mathcal{R}_2) = \mathcal{M}(\mathcal{R}_1) \cap \mathcal{M}(\mathcal{R}_2).$$

Hence, when $\mathcal{R}_1 \cup \mathcal{R}_2$ avoids partial loss, then the natural extension of $\mathcal{R}_1 \cup \mathcal{R}_2$ is in correspondence with the credal set $\mathcal{M}(\mathcal{R}_1) \cap \mathcal{M}(\mathcal{R}_2)$. Note that this credal set can generally have extreme points that belong to neither $\mathcal{M}(\mathcal{R}_1)$ nor $\mathcal{M}(\mathcal{R}_2)$. However, when $\mathcal{R}_1 \cup \mathcal{R}_2$ is coherent we can go one step further.

**Proposition 20.** *Let $\underline{P}_1, \underline{P}_2$, defined on $\mathcal{L}$, be the coherent lower previsions derived from the respective coherent sets of desirable gambles $\mathcal{R}_1, \mathcal{R}_2$. Assume that $\mathcal{R}_1 \cup \mathcal{R}_2$ avoids partial loss. Then*

$$\mathcal{R}_1 \cup \mathcal{R}_2 \textit{ coherent } \Rightarrow \underline{P}(f) = \max\{\underline{P}_1(f), \underline{P}_2(f)\} \quad \forall f \in \mathcal{L}, \tag{A.2}$$

*where $\underline{P}$ is the coherent lower prevision derived from* $\mathrm{posi}(\mathcal{R}_1 \cup \mathcal{R}_2)$ *by* (16). *If moreover $\mathcal{R}_1, \mathcal{R}_2$ are coherent sets of strictly desirable gambles, then the converse also holds.*

*Proof.* For the first statement, consider that for any gamble $f$ it holds that

$$\underline{P}(f) = \sup\{\mu : f - \mu \in \mathrm{posi}(\mathcal{R}_1 \cup \mathcal{R}_2)\} = \sup\{\mu : f - \mu \in \mathcal{R}_1 \cup \mathcal{R}_2\} = \max\{\underline{P}_1(f), \underline{P}_2(f)\},$$

because $\mathrm{posi}(\mathcal{R}_1 \cup \mathcal{R}_2) = \mathcal{R}_1 \cup \mathcal{R}_2$ when the latter is coherent.

To see that the converse holds when $\mathcal{R}_1, \mathcal{R}_2$ are coherent sets of strictly desirable gambles, it suffices to show that Eq. (A.2) implies (A.1). Take $f \in \mathcal{R}_1, g \in \mathcal{R}_2$. If both of them are positive gambles, then so is $f + g$, whence $f + g \in \mathcal{R}_1 \cap \mathcal{R}_2 \subseteq \mathcal{R}_1 \cup \mathcal{R}_2$.

On the other hand, if for instance $f$ has a negative part, then we deduce from the definition of strictly desirable gambles that $\underline{P}_1(f) > 0$, whence also $\underline{P}(f) \geq \underline{P}_1(f) > 0$. On the other hand, it holds that $\underline{P}(g) \geq \underline{P}_2(g) \geq 0$, and using the coherence of $\underline{P}$ we deduce that

$$\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g) > 0.$$

As a consequence, either $\underline{P}_1(f + g) > 0$ or $\underline{P}_2(f + g) > 0$, and therefore $f + g \in \mathcal{R}_1 \cup \mathcal{R}_2$. Applying Lemma 3 we deduce that $\mathcal{R}_1 \cup \mathcal{R}_2$ is coherent. ☐

**Theorem 6.** *Consider two credal sets $\mathcal{M}_1, \mathcal{M}_2$. The following are equivalent:*

(a) *$\mathcal{M}_1 \cup \mathcal{M}_2$ is convex.*

(b) *For every $P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$ there is some $\alpha \in [0, 1]$ such that the linear prevision $\alpha P_1 + (1 - \alpha) P_2$ belongs to $\mathcal{M}_1 \cap \mathcal{M}_2$.*

(c) *The lower envelope of $\mathcal{M}_1 \cap \mathcal{M}_2$ is the maximum of the lower envelopes $\underline{P}_1, \underline{P}_2$ of $\mathcal{M}_1, \mathcal{M}_2$.*

*Proof.*

(a) ⇒ (b) Consider $P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$, and let us define

$$A_1 := \{\alpha \in [0,1] : \alpha P_1 + (1-\alpha)P_2 \in \mathcal{M}_1\}$$
$$A_2 := \{\alpha \in [0,1] : \alpha P_1 + (1-\alpha)P_2 \in \mathcal{M}_2\}.$$

Since $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex, we deduce that $A_1 \cup A_2 = [0,1]$. Moreover, both these sets are non-empty, because $0 \in A_2$ and $1 \in A_1$. Let $x_1$ be the infimum of $A_1$ and let $x_2$ be the supremum of $A_2$. Then $x_1, x_2$ are a minimum and a maximum, respectively, because $\mathcal{M}_1, \mathcal{M}_2$ are closed sets. Moreover, given $1 \geq z > x_1$, then also $z \in A_1$, because

$$zP_1 + (1-z)P_2 = \alpha P_1 + (1-\alpha)(x_1 P_1 + (1-x_1)P_2) \text{ for } \alpha := \frac{z - x_1}{1 - x_1}.$$

Similarly, given $z < x_2$ then also $z \in A_2$. Thus, $A_1 = [x_1, 1], A_2 = [0, x_2]$ and $A_1 \cup A_2 = [0,1]$, whence $x_2 \geq x_1$. This implies that for every $\alpha \in [x_1, x_2]$ it holds that $\alpha P_1 + (1-\alpha)P_2 \in \mathcal{M}_1 \cap \mathcal{M}_2$.

(b) ⇒ (a) Take $P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$. Then there is some $\alpha \in [0,1]$ such that $\alpha P_1 + (1-\alpha)P_2 \in \mathcal{M}_1 \cap \mathcal{M}_2$. Since $\mathcal{M}_1$ is convex, we deduce that for every $x > \alpha$ it holds that

$$xP_1 + (1-x)P_2 = \lambda P_1 + (1-\lambda)(\alpha P_1 + (1-\alpha)P_2) \in \mathcal{M}_1 \text{ where } \lambda := \frac{x-\alpha}{1-\alpha},$$

and similarly for every $y < \alpha$ it holds that $yP_1 + (1-y)P_2 \in \mathcal{M}_2$. Hence, $\gamma P_1 + (1-\gamma)P_2 \in \mathcal{M}_1 \cup \mathcal{M}_2$ for every $\gamma \in [0,1]$. We conclude that $\mathcal{M}_1 \cup \mathcal{M}_2$ is convex.

(b) ⇒ (c) Let us consider a gamble $f \in \mathcal{L}$. Then there are linear previsions $P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$ such that $P_1(f) = \underline{P}_1(f)$ and $P_2(f) = \underline{P}_2(f)$. Applying (b), there is some $\alpha \in [0,1]$ such that $\alpha P_1 + (1-\alpha)P_2 \in \mathcal{M}_1 \cap \mathcal{M}_2$, whence

$$\underline{P}(f) := \min_{P \in \mathcal{M}_1 \cap \mathcal{M}_2} P(f) \leq \alpha P_1(f) + (1-\alpha)P_2(f) = \alpha \underline{P}_1(f) + (1-\alpha)\underline{P}_2(f) \leq \max\{\underline{P}_1(f), \underline{P}_2(f)\},$$

and since the converse inequality always holds we deduce the equality $\underline{P}(f) = \max\{\underline{P}_1(f), \underline{P}_2(f)\}$. Since we can do this for every $f$, we deduce that (c) holds.

(c) ⇒ (b) Assume ex-absurdo that (b) does not hold. Then there are $P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2$ such that $\alpha P_1 + (1-\alpha)P_2$ does not belong to $\mathcal{M}_1 \cap \mathcal{M}_2$ for every $\alpha \in [0,1]$. Since both $\mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{V} := \{\alpha P_1 + (1-\alpha)P_2 : \alpha \in [0,1]\}$ are compact convex sets of linear previsions, we can apply [68, Appendix E3] to conclude that there is some continuous linear functional $\Lambda$, $\lambda \in \mathbb{R}$ and $\delta > 0$ such that $\Lambda(P) \leq \lambda - \delta$ for every $P \in \mathcal{V}$ and $\Lambda(P) \geq \lambda + \delta$ for every $P \in \mathcal{M}_1 \cap \mathcal{M}_2$. Since from [68, Appendix D3] continuous linear functionals are always evaluation functionals, there is some gamble $f$ such that $P(f) \leq \lambda - \delta$ for every $P \in \mathcal{V}$. In particular, $P_1(f), P_2(f) \leq \lambda - \delta$, whence $\max\{\underline{P}_1(f), \underline{P}_2(f)\} \leq \lambda - \delta$, while

$$\min_{P \in \mathcal{M}_1 \cap \mathcal{M}_2} P(f) \geq \lambda + \delta,$$

a contradiction with (c). □

We are finally ready to report the most important result for a geometrical interpretation of the coherence of $\mathcal{R}_1 \cup \mathcal{R}_2$.

**Corollary 6.** *Consider two coherent sets of desirable gambles, $\mathcal{R}_1$ and $\mathcal{R}_2$, such that $\mathcal{R}_1 \cup \mathcal{R}_2$ avoids partial loss. Then for $\mathcal{R}_1 \cup \mathcal{R}_2$ to be coherent it is necessary that $\mathcal{M}(\mathcal{R}_1) \cup \mathcal{M}(\mathcal{R}_2)$ be convex. This condition is also sufficient if $\mathcal{R}_1$ and $\mathcal{R}_2$ are sets of strictly desirable gambles.*

*Proof.* Necessity and sufficiency follow from Proposition 20 and Theorem 6. □

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