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Updating coherent previsions on finite spaces

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Abstract

We compare the different notions of coherence within the behavioural theory of imprecise probabilities when all the spaces are finite. We show that the differences between the notions are due to conditioning on sets of (lower, and in some cases upper) probability zero. Next, we characterise the range of coherent extensions in the finite case, proving that the greatest coherent extensions can always be calculated using the notion of regular extension, and we discuss the extensions of our results to infinite spaces.

Keywords: Lower previsions, avoiding partial loss, weak and strong coherence, regular extension, natural extension.

1 Introduction

This paper is devoted to the study of the different notions of coherence within the theory of conditional lower previsions. This theory, established mainly in [17], provides a behavioural interpretation of probability in terms of acceptable buying and selling prices for gambles. It includes as particular cases most of the other uncertainty models present in the literature such as probability charges [1], 2- and n-monotone set functions [2], possibility measures [6, 21], and p-boxes [9]. They have also been linked to various theories of integration, such as Choquet integration [16, p. 53] and Lebesgue integration [17, p. 132].

The behavioural consistency of the acceptable buying and selling prices represented in a lower prevision is modelled in the unconditional case by means of the notion of *coherence*. This notion means basically that our supremum acceptable buying prices for a random variable should not be affected by our assessments for other variables, and also that a combination of acceptable transactions should never result in a sure loss. Coherent lower previsions can be given a sensitivity analysis interpretation as lower envelopes of sets or (precise) previsions, and this serves as a connection between imprecise probabilities and robust Bayesian analysis [15].

When we want to update a coherent lower prevision taking into account the observation of the values attained by some variables, there is not a unique way of extending the notion of coherence. In this paper, we consider two alternatives put forward by Walley in [17]: *weak coherence* and *(strong) coherence*. We are going to compare both of them and to establish sufficient conditions for their equivalence. This is interesting because weak coherence is much more manageable than coherence for practical purposes, as it essentially only depends of local considerations. We are also going to compare these two conditions with the notion of *avoiding partial loss*, which is equivalent to coherence in the precise case.

We shall deduce from the results in this paper that in the precise case there is often a unique way of updating a prevision that satisfies the property of coherence. This is not the case when we deal with lower and upper previsions. In this paper, we are going to study the set of conditional lower previsions that we can derive from some unconditional model which satisfy the properties of weak or strong coherence. We shall establish the smallest and the greatest models with this property.

In this paper we restrict ourselves to the case where all the referential spaces are finite. As we shall see, this assumption has a number of technical advantages. One of the most important is that in that case we still can give our updated models a sensitivity analysis interpretation, so a number of weakly coherent (resp., coherent) conditional lower previsions can be seen as a model for the imprecise knowledge of a number of weakly coherent (resp., coherent) conditional linear previsions. Such an interpretation does not hold in general when we deal with infinite spaces, as we will also show.

In addition to these advantages, the finite case is also the one used in a number of applications, for instance with credal (or Bayesian) networks [3, 4].

On the other hand, to make our treatment as complete as possible, we shall also discuss in detail in Section 5 to which extent our results can be extended to the case where some of the referential spaces are infinite. We shall see that most of them do not hold in that case.

The paper is organised as follows: in Section 2, we give a brief introduction to the behavioural theory of conditional lower previsions; in section 3, we compare the notions of weak and strong coherence, and avoiding partial loss; in section 4 we provide the smallest and the greatest conditional lower previsions with are coherent with some joint; in section 5 we discuss the extension of our results towards infinite spaces; and in section 6 we give some further comments on the subject. We have gathered all the proofs in an appendix.

2 Coherence notions on finite spaces

Let us give a short introduction to the concepts and results from the behavioural theory of imprecise probabilities that we shall use in the rest of the paper. We refer to [17] for an in-depth study of these and other properties, and to [14] for a brief survey.

Given a possibility space Ω, a *gamble* is a bounded real-valued function on Ω. This function represents a random reward $f(\omega)$, which depends on the a priori unknown value ω of Ω. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω. A *lower prevision* $\underline{P}$ is a real functional defined on some set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. It is used to represent a subject's supremum acceptable buying prices for these gambles, in the sense that for any $\epsilon > 0$ and any f in $\mathcal{K}$ the subject is disposed to accept the uncertain reward $f - \underline{P}(f) + \epsilon$.

We can also consider the supremum buying prices for a gamble, *conditional* on a subset of Ω. Given such a set B and a gamble f on Ω, the lower prevision $\underline{P}(f|B)$ represents the subject's supremum acceptable buying price for the gamble f, updated after coming to know that the unknown value ω belongs to B, and nothing else. If we consider a partition $\mathcal{B}$ of Ω (for instance a set of categories), then we shall represent by $\underline{P}(f|\mathcal{B})$ the gamble on Ω that takes the value $\underline{P}(f|B)$ if and only if $\omega \in B$. The functional $\underline{P}(\cdot|\mathcal{B})$ that maps any gamble f on its domain into the gamble $\underline{P}(f|\mathcal{B})$ is called a *conditional lower prevision*.

Let us now re-formulate the above concepts in terms of random variables, which are the focus of our attention in this paper. Consider random variables $X_1, \dots, X_n$, taking values in respective *finite* sets $\mathcal{X}_1, \dots, \mathcal{X}_n$. For any subset $J \subseteq \{1, \dots, n\}$ we shall denote by X_J the (new) random variable

$$X_J := (X_j)_{j \in J},$$

which takes values in the product space

$$\mathcal{X}_J := \times_{j \in J} \mathcal{X}_j.$$

We shall also use the notation $\mathcal{X}^n$ for $\mathcal{X}_{\{1,\dots,n\}}$. In the current formulation made by random variables, $\mathcal{X}^n$ is just the definition of the possibility space Ω.

Definition 1. Let J be a subset of $\{1, \ldots, n\}$, and let $\pi_J : \mathcal{X}^n \to \mathcal{X}_J$ be the so-called *projection operator*, i.e., the operator that drops the elements of a vector in $\mathcal{X}^n$ that do not correspond to indexes in J. A gamble f on $\mathcal{X}^n$ is called $\mathcal{X}_J$-*measurable* when for any $x, y \in \mathcal{X}^n$, $\pi_J(x) = \pi_J(y)$ implies that $f(x) = f(y)$.

There is a one-to-one correspondence between the gambles on $\mathcal{X}^n$ that are $\mathcal{X}_J$-measurable and the gambles on $\mathcal{X}_J$. We shall denote by $\mathcal{K}_J$ the set of $\mathcal{X}_J$-measurable gambles.

Consider two disjoint subsets O, I of $\{1, \ldots, n\}$. $\underline{P}(X_O|X_I)$ represents a subject's behavioural dispositions about the gambles that depend on the outcome of the variables $\{X_k, k \in O\}$, after coming to know the outcome of the variables $\{X_k, k \in I\}$. As such, it is defined on the set of gambles that depend on the values of the variables in $O \cup I$ only, i.e., on the set $\mathcal{K}_{O\cup I}$ of the $\mathcal{X}_{O\cup I}$-measurable gambles on $\mathcal{X}^n$. Given such a gamble f and $x \in \mathcal{X}_I$, $\underline{P}(f|X_I = x)$ represents a subject's supremum acceptable buying price for the gamble f, if he came to know that the variable X_I took the value x (and nothing else). Under the notation we gave above for lower previsions conditional on events and partitions, this would be $\underline{P}(f|B)$, where $B := \pi_I^{-1}(x)$. When there is no possible confusion about the variables involved in the lower prevision, we shall use the notation $\underline{P}(f|x)$ for $\underline{P}(f|X_I = x)$. The sets $\{\pi_I^{-1}(x) : x \in \mathcal{X}_I\}$ form a partition of $\mathcal{X}^n$. Hence, we can define the gamble $\underline{P}(f|X_I)$, which takes the value $\underline{P}(f|x)$ on $x \in \mathcal{X}_I$. This is a conditional lower prevision.

These assessments can be made for any disjoint subsets O, I of $\{1, \ldots, n\}$, and therefore it is not uncommon to model a subject's beliefs using a finite number of different conditional previsions. We should verify then that all the assessments modelled by these conditional previsions are coherent with each other. The first requirement we make is that for any disjoint $O, I \subseteq \{1, \ldots, n\}$, the conditional lower prevision $\underline{P}(X_O|X_I)$ defined on $\mathcal{K}_{O\cup I}$ should be *separately coherent*. In this case, where the domain is a linear set of gambles, separate coherence holds if and only if the following conditions are satisfied for any $x \in \mathcal{X}_I$, $f, g \in \mathcal{K}_{O\cup I}$, and $\lambda > 0$:

$$\underline{P}(f|x) \geq \min_{\omega \in \pi_I^{-1}(x)} f(\omega). \tag{SC1}$$

$$\underline{P}(\lambda f|x) = \lambda \underline{P}(f|x). \tag{SC2}$$

$$\underline{P}(f + g|x) \geq \underline{P}(f|x) + \underline{P}(g|x). \tag{SC3}$$

It is also be useful for this paper to consider the particular case where $I = \emptyset$, that is, when we have (unconditional) information about the variables X_O. We have then an *(unconditional) lower prevision* $\underline{P}(X_O)$ on the set $\mathcal{K}_O$ of $\mathcal{X}_O$-measurable gambles. Separate coherence is called then simply *coherence*, and it holds if and only if the following three conditions hold for any $f, g \in \mathcal{K}_O$, and $\lambda > 0$:

$$\underline{P}(f) \geq \min f. \tag{C1}$$

$$\underline{P}(\lambda f) = \lambda \underline{P}(f). \tag{C2}$$

$$\underline{P}(f + g) \geq \underline{P}(f) + \underline{P}(g). \tag{C3}$$

In general, separate coherence is not enough to guarantee the consistency of the lower previsions: conditional lower previsions can be conditional on the values of many different variables, and still we should verify that the assessments they provide are consistent not only separately, but also with each other. Formally, we are going to consider what we shall call *collections* of conditional lower previsions.

Definition 2. Let $\{\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})\}$ be conditional previsions with respective domains $\mathcal{K}^1, \dots, \mathcal{K}^m \subseteq \mathcal{L}(\mathcal{X}^n)$, where $\mathcal{K}^j$ is the set of $\mathcal{X}_{O_j \cup I_j}$-measurable gambles,[1] for $j = 1, \dots, m$. This is called a *collection* on X^n when for each $j_1 \neq j_2$ in $\{1, \dots, m\}$, either $O_{j_1} \neq O_{j_2}$ or $I_{j_1} \neq I_{j_2}$.

This means that we do not have two different conditional lower previsions giving information about the same set of variables X_O, conditional on the same set of variables X_I. Given a collection $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ of conditional lower previsions, there are different ways in which we can guarantee their consistency[2]. The first one is called avoiding partial loss.

The $\mathcal{X}_I$-*support* $S(f)$ of a gamble f in $\mathcal{K}_{O \cup I}$ is given by

$$S(f) := \{\pi_I^{-1}(x) : x \in \mathcal{X}_I, f\mathbb{I}_{\pi_I^{-1}(x)} \neq 0\}, \tag{1}$$

i.e., it is the set of conditioning events for which the restriction of f is not identically zero. We shall also use the notations

$$G(f|x) = \mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x)),\ G(f|X_I) = \sum_{x \in \mathcal{X}_I} G(f|x) = f - \underline{P}(f|X_I)$$

for any $f \in \mathcal{K}_{O \cup I}$ and any $x \in \mathcal{X}_I$.

Definition 3. Consider separately coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$. We say that they *avoid partial loss* when for any $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$,

$$\max_{\omega \in A_{f_1, \dots, f_m}} \left[\sum_{j=1}^{m} G_j(f_j|X_{I_j}) \right](\omega) \geq 0,$$

where $A_{f_1, \dots, f_m}$ is the set of elements that belong to some $B \in S_i(f_i)$ for some $i = 1, \dots, m$.

The idea behind this notion is that a combination of transactions that are acceptable for our subject should not make him lose utiles. It is based on the rationality requirement that a gamble $f \leq 0$ such that $f < 0$ on some set A should not be desirable.

Definition 4. Consider separately coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$. We say that they are *weakly coherent* when for any $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$,

[1] We use $\mathcal{K}^j$ instead of $\mathcal{K}_{O_j \cup I_j}$ in order to alleviate the notation when no confusion is possible about the variables involved.

[2] We give the particular definitions of these notions for finite spaces. See [12, 17] for the general definitions of these notions on infinite spaces and non-linear domains.

$j_0 \in \{1, \ldots, m\}$, $f_0 \in \mathcal{K}^{j_0}, x_0 \in \mathcal{X}_{I_{j_0}}$,

$$\max_{\omega \in \mathcal{X}^n} \left[\sum_{j=1}^{m} G_j(f_j|X_{I_j}) - G_{j_0}(f_0|x_0) \right](\omega) \geq 0.$$

With this condition we require that our subject should not be able to raise his supremum acceptable buying price $\underline{P}_{j_0}(f_0|x_0)$ for a gamble f_0 contingent on x_0 by taking into account other conditional assessments. However, under the behavioural interpretation, a number of weakly coherent conditional lower previsions can still present some forms of inconsistency with each other; see [17, Example 7.3.5] for an example and [17, Chapter 7] and [19] for some discussion. On the other hand, weak coherence neither implies or is implied by the notion of avoiding partial loss. Because of these two facts, we consider another notion which is stronger than both, and which is called (*joint* or *strong*) *coherence*:[3]

Definition 5. Consider separately coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$. We say that they are *coherent* when for every $f_j \in \mathcal{K}^j$, $j = 1, \ldots, m$, $j_0 \in \{1, \ldots, m\}$, $f_0 \in \mathcal{K}^{j_0}, x_0 \in \mathcal{X}_{I_{j_0}}$,

$$\left[\sum_{j=1}^{m} G_j(f_j|X_{I_j}) - G_{j_0}(f_0|x_0) \right](\omega) \geq 0 \tag{2}$$

for some $\omega \in A_{f_0,f_1,\ldots,f_m}$, where $A_{f_0,f_1,\ldots,f_m}$ is the set of elements that belong to $\pi^{-1}_{I_{j_0}}(x_0)$ or to some $B \in S_i(f_i)$ for some $i = 1, \ldots, m$.

Because we are dealing with finite spaces, this notion coincides with the one given by Williams in [20]. The coherence of a collection of conditional lower previsions implies their weak coherence; although the converse does not hold in general, it does in the particular case when we only have a conditional and an unconditional lower prevision.

It is important at this point to introduce a particular case of conditional lower previsions that will be of special interest for us: that of *conditional linear previsions*. We say that a conditional lower prevision $\underline{P}(X_O|X_I)$ on the set $\mathcal{K}_{O\cup I}$ is linear if and only if it is separately coherent and moreover $\underline{P}(f + g|x) = \underline{P}(f|x) + \underline{P}(g|x)$ for any $x \in \mathcal{X}_I$ and $f, g \in \mathcal{K}_{O\cup I}$. Conditional linear previsions correspond to the case where a subject's supremum acceptable buying price (lower prevision) coincides with his infimum acceptable selling price (or *upper prevision*) for any gamble on the domain. When a separately coherent conditional lower prevision $\underline{P}(X_O|X_I)$ is linear we shall denote it by $P(X_O|X_I)$; in the unconditional case, we shall use the notation $P(X_O)$.

Conditional linear previsions correspond to conditional expectations with respect to a finitely additive probability. In particular, an unconditional linear prevision P is the expectation with respect to the finitely additive probability

[3] The distinction between this and the unconditional notion of coherence mentioned above will always be clear from the context.

which is the restriction of P to events. One of the nice features of the notion of coherence is that it can be given a Bayesian sensitivity analysis interpretation: a coherent (unconditional) lower prevision $\underline{P}$ is always the lower envelope of some set of linear previsions, and as such can be seen as a model for the imprecise knowledge of some finitely additive probability P. Conversely, the lower envelope of a closed and convex set of linear previsions is always a coherent lower prevision.

The situation is slightly more complicated for conditional lower previsions. In [17] Walley proved that in the context of this paper, where we deal with finite spaces, coherent conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are always the envelope of a set $\{P_\gamma(X_{O_1}|X_{I_1}), \dots, P_\gamma(X_{O_m}|X_{I_m}) : \gamma \in \Gamma\}$ of dominating conditional linear previsions. However, this does not extend to the case where we have infinite spaces involved. We shall prove later on that a similar property can be established for weak coherence. Because of this, the results we shall establish could also all be formulated in terms of sets of conditional linear previsions.

Another interesting particular case is that where we are given only an unconditional lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ and a conditional lower prevision $\underline{P}(X_O|X_I)$ on $\mathcal{K}_{O\cup I}$. Then weak and strong coherence are equivalent, and they both hold if and only if, for any $\mathcal{X}_{O\cup I}$-measurable f and any $x \in \mathcal{X}_I$,

$$\underline{P}(G(f|x)) = 0. \tag{GBR}$$

This is called the Generalised Bayes' Rule (GBR). When $\underline{P}(x) > 0$, GBR can be used to determine the value $\underline{P}(f|x)$: it is then the *unique* value for which $\underline{P}(G(f|x)) = \underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))) = 0$ holds.

If P and $P(X_O|X_I)$ are linear previsions, they are coherent if and only if for any $\mathcal{X}_{O\cup I}$-measurable f, $P(f) = P(P(f|X_I))$. This is equivalent to requiring that $P(f|x) = \frac{P(f\mathbb{I}_{\pi_I^{-1}(x)})}{P(x)}$ for all $f \in \mathcal{K}_{O\cup I}$ and all $x \in \mathcal{X}_I$ with $P(x) > 0$.

3 Relationships between weak and strong coherence

Let us study in more detail the notions of avoiding sure loss, weak coherence and strong coherence. We start by recalling a recent characterisation of weak coherence:

Theorem 1. *[13, Theorem 1] $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are weakly coherent if and only if there is a lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ that is pairwise coherent with each conditional lower prevision $\underline{P}_j(X_{O_j}|X_{I_j})$. In particular, given linear conditional previsions $P_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$, they are weakly coherent if and only if there is linear prevision P which is coherent with each $P_j(X_{O_j}|X_{I_j})$.*

This theorem shows one of the differences between weak and strong coherence: weak coherence is equivalent to the existence of a joint which is coherent with each of the assessments; coherence on the other hand is equivalent to the existence of a joint which is coherent with all the assessments, taken together.

Weakly coherent conditional previsions can be given the following sensitivity analysis interpretation; a similar result for coherent ones has been established in [17, Theorem 8.1.9].

Theorem 2. *Any weakly coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are the lower envelope of a family of weakly coherent conditional linear previsions.*

We see from Theorem 1 that a number of weakly coherent conditional lower previsions always have a compatible joint $\underline{P}$, meaning that $\underline{P}$ is a coherent lower prevision on all gambles which is coherent with each of the conditional previsions. Our following result establishes the smallest such joint:

Theorem 3. *Consider weakly coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$, and let $\underline{E}$ be given on $\mathcal{L}(\mathcal{X}^n)$ by*

$$\underline{E}(f) := \sup\{\alpha : \exists f_j \in \mathcal{K}^j, j = 1, \ldots, m, \max_{\omega \in \mathcal{X}^n} [\sum_{j=1}^{m} G(f_j|X_{I_j}) - (f - \alpha)](\omega) < 0\}. \tag{3}$$

$\underline{E}$ is the smallest coherent lower prevision which is coherent with $\underline{P}_j(X_{O_j}|X_{I_j})$ for $j = 1, \ldots, m$.

Using this result and Theorem 2, we can also give a sensitivity analysis interpretation to $\underline{E}$ in the precise case.

Corollary 1. *Let $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$ be weakly coherent. Then, the functional $\underline{E}$ defined in (3) is the lower envelope of the set $\mathcal{M}$ of linear previsions which are coherent with each $P_j(X_{O_j}|X_{I_j}), j = 1, \ldots, m$.*

Let us focus now on the relationship between weak and strong coherence and avoiding partial loss. We start by considering this problem in the precise case. Let us consider separately coherent $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$, with respective domains $\mathcal{K}^1, \ldots, \mathcal{K}^m$. It follows that in this case coherence is equivalent to avoiding partial loss, and is in general greater than weak coherence; see [17, Example 7.3.5] for an example of weakly coherent conditional previsions that incur a sure loss. We are going to show next that when a number of conditional previsions are weakly coherent but not coherent, this is due to the definition of the conditional previsions on some sets of probability zero.

Theorem 4. *Let $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$ be weakly coherent conditional linear previsions, and let $\overline{E}$ be the conjugate of the functional $\underline{E}$ defined in (3). They are coherent if and only if for all gambles $f_i \in \mathcal{K}^i$, $i = 1, \ldots, m$ with $\overline{E}(A_{f_1,\ldots,f_m}) = 0$, $\max_{\omega \in A_{f_1,\ldots,f_m}} \sum_{i=1}^{m} [f_i - P(X_{O_i}|X_{I_i})](\omega) \geq 0$.*

Taking into account this theorem and the envelope result established in Theorem 2, we can characterise the difference between weak coherence and avoiding partial loss for conditional lower previsions:

Corollary 2. *Let* $\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ *be weakly coherent lower previsions. They avoid partial loss if and only if for all* $f_j \in \mathcal{K}^j$*,* $j = 1,\dots,m$ *with* $\overline{E}(A_{f_1,\dots,f_m}) = 0$*,* $\max_{\omega\in A_{f_1,\dots,f_m}} \sum_{j=1}^m [f_j - \underline{P}_j(X_{O_j}|X_{I_j})](\omega) \geq 0$*, where* $\overline{E}$ *is the conjugate of the functional defined in* (3)*.*

Hence, if a number of weakly coherent lower previsions incur sure loss, this incoherent behaviour is due to the definition of the conditional previsions on some sets of zero upper probability. It may be argued, specially since we are dealing with finite spaces, that we may modify the definition of these conditional lower previsions on these sets in order to avoid partial loss without further consequences, in the sense that this will not affect their weak coherence: they will still be weakly coherent with the same unconditional lower previsions.

So let us consider a number of weakly coherent conditional lower previsions that avoid partial loss. Our next example shows that, unlike for the precise case, this is not sufficient for coherence. Hence, Theorem 4 does not extend to the imprecise case. This is because the characterisation of avoiding partial loss in Corollary 2 does not hold for coherence, in the sense that the union of the supports of a number of gambles producing incoherence may have positive upper probability:

Example 1. Consider two random variables X_1, X_2 taking values in the finite space $\mathcal{X} := \{1,2,3\}$, and let us define conditional lower previsions $\underline{P}(X_2|X_1)$ and $\underline{P}(X_1|X_2)$ by

$$\begin{aligned}
&\underline{P}(f|X_1 = 1) = f(1,1)\\
&\underline{P}(f|X_1 = 2) = f(2,3)\\
&\underline{P}(f|X_1 = 3) = \min\{f(3,2), f(3,3)\}\\
&\underline{P}(f|X_2 = 1) = f(2,1)\\
&\underline{P}(f|X_2 = 2) = \min\{f(1,2), f(2,2), f(3,2)\}\\
&\underline{P}(f|X_2 = 3) = \min\{f(1,3), f(2,3), f(3,3)\},
\end{aligned}$$

for any gamble f in $\mathcal{L}(\mathcal{X}^2)$.

Let us consider the unconditional lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^2)$ given by $\underline{P}(f) = \min\{f(3,2), f(3,3)\}$. Using Theorem 1, we can see that $\underline{P}, \underline{P}(X_1|X_2)$ and $\underline{P}(X_2|X_1)$ are weakly coherent.

To see that $\underline{P}(X_1|X_2)$ and $\underline{P}(X_2|X_1)$ avoid partial loss, we apply Corollary 2 and consider any $f_1, f_2 \in \mathcal{L}(\mathcal{X}^2)$ such that $\overline{P}(A_{f_1,f_2}) = 0$. Let us prove that

$$\max_{\omega\in A_{f_1,f_2}} [G(f_1|X_2) + G(f_2|X_1)](\omega) \geq 0. \tag{4}$$

Assume $f_1 \neq 0 \neq f_2$; the other cases are similar (and easier). Since $\overline{P}(A_{f_1,f_2}) = 0$ for any coherent lower prevision that is weakly coherent with $\underline{P}(X_1|X_2)$ and $\underline{P}(X_2|X_1)$, we deduce that neither $(3,2)$ nor $(3,3)$ belong to A_{f_1,f_2}, and consequently $f_1(x,2) = f_1(x,3) = 0$ for $x = 1,2,3$. If $(X_1 = 2) \in S_1(f_2)$, then $[G(f_1|X_2) + G(f_2|X_1)](2,3) = 0 + 0 = 0$, and therefore Equation (4) holds. If

$(X_1 = 2) \notin S_1(f_2)$, then $[G(f_1|X_2) + G(f_2|X_1)](2,1) = 0 + 0 = 0$ (note that $(X_2 = 1)$ belongs to A_{f_1,f_2} or it would be $f_1 = 0$).

Let us prove finally that $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$ are not coherent. Let $f_1 = -I_{\{(1,1),(3,1)\}}, f_2 = -I_{\{(1,2),(1,3),(2,1),(2,2)\}}$ and $f_3 = I_{\{(2,3),(3,3)\}}$, and let us show that

$$[G(f_1|X_2) + G(f_2|X_1) - G(f_3|X_2 = 3)](\omega) < 0$$

for all $\omega \in A_{f_1,f_2,f_3}$. $S_2(f_1) = \{X_2 = 1\}$ and $S_1(f_2) = \{X_1 = 1, X_1 = 2\}$, whence $A_{f_3,f_1,f_2} := S_2(f_1) \cup S_1(f_2) \cup \{X_2 = 3\} = \mathcal{X}^2 \setminus \{(3,2)\}$. On the other hand, we have that

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
$G(f_1\|X_2)$	-1	0	0	0	0	0	-1	0	0
$G(f_2\|X_1)$	0	-1	-1	-1	-1	0	0	0	0
$-G(f_3\|X_2 = 3)$	0	0	0	0	0	-1	0	0	-1

and therefore the gamble $g := G(f_1|X_2) + G(f_2|X_1) - G(f_3|X_2 = 3)$ satisfies $g(\omega) = -1$ for all $\omega \in A_{f_3,f_1,f_2}$. This shows that $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$ are not coherent. However, $\overline{E}(A_{f_3,f_1,f_2}) = 1$ because $(3,3) \in A_{f_3,f_1,f_2}$. ♦

Hence, when a number of conditional lower previsions are weakly coherent but not coherent, the behaviour causing a contradiction can be caused by conditioning on sets of positive upper probability. It is interesting then to look for conditions under which it suffices to check the weak coherence of a number of previsions to be able to deduce their coherence. One such condition was established, in a different context, in [13].

In the case of conditional linear previsions, Theorem 4 allows us to derive immediately the following result:

Lemma 1. *Consider weakly coherent $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$, and let P be a coherent prevision such that $P, P_j(X_{O_j}|X_{I_j})$ are coherent for $j = 1, \ldots, m$. If $P(x) > 0$ for any $x \in \mathcal{X}_{I_j}$, $j = 1, \ldots, m$, then the conditional previsions $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$ are coherent.*

From this result, we can easily derive a similar condition for conditional lower previsions.

Theorem 5. *Consider weakly coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$, and let $\underline{P}$ be a coherent prevision such that $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ are coherent for $j = 1, \ldots, m$. If $\underline{P}(x) > 0$ for all $x \in \mathcal{X}_{I_j}$ and all $j = 1, \ldots, m$, then the conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are coherent.*

We can deduce from the proof of this theorem that if a number of weakly coherent conditional lower previsions avoid partial loss but are not coherent, for any gambles $f_0, \ldots, f_m$ violating Definition 5 it must be $\underline{E}(A_{f_0,f_1,\ldots,f_m}) = 0$ (although, as Example 1 shows, it can be $\overline{E}(A_{f_0,f_1,\ldots,f_m}) > 0$).

Note that when the conditioning events have all positive lower probability, the conditional lower previsions are uniquely determined by the joint $\underline{P}$ and by (GBR). Hence, in that case $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are the only conditional previsions which are coherent with $\underline{P}$.

4 Coherent updating

Although our last result is interesting, it is fairly common in situations of imprecise information to be conditioning on some sets of lower probability zero and positive upper probability. In that case, there is an infinite number of conditional lower previsions which are coherent with the unconditional $\underline{P}$. In this section, we characterise them by determining the smallest and the greatest coherent extensions.

4.1 Updating with the regular extension

The first updating rule we are going to consider is called the *regular extension.* Consider an unconditional lower prevision $\underline{P}$ and disjoint O, I in $\{1, \ldots, n\}$. The conditional lower prevision $\underline{R}(X_O|X_I)$ defined by regular extension is given, for any $f \in \mathcal{K}_{O\cup I}$ and any $x \in \mathcal{X}_I$ by

$$\underline{R}(f|x) := \inf \left\{ \frac{P(f\mathbb{I}_{\pi_I^{-1}(x)})}{P(x)} : P \geq \underline{P}, P(x) > 0 \right\}.$$

For this definition to be applicable, we need that $\overline{P}(x) > 0$ for any $x \in \mathcal{X}_I$. The regular extension is the lower envelope of the updated linear previsions using Bayes's rule. It has been used as an updating rule in a number of works in the literature [5, 7, 8, 10, 11, 18].

The conditional lower prevision defined using regular extension is not in general coherent with the unconditional lower prevision it is defined from, as it is discussed in Section 5 further on and in [17, Appendix J]. The following lemma shows that it is coherent in the context considered in this paper:

Lemma 2. *Let $\underline{P}, \underline{P}(X_O|X_I)$ be coherent unconditional and conditional previsions, with $\mathcal{X}_I$ finite. Assume that $\overline{P}(x) > 0$ for all $x \in \mathcal{X}_I$, and define $\underline{R}(X_O|X_I)$ from $\underline{P}$ using regular extension. Then:*

1. *$\underline{P}, \underline{R}(X_O|X_I)$ are coherent.*
2. *$\underline{R}(X_O|X_I) \geq \underline{P}(X_O|X_I)$.*
3. *For any $P \geq \underline{P}$, there exists some $P(X_O|X_I)$ which is coherent with P and dominates $\underline{P}(X_O|X_I)$.*

From this lemma, we deduce that if we use regular extension to define conditional lower previsions $\underline{R}_1(X_{O_1}|X_{I_1}), \ldots, \underline{R}_m(X_{O_m}|X_{I_m})$ from an unconditional $\underline{P}$, then $\underline{P}, \underline{R}_1(X_{O_1}|X_{I_1}), \ldots, \underline{R}_m(X_{O_m}|X_{I_m})$ are weakly coherent. Moreover, if we consider any other weakly coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$, it must hold that $\underline{R}_j(X_{O_j}|X_{I_j}) \geq \underline{P}_j(X_{O_j}|X_{I_j})$ for $j = 1, \ldots, m$. Hence, the procedure of regular extension provides the greatest, or more informative, updated lower previsions that are weakly coherent with $\underline{P}$. In the following theorem we prove that they are also coherent.

Theorem 6. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}^n)$, and consider disjoint O_j, I_j for $j = 1, \dots, m$. Assume that $\overline{P}(x) > 0$ for all $x \in \mathcal{X}_{I_j}$, and let us define $\underline{R}_j(X_{O_j}|X_{I_j})$ using regular extension for $j = 1, \dots, m$. Then the lower previsions $\underline{P}, \underline{R}_1(X_{O_1}|X_{I_1}), \dots, \underline{R}_m(X_{O_m}|X_{I_m})$ are coherent.*

When $\overline{P}(x) = 0$ for some $x \in \mathcal{X}_{I_j}$, $j = 1, \dots, m$, we cannot apply the procedure of regular extension to define $\underline{R}_j(X_{O_j}|x)$. It can be checked that we could use any separately coherent conditional lower prevision and still we would have weak coherence with $\underline{P}$. However, in that case we cannot guarantee the strong coherence, as we show in the following example:

Example 2. Let us consider $\mathcal{X}_1 = \mathcal{X}_2 = \{1, 2, 3\}$, and $P(X_1), P(X_2|X_1)$ given by $P(X_1 = 3) = 1$, and $P(X_2 = x|X_1 = x) = 1$ for $x = 1, 2, 3$. It follows from the marginal extension theorem that $P(X_1), P(X_2|X_1)$ are coherent. However, if we define arbitrarily $P(X_1|X_2 = x)$ when $P(X_2 = x) = 0$ (that is, for $x = 1, 2$), then $P(X_1|X_2)$ and $P(X_2|X_1)$ may not be coherent: make it for instance $P(X_1 = 1|X_2 = 2) = 1 = P(X_1 = 2|X_2 = 1) = P(X_1 = 3|X_2 = 3)$. Then it has been shown in [17, Example 7.3.5] that $P(X_1|X_2)$ and $P(X_2|X_1)$ are not coherent. ♦

From now on, we shall assume that the unconditional lower prevision $\underline{P}$ satisfies $\overline{P}(x) > 0$ for any conditioning event x, and that as a consequence we can use the procedure of regular extension to provide the most informative coherent extensions.

4.2 Updating with the natural extension

Next, we introduce the notion of *natural extension*. Let us consider conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{O_m})$ defined on respective linear spaces $\mathcal{H}^1, \dots, \mathcal{H}^m$ and avoiding partial loss. Given $j_0 \in \{1, \dots, m\}$, a gamble f on $\mathcal{X}^n$ and an element x_0 of $\mathcal{X}_{I_{j_0}}$, the natural extension $\underline{E}_{j_0}(f|x_0)$ is defined as the supremum α for which there are $f_j \in \mathcal{H}^j$, $j = 1, \dots, m$ such that

$$\left[\sum_{j=1}^{m} G(f_j|X_{I_j}) - I_{\pi^{-1}_{I_{j_0}}(x_0)}(f - \alpha)\right](\omega) < 0$$

for all $\omega \in A_{f_0,\dots,f_m}$. It is proven in [17, Theorem 8.1.9] that the lower previsions $\underline{E}_1(X_{O_1}|X_{I_1}), \dots, \underline{E}_m(X_{O_m}|X_{I_m})$ obtained in this way are the smallest coherent conditional previsions that dominate $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ on their domains.

Given disjoint subsets O_j, I_j of $\{1, \dots, n\}$ for $j = 1, \dots, m$, we can define separately coherent $\underline{P}_j(X_{O_j}|X_{I_j})$ on the set of constant gambles by $\underline{P}_j(\mu|x) = \mu$ for all $x \in \mathcal{X}_j$, $j = 1, \dots, m$. Then, given any coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X}^n)$ the lower previsions $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ avoid partial loss. We can thus consider their natural extensions $\underline{P}, \underline{E}_1(X_{O_1}|X_{I_1}), \dots, \underline{E}_m(X_{O_m}|X_{I_m})$ using the above definition. We deduce the following:

Theorem 7. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}^n)$. Consider disjoint O_j, I_j for $j = 1, \dots, m$, and let us define $\underline{E}_j(X_{O_j}|X_j)$, $j = 1, \dots, m$ using natural extension. Then $\underline{P}, \underline{E}_1(X_{O_1}|X_{I_1}), \dots, \underline{E}_m(X_{O_m}|X_{I_m})$ are coherent.*

Hence, the procedure of natural extension provides the smallest conditional lower previsions which are coherent together with $\underline{P}$. For any $j = 1, \dots, m$, $\underline{E}_j(X_{O_j}|X_{I_j})$ is uniquely determined by the (GBR) when $\underline{P}(x) > 0$ and are vacuous when $\underline{P}(x) = 0$, being then defined by $\underline{E}_j(f|x) = \min_{\omega \in \pi_{I_j}^{-1}(x)} f(\omega)$. Hence, in that respect the natural extensions can be calculated more easily than the regular extensions.

We showed before that the conditional previsions defined by regular extension were also the greatest conditional lower previsions that are weakly coherent with the unconditional lower prevision $\underline{P}$. Using Theorem 1 and the results in [17, Chapter 6], it is not difficult to show that the natural extensions are the smallest weakly coherent extensions:

Theorem 8. *Let $\underline{P}$ be coherent on $\mathcal{L}(\mathcal{X}^n)$, and define conditional lower previsions $\underline{E}_1(X_{O_1}|X_{I_1}), \dots, \underline{E}_m(X_{O_m}|X_{I_m})$ using natural extension in the manner described above. Then $\underline{P}, \underline{E}_1(X_{O_1}|X_{I_1}), \dots, \underline{E}_m(X_{O_m}|X_{I_m})$ are weakly coherent and any other conditional previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ which are weakly coherent with $\underline{P}$ satisfy that $\underline{P}_j(X_{O_j}|X_{I_j}) \geq \underline{E}_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$.*

4.3 On the range of coherent extensions

We see then that given an unconditional lower prevision $\underline{P}$ on $\mathcal{X}^n$, the smallest conditional lower previsions that are coherent with it are given by the natural extensions, and the greatest conditional lower previsions that are coherent with it are given by the regular extensions. They are also the smallest and greatest weakly coherent updated previsions. In general, these are not the only possibilities to update in a coherent way, so it is interesting to study the set of possible updated previsions.

In the following theorem, we prove that any conditional lower previsions that lie between the natural and the regular extensions are weakly coherent with $\underline{P}$ and avoid partial loss:

Theorem 9. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}^n)$. Consider disjoint O_j, I_j for $j = 1, \dots, m$, and let us consider separately coherent $\underline{P}_j(X_{O_j}|X_{I_j})$ such that $\underline{E}_j(X_{O_j}|X_{I_j}) \leq \underline{P}_j(X_{O_j}|X_{I_j}) \leq \underline{R}_j(X_{O_j}|X_{I_j})$. Then the lower previsions $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are weakly coherent and avoid partial loss.*

This theorem can be used as a test to verify the weak coherence of a number of conditional lower previsions with some unconditional $\underline{P}$: it suffices to check whether they lie between the natural and the regular extensions that can derive from $\underline{P}$.

On the other hand, since there can be conditional lower previsions that avoid partial loss but are not coherent, we deduce that not all of them are bounded between the natural and the regular extensions. To see one example, define $\underline{P}_j(X_{O_j}|X_{I_j})$ as $\underline{P}_j(f|x) = \min_{\omega \in \pi^{-1}_{I_j}(x)}$ for all $f \in \mathcal{K}^j, x \in \mathcal{X}_{I_j}$. Then, $\underline{P}_j(X_{O_j}|X_{I_j})$ are separately coherent and moreover $G_j(f|X_{I_j}) \geq 0$ for all $f \in \mathcal{K}^j$ for all j. From this we deduce that $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ avoid partial loss.

However, not any choice of conditional lower previsions between the natural and the regular extensions are coherent, as we show in the following example:

Example 3. Consider $\mathcal{X}_1 = \mathcal{X}_2 = \{1,2,3\}$, and let $\mathcal{M}$ be the set of probability mass functions on $\mathcal{X}_1 \times \mathcal{X}_2$ satisfying $P(1,2) = P(2,2) = P(3,1) = 0, P(1,1) = P(2,1), P(1,1) \geq P(1,3), P(2,1) \leq P(2,3)$, where the first index denotes the value of X_1 and the second the value of X_2. Let $\underline{P}$ be the lower envelope of the set $\mathcal{M}$. It is a coherent lower prevision, and satisfies moreover $\overline{P}(X_1 = x) > 0$ for any $x \in \mathcal{X}_1, \overline{P}(X_2 = x) > 0$ for any $x \in \mathcal{X}_2$. Consider $\underline{P}(X_2|X_1)$ be defined from $\underline{P}$ using regular extension, and let $\underline{P}(X_1|X_2 = x)$ be defined from $\underline{P}$ by natural extension if $x = 3$ and by regular extension otherwise. It follows from Theorem 9 that $\underline{P}(X_2|X_1), \underline{P}(X_1|X_2)$ are weakly coherent and avoid partial loss. Let us show that they are not coherent.

Consider the gambles f_1, f_2, f_3 on $\mathcal{X}_1 \times \mathcal{X}_2$ given by the following table:

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
f_1	1	0	0	0	0	3	0	0	0
f_2	0	0	0	2	0	0	0	0	0
f_3	0	0	0	0	0	2	0	0	2

Then $G(f_1|X_1), G(f_2|X_2)$ and $-G(f_3|X_2 = 3)$ are given by

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
$G(f_1\|X_1)$	0.5	-0.5	-0.5	-1.5	-1.5	1.5	0	0	0
$G(f_2\|X_2)$	-1	0	0	1	0	0	-1	0	0
$-G(f_3\|X_2 = 3)$	0	0	0	0	0	-2	0	0	-2

As a consequence, $[G(f_1|X_1) + G(f_2|X_2) - G(f_3|X_2 = 3)](x) < 0$ for any $x \in A_{f_3,f_1,f_2} = (\mathcal{X}_1 \times \mathcal{X}_2) \setminus (3,2)$. ♦

It also follows from Theorem 9 that since a number of conditional lower previsions are weakly coherent with an unconditional lower prevision $\underline{P}$ if and only if they lie between the natural and the regular extensions, a convex combination of weakly coherent conditional lower previsions is again weakly coherent with $\underline{P}$. It is not very difficult to show that this property does not hold for the stronger notion of coherence.

On the other hand, we can prove that, even if not all the conditional lower previsions bounded between the natural and the regular extensions are coherent, these can be used to determine the set of updated previsions for any particular gamble. This is detailed in the following result:

Theorem 10. *Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}^n)$, and consider disjoint O_j, I_j for $j = 1, \dots, m$. For any $j \in \{1, \dots, m\}$, $f \in \mathcal{K}^j, x \in \mathcal{X}_{I_j}$ and any $a \in [\underline{E}_j(f|x), \underline{R}_j(f|x)]$, there are coherent $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ such that $\underline{P}_j(f|x) = a$.*

We conclude this section by remarking that, in the case where we only have a conditional and an unconditional lower prevision, weak and strong coherence are equivalent, and therefore they will be coherent if and only if the conditional lower prevision is bounded between the natural and the regular extensions determined by the unconditional lower prevision.

5 Going from finite to infinite spaces

The results in the previous sections provide us with tools for updating a lower prevision in a coherent way. In this section, we are going to discuss which of the properties we have established hold when some of the spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$ are infinite.

The first thing we have to remark is that the definitions of avoiding partial loss and coherence and strong coherence are slightly different: we say that a number of conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ avoid partial loss when for any $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$, there is some $B \in \cup_{j=1}^m S_j(f_j)$ such that

$$\sup_{\omega \in B} \left[\sum_{j=1}^m G_j(f_j|X_{I_j}) \right](\omega) \geq 0, \tag{5}$$

and we say that they are coherent when for every $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$, $j_0 \in \{1, \dots, m\}$, $f_0 \in \mathcal{K}^{j_0}, x_0 \in \mathcal{X}_{I_{j_0}}$ there is some $B \in \pi_{I_{j_0}}^{-1}(x_0) \cup \cup_{j=1}^m S_j(f_j)$ such that

$$\sup_{\omega \in B} \left[\sum_{j=1}^m G_j(f_j|X_{I_j}) - G_{j_0}(f_0|x_0) \right](\omega) \geq 0.$$

It is easy to see that when $\mathcal{X}_1, \dots, \mathcal{X}_n$ are all finite these conditions agree with the ones given in Definitions 3 and 5, respectively. If in particular we have an unconditional lower prevision $\underline{P}$ on $\mathcal{X}^n$ and a conditional lower prevision $\underline{P}(X_O|X_I)$ on $\mathcal{K}_{O \cup I}$, the Generalised Bayes' Rule (GBR) is only necessary for coherence, and we need to require moreover that $\underline{P}(G(f|X_I)) \geq 0$ for any $f \in \mathcal{K}_{O \cup I}$.

Let us see whether the relationships established in this paper also hold in the case of infinite spaces. The characterisation of weak coherence given in Theorem 1 holds irrespective of the cardinality of the spaces, as it is established in [13]. On the other hand, in general weakly coherent conditional lower previsions are not necessarily the envelopes of a family of weakly coherent linear previsions: this follows because in the case of a conditional and an unconditional lower prevision weak and strong coherence are equivalent, and Walley gives in [17, Example 6.6.10] and example of coherent $\underline{P}, \underline{P}(X|\mathcal{B})$ which are not dominated by any coherent linear $P, P(X|\mathcal{B})$.

It is easy to see that the proof of Theorem 3, given in Section A.3 of the Appendix, also holds in the case of infinite spaces. As a consequence, given a number of weakly coherent conditional lower previsions there is always a smallest coherent lower prevision which is coherent with each of them, and it is given by Equation (3).

Let us comment next on the relationships between weak coherence and avoiding partial loss. Let $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ be weakly coherent conditional lower previsions. If they incur partial loss, there are gambles $f_1, \ldots, f_m$ for which Equation (5) does not hold. Using a proof similar to that of Corollary 2 (see Section A.6 in the Appendix), it can be checked that $\overline{E}(B) = 0$ for all $B \in S_i(f_i)$ and all $i = 1, \ldots, m$. In the finite case, this and the subadditivity of $\overline{E}$ imply that $\overline{E}(A_{f_1,\ldots,f_m}) = 0$. However, this is not the case when the spaces are infinite; see [17, Example 7.4.4] for a counterexample.

With respect to the sufficient conditions for weak coherence to guarantee coherence, note that Theorem 5 also holds in the infinite case. This is established is the following theorem, which also provides an alternative proof for Theorem 5:

Theorem 11. *Consider weakly coherent* $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$*, and let* $\underline{P}$ *be a coherent prevision such that* $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ *are coherent for* $j = 1, \ldots, m$*. If* $\underline{P}(x) > 0$ *for all* $x \in \mathcal{X}_{I_j}$ *and all* $j = 1, \ldots, m$*, then the conditional lower previsions* $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ *are coherent.*

Note however that the interest of this theorem is limited because there can only be an countable number of x for which $\underline{P}(x) > 0$, so in order to apply this result all the conditioning spaces must be countable.

Let us turn now to the range of coherent extensions. First of all, in the case of infinite spaces, the conditional lower previsions defined by regular extension are not necessarily coherent with the unconditional lower prevision $\underline{P}$; in fact, this may happen even if there are conditional lower previsions which are coherent with $\underline{P}$. This follows from the discussion in [17, Appendix J]. Because of this, it follows that Theorem 6 does not extend to the infinite case and as a consequence the regular extensions are not the greatest coherent extensions. We can prove nonetheless that they are a bound of any coherent extensions:

Theorem 12. *Let* $\underline{P}$ *be a coherent lower prevision on* $\mathcal{L}(\mathcal{X}^n)$*. Consider disjoint* O_j, I_j *for* $j = 1, \ldots, m$*, and separately coherent lower previsions* $\underline{P}_j(X_{O_j}|X_{I_j})$ *such that* $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ *are weakly coherent. Then for all* $j = 1, \ldots, m$ $\underline{P}_j(X_{O_j}|X_{I_j}) \leq \underline{R}_j(X_{O_j}|X_{I_j})$*.*

Similarly, if follows from [17, Section 8.1] that the conditional lower previsions defined by natural extension are not necessarily coherent, and that they are only a lower bound of any coherent extensions. Again, we can easily see that they provide a lower bound of any weakly coherent extensions:

Theorem 13. *Let* $\underline{P}$ *be a coherent lower prevision on* $\mathcal{L}(\mathcal{X}^n)$*. Consider disjoint* O_j, I_j *for* $j = 1, \ldots, m$*, and separately coherent lower previsions* $\underline{P}_j(X_{O_j}|X_{I_j})$

such that $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ *are weakly coherent. Then for all* $j = 1, \dots, m$ $\underline{P}_j(X_{O_j}|X_{I_j}) \geq \underline{E}_j(X_{O_j}|X_{I_j})$.

Note that this applies even in the case where we only consider a conditional and an unconditional lower prevision, and from this we can deduce that neither the regular nor the natural extensions may be weakly coherent with the joint $\underline{P}$. In this respect, it is also worth noting that in the infinite case a lower prevision $\underline{P}$ may not have conditional lower previsions which are coherent with it. For the existence of these conditional lower prevision, $\underline{P}$ needs to satisfy the condition of *conglomerability*, which is discussed in detail in [17, Section 6.8] and which is trivial in the finite case.

6 Conclusions

In this paper we have studied the difference between weak and strong coherence in the case of finite spaces, and established the smallest and greatest updated previsions. Although weak and strong coherence are not equivalent, it follows from our results that the smallest and greatest weakly coherent updated previsions coincide with the smallest and greatest coherent updated previsions, and are given by the natural and regular extensions, respectively.

The results we have established are valid for (unconditional) coherent lower previsions, and in particular may be applicable to the precise case, when we want to update a finitely additive probability. Let us discuss this in more detail. Consider an unconditional prevision (a finitely additive probability) on $\mathcal{L}(\mathcal{X}^n)$, and assume that we want to define conditional linear previsions $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$. If $P(z) > 0$ for any $z \in \mathcal{X}_{I_j}$ and for all $j = 1, \dots, m$, then the conditional linear previsions are uniquely determined from P by Bayes's Rule. They are the only conditional (lower, and therefore linear) previsions to be weakly coherent with P, and it follows from the results in Section 4.3 that they are also coherent. Hence, in that case there is a unique updating rule, and both natural and regular extensions coincide with the conditional previsions that we can define using Bayes's rule.

When $P(z) = 0$ for some $z \in \mathcal{X}_{I_j}$ and some $j = 1, \dots, m$, then we cannot apply Bayes' rule and any separately coherent conditional prevision $P_j(X_{O_j}|z)$ will be coherent with P. Note nevertheless that if we use this procedure to define $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$, i.e., considering arbitrarily defined conditional previsions when the conditioning event has probability zero, the previsions we end up with may not satisfy the property of coherence, as we can see from Theorem 4.

On the other hand, using the results in this paper we can provide weak coherent lower previsions with a sensitivity analysis interpretation: given an unconditional lower prevision $\underline{P}$ modelling our imprecise knowledge about some linear prevision P, it is equivalent to update it to a number of conditional lower previsions or to take the lower envelopes of a family of updated conditional linear previsions. This is the case both if we work with weak coherence or with coherence. Note moreover that such a property does not extend to the case of infinite spaces.

Finally, we would like to remark that one important assumption we have made throughout is that the domain of the conditional prevision $\underline{P}_j(X_{O_j}|X_{I_j})$ is the set $\mathcal{K}^j$ of the gambles that depend on the values that the variables in $O_j \cup I_j$ take, i.e., the set of $\mathcal{X}_{O_j \cup I_j}$-measurable gambles. An open problem would be to study the same problem when we want to define coherent conditional lower previsions on some smaller domains. Another open problem would be to establish some sufficient conditions for the properties discussed in Section 5 to hold on infinite spaces.

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A Proofs of Theorems

A.1 Proof of Lemma 2

Because we are dealing with finite spaces, the coherence of $\underline{P}, \underline{R}(X_O|X_I)$ is equivalent to $\underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{R}(f|x))) = 0$ for any $x \in \mathcal{X}_I$, and this condition holds because of [17, Appendix (J3)].

For the second statement, consider some x in $\mathcal{X}_I$ and $f \in \mathcal{K}_{O \cup I}$. Assume ex-absurdo that $\underline{R}(f|x) < \underline{P}(f|x)$. It follows from the definition of the regular extension that there is some $P \geq \underline{P}$ such that $P(x) > 0$ and $P(f|x) < \underline{P}(f|x)$. Since $P(x) > 0$, it follows from the Generalised Bayes Rule that $P(f|x)$ is the *unique* value satisfying $0 = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x)))$. As a consequence, given $\underline{P}(f|x) > P(f|x)$, we have that $\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x)) \geq \mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))$, whence

$$\begin{aligned}0 = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x))) &\geq P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))) \\ &\geq \underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x)) = 0,\end{aligned}$$

using the coherence of $\underline{P}, \underline{P}(X_O|X_I)$. But this implies that $P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x))) = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))) = 0$, and then there are two different values of μ for which $P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \mu)) = 0$. This is a contradiction.

Let us finally establish the third statement. Consider $P \geq \underline{P}$, and $x \in \mathcal{X}_I$. If $P(x) > 0$, then for any $f \in \mathcal{K}_{O \cup I}$ $P(f|x)$ is uniquely determined by the Generalised Bayes Rule and dominates the regular extension $\underline{R}(f|x)$. Hence, $P(f|x) \geq \underline{R}(f|x) \geq \underline{P}(f|x)$, where the last inequality follows from the second statement. Finally, if $P(x) = 0$, taking any element $P(X_O|x)$ of $\mathcal{M}(\underline{P}(X_O|x))$

we have that $P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x))) = 0$ for any $f \in \mathcal{K}_{O \cup I}$. This completes the proof.

A.2 Proof of Theorem 2

From Theorem 1, there exists a coherent lower prevision $\underline{P}$ on $\mathcal{X}^n$ which is coherent with $\underline{P}_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$. Consider $j \in \{1, \dots, m\}$, $x \in \mathcal{X}_{I_j}$ and $f \in \mathcal{K}^j$. If $\underline{P}(z) > 0$, then $\underline{P}_j(f|x)$ is uniquely determined by the Generalised Bayes Rule, and from [17, Section 6.4.2], it coincides with the regular extension. Hence, for any $\epsilon > 0$ there exists some $P \geq \underline{P}$ such that $P(x) > 0$ and $P(f|x) - \underline{P}_j(f|x) < \epsilon$. Given this P, we can apply Lemma 2 to define conditional previsions $P_i(X_{O_i}|x')$ for $i \neq j, x' \in \mathcal{X}_{I_i}$ and for $i = j$, $x' \in \mathcal{X}_{I_i}, x' \neq x$ such that P and $P_i(X_{O_i}|X_{I_i})$ are coherent for $i = 1, \dots, m$.

If $\underline{P}(x) = 0$, we consider some $P \geq \underline{P}$ such that $P(x) = 0$, and take $P(X_{O_j}|x) \in \mathcal{M}(X_{O_j}|x)$ such that $P(f|x) = \underline{P}(f|x)$. For any $x' \in \mathcal{X}_{I_i}, x' \neq x$ and any $x' \in \mathcal{X}_{I_j}, j \neq i$ for which $P(x') = 0$, we consider an arbitrary $P(X_{O_j}|x')$ in the credal set $\mathcal{M}(\underline{P}(X_{O_j}|x'))$. Finally, for any other x' we can apply Lemma 2 to define conditional previsions $P_i(X_{O_i}|x')$ for $i \neq j, x' \in \mathcal{X}_{I_i}$ and for $i = j$, $x' \in \mathcal{X}_{I_i}, x' \neq x$ such that P and $P_i(X_{O_i}|X_{I_i})$ are coherent for $i = 1, \dots, m$.

In any of the two cases, we end up with a family of conditional previsions $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ which are weakly coherent (P is a compatible joint), dominate $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ and s.t. $P_j(f|x) - \underline{P}_j(f|x) < \epsilon$. This shows that $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are the lower envelope of a family of weakly coherent conditional previsions.

A.3 Proof of Theorem 3

We prove in [13, Theorem 1] that $\underline{E}$ is the a coherent lower prevision that is also coherent with $\underline{P}_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$. Let $\underline{P}_1$ be another coherent lower prevision with this property. Assume that there is some gamble f such that $\underline{P}_1(f) = \underline{E}(f) - \delta$ for some $\delta > 0$. It follows from the definition of $\underline{E}$ that there exist $f_j \in \mathcal{K}^j$ for $j = 1, \dots, m$ such that

$$\sup_{x \in \mathcal{X}^n} \left[\sum_{j=1}^{m} G(f_j|X_{I_j}) - (f - (\underline{P}_1(f) + \frac{\delta}{2})) \right](x) < 0,$$

whence

$$\sup_{x \in \mathcal{X}^n} \left[\sum_{j=1}^{m} G(f_j|X_{I_j}) - (f - \underline{P}_1(f)) \right](x) < -\frac{\delta}{2},$$

contradicting the weak coherence of $\underline{P}_1, \underline{P}_j(X_{O_j}|X_{I_j})$, $j = 1, \dots, m$. Hence, $\underline{E}$ is the smallest coherent lower prevision s.t. $\underline{E}, \underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are coherent.

A.4 Proof of Corollary 1

From Theorem 3, we see that $\underline{E}$ is the smallest coherent lower prevision such that $\underline{E}, P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ are weakly coherent. From Theorem 2, the previsions $\underline{E}, P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ are the lower envelope of a class of dominating weakly coherent linear previsions. But since our conditional previsions are all linear, this means that $\underline{E}$ is the lower envelope of a class $\mathcal{M}$ of linear previsions P which are weakly coherent with the conditional previsions $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$.

Assume the existence of a linear prevision P which is weakly coherent with $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ and such that $P(f) < \underline{E}(f)$ for some gamble f'. If we define the coherent lower prevision $\underline{P}_1 := \min\{P, \underline{E}\}$, we would deduce that $\underline{P}_1, P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ are also weakly coherent, thus contradicting that $\underline{E}$ is the smallest coherent lower prevision which is weakly coherent with $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$. Therefore, $\underline{E}$ is the lower envelope of the set of linear previsions which are coherent with $P_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$.

A.5 Proof of Theorem 4

Because we are dealing with conditional linear previsions, coherence is equivalent to avoiding partial loss. Hence, we must verify whether for any $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$,

$$\max_{\omega \in A_{f_1,\dots,f_m}} \sum_{j=1}^{m} [f_j - P(f_j|X_{I_j})](\omega) \geq 0, \tag{6}$$

where $A_{f_1,\dots,f_m} := \cup\{B : B \in \cup_{j=1}^m S_j(f_j)\}$.

It is clear that if Equation (6) holds for any $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$, it also holds for any gambles $f_1, \dots, f_m$ satisfying $\overline{E}(A_{f_1,\dots,f_m}) = 0$. Conversely, assume that this condition holds. If $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ are not coherent, there must be $f_j \in \mathcal{K}^j$, $j = 1, \dots, m$, such that $\overline{E}(A_{f_1,\dots,f_m}) > 0$ and

$$\max_{\omega \in A_{f_1,\dots,f_m}} \sum_{j=1}^{m} [f_j - P(f_j|X_{I_j})](\omega) \leq -\delta < 0$$

for some $\delta > 0$. Applying Corollary 1, there is some linear prevision P which is coherent with $P_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$ and such that $P(A_{f_1,\dots,f_m}) > 0$.

Let us define $g := \sum_{j=1}^{m} [f_j - P(f_j|X_{I_j})]$. The coherence of $P, P_j(X_{O_j}|X_{I_j})$ for $j = 1, \dots, m$ implies that $P(f_j) = P(P_j(f_j|X_{I_j}))$ for $j = 1, \dots, m$, and the linearity of P implies then that

$$P(g) = \sum_{j=1}^{m} P_j(f_j - P(f_j|X_{I_j})) = 0.$$

But on the other hand we have that

$$P(g) = P(gI_{A_{f_1,\dots,f_m}}) \leq P(-\delta I_{A_{f_1,\dots,f_m}}) = -\delta P(A_{f_1,\dots,f_m}) < 0.$$

This is a contradiction. Therefore, it suffices to verify the coherence condition on those gambles whose union of supports have upper probability zero.

A.6 Proof of Corollary 2

$\underline{P}_1(X_{O_1}|X_{I_1}),\dots,\underline{P}_m(X_{O_m}|X_{I_m})$ avoid partial loss if and only if for any $f_j \in \mathcal{K}^j$, $j=1,\dots,m$,

$$\max_{\omega \in A_{f_1,\dots,f_m}} \sum_{j=1}^m [f_j - \underline{P}_j(f_j|X_{I_j})](\omega) \geq 0.$$

It is clear that if this condition holds it also holds in particular for gambles $f_1,\dots,f_m$ with $\overline{P}(A_{f_1,\dots,f_m})=0$. Conversely, assume that this holds but that there are $f_1,\dots,f_m$ such that $\overline{P}(A_{f_1,\dots,f_m})>0$ and

$$\max_{\omega \in A_{f_1,\dots,f_m}} \sum_{j=1}^m [f_j - \underline{P}_j(f_j|X_{I_j})](\omega) \leq -\delta < 0.$$

Let us define $g := \sum_{j=1}^m [f_j - \underline{P}_j(f_j|X_{I_j})]$. Since $\underline{E}$ and $\underline{P}_j(X_{O_j}|X_{I_j})$ are coherent for $j=1,\dots,m$, we deduce that $\underline{E}(f_j - \underline{P}_j(f_j|X_{I_j})) \geq 0$ for $j=1,\dots,m$. The super-additivity of the coherent lower prevision $\underline{E}$ implies then that

$$\underline{E}(g) = \underline{E}(\sum_{j=1}^m [f_j - \underline{P}_j(f_j|X_{O_j})]) \geq \sum_{j=1}^m \underline{E}(f_j - \underline{P}_j(f_j|X_{O_j})) \geq 0.$$

But on the other hand, we have that

$$\underline{E}(g) = \underline{E}(gI_{A_{f_1,\dots,f_m}}) \leq \underline{E}(-\delta I_{A_{f_1,\dots,f_m}}) = -\delta \overline{E}(A_{f_1,\dots,f_m}) < 0.$$

This is a contradiction. Therefore, it suffices to verify the avoiding partial loss condition on those gambles whose union of supports has upper probability zero.

A.7 Proof of Lemma 1

Since all the conditional previsions are linear, coherence is equivalent to avoiding partial loss. Consider then $f_j \in \mathcal{K}^j$ for $j=1,\dots,m$, and let us prove the existence of some $x^* \in A_{f_1,\dots,f_m}$ such that

$$\sum_{j=1}^m G_j(f_j|X_{O_j})(x^*) \geq 0.$$

Since $P(x)>0$ for any $x \in \mathcal{X}_{I_j}$ and $j=1,\dots,m$, it follows that $P(A_{f_1,\dots,f_m})>0$. From the coherence of P and $P_j(X_{O_j}|X_{I_j})$ we deduce that $P(G_j(f_j|X_{O_j}))=0$ for $j=1,\dots,m$, and consequently given $g := \sum_{j=1}^m G_j(f_j|X_{O_j})$, we have $P(g)=0$.

If there was some $\delta>0$ such that $g(x) < -\delta$ for all $x \in A_{f_1,\dots,f_m}$, then $P(g) = P(gI_{A_{f_1,\dots,f_m}}) \leq -\delta P(A_{f_1,\dots,f_m}) < 0$, a contradiction. Since $A_{f_1,\dots,f_m}$ is finite, this implies the existence of some $x^* \in A_{f_1,\dots,f_m}$ such that $g(x^*) \geq 0$. This completes the proof.

A.8 Proof of Theorem 6

The conditional previsions defined by regular extension are all coherent with $\underline{P}$ from Lemma 2, and therefore the assumption of the theorem is compatible with weak coherence. Consider $f_j \in \mathcal{K}^j$ for $j = 1, \ldots, m$, $j_0 \in \{1, \ldots, m\}, x_0 \in \mathcal{X}_{I_{j_0}}$, $f_0 \in \mathcal{K}^{j_0}$. Then, for any $\epsilon > 0$ there is some $P \geq \underline{P}$ such that $P(x_0) > 0$ and $P_{j_0}(f_0|x_0) - \underline{P}_{j_0}(f_0|x_0) < \epsilon$. From Lemma 2, we can consider conditional linear previsions $P_1(X_{O_1}|X_{I_1}), \ldots, P_m(X_{O_m}|X_{I_m})$ such that $P_j(X_{O_j}|X_{I_j})$ dominates $\underline{P}_j(X_{O_j}|X_{I_j})$ and is coherent with P for all j, and such that moreover $P_{j_0}(f|x_0) - \underline{P}_{j_0}(f|x_0) < \epsilon$. As a consequence,

$$\sum_{j=1}^{m}(f_j - \underline{P}_j(f_j|X_{O_j})) - G_{j_0}(f_0|x_0) \geq \sum_{j=1}^{m}(f_j - P_j(f_j|X_{O_j})) - G_{j_0}(f_0|x_0) - \epsilon,$$

and if we let $g := \sum_{j=1}^{m}(f_j - P_j(f_j|X_{O_j})) - G_{j_0}(f_0|x_0)$ then it follows from the coherence of P and $P_j(X_{O_j}|X_{I_j})$ for all j that $P(g) = 0$.

Assume that there is some $\delta > 0$ such that for all $x \in A_{f_0,f_1,\ldots,f_m}$

$$[\sum_{j=1}^{m}(f_j - \underline{P}_j(f_j|X_{O_j})) - G_{j_0}(f_0|x_0)](x) < -\delta.$$

We deduce that $g(x) < -\delta$ for all $x \in A_{f_0,f_1,\ldots,f_m}$. The definition of the supports implies moreover that $g(x) = 0$ for any $x \notin A_{f_0,f_1,\ldots,f_m}$. Hence, $P(g) = P(gI_{A_{f_0,f_1,\ldots,f_m}}) < -\epsilon P(A_{f_0,f_1,\ldots,f_m}) < 0$, because $P(A_{f_0,f_1,\ldots,f_m}) \geq P(x_0) > 0$. This is a contradiction. As a consequence, there is some $x^* \in A_{f_0,f_1,\ldots,f_m}$ such that

$$[\sum_{j=1}^{m}(f_j - \underline{P}_j(f_j|X_{O_j})) - G_{j_0}(f_0|x_0)](x) \geq -\delta,$$

and since we can do this for any $\delta > 0$ this implies that the conditional lower previsions $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are coherent.

A.9 Proof of Theorem 8

Since $\underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are coherent because of Theorem 7, they are also weakly coherent. Consider $j \in \{1, \ldots, m\}$ and $\underline{P}_j(X_{O_j}|X_{I_j})$ which is coherent with $\underline{P}$. For any $x \in \mathcal{X}_{I_j}$, there are two possibilities: either $\underline{P}(x) > 0$, and then $\underline{P}_j(X_{O_j}|x)$ is uniquely determined by (GBR), whence $\underline{P}_j(X_{O_j}|x) = \underline{E}_j(X_{O_j}|x)$; or $\underline{P}(x) = 0$, and the separate coherence of $\underline{P}_j(X_{O_j}|x)$ implies that $\underline{P}_j(f|x) \geq \max_{\omega \in \pi_{I_j}^{-1}(x)} f(\omega) = \underline{E}_j(f|x)$ for any $f \in \mathcal{K}^j$.

Hence, for all $j = 1, \ldots, m$, any conditional lower prevision $\underline{P}_j(X_{O_j}|X_{I_j})$ which is coherent with $\underline{P}$ dominates the natural extension $\underline{E}_j(X_{O_j}|X_{I_j})$. Applying Theorem 1, we deduce that $\underline{E}_1(X_{O_1}|X_{I_1}), \ldots, \underline{E}_m(X_{O_m}|X_{I_m})$ are the smallest weakly coherent extensions.

A.10 Proof of Theorem 9

Let us prove first of all that $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are weakly coherent. From Theorem 1, it suffices to show that $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ are coherent for $j = 1, \ldots, m$, which in turn is equivalent to verifying that $\underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}_j(f|x))) = 0$ for all $j = 1, \ldots, m$, $x \in I_j$, and $f \in \mathcal{K}^j$.

Since $\underline{E}_j(f|x) \leq \underline{P}_j(f|x) \leq \underline{R}_j(f|x)$, it follows that $\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{E}_j(f|x)) \geq \mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}_j(f|x)) \geq \mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{R}_j(f|x))$, and consequently

$$0 = \underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{E}_j(f|x))) \geq P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}_j(f|x))) \geq \underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{R}_j(f|x))) = 0.$$

Hence, $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ are coherent for $j = 1, \ldots, m$ and consequently the previsions $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ are weakly coherent.

Let us show now that they also avoid partial loss. Consider $f \in \mathcal{L}(\mathcal{X}^n)$, $f_j \in \mathcal{K}^j$ for $j = 1, \ldots, m$, and let us prove that

$$\left[G(f) + \sum_{j=1}^{m} G_j(f_j|X_{O_j})\right](x) \geq 0 \tag{7}$$

for some $x \in A_{f,f_1,\ldots,f_m}$. If $f \neq 0$, $A_{f,f_1,\ldots,f_m} = \mathcal{X}^n$, and Equation (7) follows from weak coherence. Assume then that $f = 0$. Since $\underline{P}_j(X_{O_j}|X_{I_j}) \leq \underline{R}_j(X_{O_j}|X_{I_j})$ for $j = 1, \ldots, m$, we see that

$$[\sum_{j=1}^{m} f_j - \underline{P}_j(f_j|X_{I_j})](x) \geq [\sum_{j=1}^{m} f_j - \underline{R}_j(X_{O_j}|X_{I_j})](x)$$

for all $x \in A_{f_1,\ldots,f_m}$. Since $\underline{R}_1(X_{O_1}|X_{I_1}), \ldots, \underline{R}_m(X_{O_m}|X_{I_m})$ avoid partial loss because they are coherent, we deduce that there is some $x^* \in A_{f_1,\ldots,f_m}$ such that $[\sum_{j=1}^{m} f_j - \underline{R}_j(X_{O_j}|X_{I_j})](x^*) \geq 0$, and consequently Equation (7) holds.

A.11 Proof of Theorem 10

Given such f, x and a, there exists some $\alpha \in [0,1]$ such that $a = \alpha\underline{E}_j(f|x) + (1-\alpha)\underline{R}_j(f|x)$. Let us define the conditional lower prevision $\underline{P}_j(X_{O_j}|X_{I_j})$ on $\mathcal{K}^j$ as $\underline{P}_j(X_{O_j}|X_{I_j}) = \alpha\underline{E}_j(X_{O_j}|X_{I_j}) + (1-\alpha)\underline{R}_j(X_{O_j}|X_{I_j})$, and for any $i \neq j$, let $\underline{P}_i(X_{O_i}|X_{I_i})$ be given on the class $\mathcal{H}$ of constant gambles by $\underline{P}_i(\mu|x) = \mu$ for all $x \in \mathcal{X}_{I_i}$. It follows from Theorem 9 that $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ are coherent and consequently so are $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$. If we consider then their natural extensions $\underline{P}, \underline{P}'_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}'_m(X_{O_m}|X_{I_m})$, where $\underline{P}'_i(X_{O_i}|X_{I_i})$ is defined on $\mathcal{K}^i$, it follows from [17, Theorem 8.1.2] that the previsions $\underline{P}, \underline{P}'_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}'_m(X_{O_m}|X_{I_m})$, are coherent. Since moreover we have $\underline{P}'_j(X_{O_j}|X_{I_j}) = \underline{P}_j(X_{O_j}|X_{I_j})$, we deduce that $\underline{P}'_j(f|x) = a$.

A.12 Proof of Theorem 11

We prove the result for the case where some of the spaces may be infinite; the case where $\mathcal{X}_1, \dots, \mathcal{X}_n$ are finite (Theorem 5) follows as a corollary. Let $f_i \in \mathcal{K}^i$ for $i = 1, \dots, m$, $j_0 \in \{1, \dots, m\}, z_0 \in \mathcal{X}_{j_0}$ and $f_0 \in \mathcal{K}^{j_0}$, and let us prove that for the mapping f given by

$$f(x) := \sum_{j=1}^{m} [G(f_j|X_j) - G_{j_0}(f_0|z_0)](x)$$

for all $x \in \mathcal{X}^n$, $\sup_{x \in \pi^{-1}_{I_{j_0}}(z_0)} f(x) \geq 0$.

It follows from [17, Theorem 6.4.2] that there is some $P \geq \underline{P}$ s.t. $P_{j_0}(f_0|z_0) = \underline{P}_{j_0}(f_0|z_0)$. Let $P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ be conditional linear previsions defined from P by Bayes' rule. Then $P, P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ are weakly coherent and moreover $P(z_0) > 0$.

Assume ex-absurdo that $f(x) < -\delta < 0$ for all $x \in \pi^{-1}_{I_{j_0}}(z_0)$. Since for all $j = 1, \dots, m$ $P_j(X_{O_j}|X_{I_j}) \geq \underline{P}_j(X_{O_j}|X_{I_j})$, it follows that

$$\sum_{j=1}^{m} [f_j - P_j(f_j|X_j) - G_{j_0}(f_0|z_0)](x) \leq \sum_{j=1}^{m} [f_j - \underline{P}_j(f_j|X_j) - G_{j_0}(f_0|z_0)](x)$$

for all x. If we denote $g := \sum_{j=1}^{m} [f_j - P_j(f_j|X_j) - G_{j_0}(f_0|z_0)]$, it follows from the linearity of P and the weak coherence of $P, P_1(X_{O_1}|X_{I_1}), \dots, P_m(X_{O_m}|X_{I_m})$ that $P(g) = 0$. But on the other hand we have that $P(g) \leq P(gI_{\pi^{-1}_{I_{j_0}}(z_0)}) \leq P(-\delta z_0) = -\delta P(z_0) < 0$. This is a contradiction. Hence, $\sup_{x \in \pi^{-1}_{I_{j_0}}(z_0)} f(x) \geq 0$, and therefore $\underline{P}_1(X_{O_1}|X_{I_1}), \dots, \underline{P}_m(X_{O_m}|X_{I_m})$ are coherent.

A.13 Proof of Theorem 12

Consider some x in $\mathcal{X}_I$, and $f \in \mathcal{K}_{O \cup I}$. Assume ex-absurdo that $\underline{R}(f|x) < \underline{P}(f|x)$. It follows from the definition of the regular extension that there is some $P \geq \underline{P}$ such that $P(x) > 0$ and $P(f|x) < \underline{P}(f|x)$. Since $P(x) > 0$, it follows from (GBR) that $P(f|x)$ is the *unique* value satisfying $0 = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x)))$. As a consequence, given $\underline{P}(f|x) > P(f|x)$, we have that $\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x)) \geq \mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x)$, whence

$$\begin{aligned} 0 = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x))) &\geq P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))) \\ &\geq \underline{P}(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x)) = 0, \end{aligned}$$

using the coherence of $\underline{P}, \underline{P}(X_O|X_I)$. But this implies that $P(\mathbb{I}_{\pi_I^{-1}(x)}(f - P(f|x))) = P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \underline{P}(f|x))) = 0$, and then there are two different values of μ for which $P(\mathbb{I}_{\pi_I^{-1}(x)}(f - \mu)) = 0$. This is a contradiction.

A.14 Proof of Theorem 13

The result follows once we remark that the weak coherence of the lower previsions $\underline{P}, \underline{P}_1(X_{O_1}|X_{I_1}), \ldots, \underline{P}_m(X_{O_m}|X_{I_m})$ implies that $\underline{P}, \underline{P}_j(X_{O_j}|X_{I_j})$ are coherent for all $j = 1, \ldots, m$. Applying [17, Theorem 8.1.2], $\underline{P}_j(X_{O_j}|X_{I_j}) \geq \underline{E}_j(X_{O_j}|X_{I_j})$ for all $j = 1, \ldots, m$.

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