On the comonotone natural extension of marginal p-boxes

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Comonotone probability measures

0. Notation

Interval notation: $[a, b]$

Interval dominance: $a \leq b$ if $a \leq b$ and $\pi \leq b$

Strict interval dominance: $a \leq b$ if $a \leq b$ and $\pi \neq b$

1. Definition [1]

$\{x, y\}, P_{X,Y}(\{(x, y)\}) > 0$ is an increasing set in $\mathbb{R}^2$

$P_{X,Y}$ comonotone $\iff$ $X = h(y)$, $h$ increasing

$F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\} \forall (x, y)$

2. Main Property

Given $X$ and $Y$ characterised by their CDFs $F_X$ and $F_Y$...

Existence ...there always exists a comonotone $P_{X,Y}$ with marginals $F_X$ and $F_Y$

Construction ...$F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$ $\rightarrow$ $P_{X,Y}$

Uniqueness ...the joint comonotone is unique

3. Construction: Example

Step 1: marginals

$F_X$ | $0.2$ | $0.5$ | $1$
|---|---|---|
$y_1$ | $1$ | $1$ | $1$
|---|---|---|
$y_2$ | $0.8$ | $0.8$ | $0.8$
|---|---|---|
$y_3$ | $0.3$ | $0.3$ | $0.3$

Step 2a: apply the min

$F_X$ | $0.2$ | $0.5$ | $1$
|---|---|---|
$y_3$ | $0.2$ | $0.5$ | $1$
|---|---|---|
$y_2$ | $0.2$ | $0.5$ | $0.8$
|---|---|---|
$y_1$ | $0.2$ | $0.5$ | $0.3$

Step 3: $P_{X,Y}$

$P_{X,Y}$ | $0.2$ | $0.3$ | $0.5$
|---|---|---|
$y_1$ | $0$ | $0$ | $0.2$
|---|---|---|
$y_2$ | $0$ | $0.2$ | $0.3$
|---|---|---|
$y_3$ | $0.2$ | $0.3$ | $0.3$

Step 2b: apply the min

$F_X$ | $0.2$ | $0.5$ | $1$
|---|---|---|
$y_3$ | $0.2$ | $0.5$ | $1$
|---|---|---|
$y_2$ | $0.2$ | $0.5$ | $0.8$
|---|---|---|
$y_1$ | $0.2$ | $0.5$ | $0.3$

$F_{X,Y}$ | $x_1$ | $x_2$ | $x_3$
|---|---|---|
$y_1$ | $0$ | $0$ | $0.2$
|---|---|---|
$y_2$ | $0$ | $0.2$ | $0.3$
|---|---|---|
$y_3$ | $0.2$ | $0.3$ | $0.3$

Comonotone lower probabilities

1. Definition [2]

$P$ comonotone $\forall P \in \mathcal{M}(P_{X,Y})$

Supp$(P_{X,Y}) = \{(x, y) \mid P_{X,Y}(\{(x, y)\}) > 0\}$ is an increasing set in $\mathbb{R}^2$

$P_{X,Y}$ comonotone lower probability $\iff P_{X,Y}(x, y) = \min\{P_X(x), P_Y(y)\}$

$F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\} \forall (x, y)$

2. Aim of the paper

$X$ and $Y$ discrete and represented by $(E_X, F_X)$ and $(E_Y, F_Y)$

Definition: $P_{X,Y}$ is comonotone extension if it is comonotone and its marginal p-boxes are $(E_X, F_X)$ and $(E_Y, F_Y)$

Aim 1: existence, construction and uniqueness of the comonotone extension

Aim 2: in case of non-uniqueness, existence and construction of the comonotone natural extension

3. Existence of a comonotone extension

Theorem: the comonotone extension exists if and only if there is interval dominance between $F_X(x)$ and $F_Y(y)$ for every $(x, y)$

4. Construction of a comonotone extension: Example

Step 1: marginals

$F_X$ | $0.2$ | $0.4$ | $1$
|---|---|---|
$y_1$ | $0.4$ | $0.8$ | $1$
|---|---|---|
$y_2$ | $0.4$ | $0.8$ | $1$
|---|---|---|
$y_3$ | $0.4$ | $0.8$ | $1$

Step 2: apply the min

$y_1$ | $0.2$, $0.2$ | $0.4$, $0.4$ | $0.4$, $0.4$
|---|---|---|
$y_2$ | $0.2$, $0.2$ | $0.4$, $0.4$ | $0.4$, $0.4$
|---|---|---|
$y_3$ | $0.2$, $0.2$ | $0.4$, $0.4$ | $0.4$, $0.4$

Step 3: set $S$

$S = \{(x, y) \mid F_{X,Y}(x, y) < F_X(x, y_1) \land F_{X,Y}(x, y_2) < F_X(x, y_2) \land F_{X,Y}(x, y_3) < F_X(x, y_3)\}$

$\rightarrow$ increasing set

Step 4: correspondence

$S \leftrightarrow Z \leftrightarrow Z$

Step 5: focal events [3]

$(F_E, F_Z) \rightarrow P_Z$ belief function $\rightarrow m_Z$

$E_1, \ldots, E_5$ $\rightarrow$ focal events

$F_i = g^{-1}(E_i)$, $i = 1, \ldots, 5$

$\downarrow$

$m_{X,Y}(F_i) = m_{X,Y}(E_i) \rightarrow E_{X,Y}$

Step 6: comonotone $P_{X,Y}$

$E_{X,Y} \rightarrow F_{X,Y}$

5. Uniqueness of the comonotone extension

Not unique!

6. Comonotone natural extension

Definition: $E_{X,Y}$ is the comonotone natural extension if it is a comonotone extension and $E_{X,Y} \leq P_{X,Y}$ for any other comonotone extension $P_{X,Y}$

Theorem: the comonotone natural extension exists if and only if for any $(x, y)$ either there is strict interval dominance between $F_X(x)$ and $F_Y(y)$ or $F_X(x) = F_X(y) = F_Y(y)$

References


Summary

Comonotone extension

Existence Not always $\times$ Characterization $\checkmark$ Belief function $\checkmark$

Construction Simple! $\checkmark$

Uniqueness No $\times$

Comonotone natural extension

Existence Not always $\times$ Characterization $\checkmark$ Belief function $\checkmark$

Construction Simple! $\checkmark$

Future work...

Continuous case

Application to finance and DM

At a glance

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