Renewal theory

¿with imprecisions?

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The goal

We want to make things work but we are not **sure** how to do so!

- The state of system depends on its environment.
- Some states are desirable, some are not - those we call failures.
- Assessment: Given uncertain models, what is the chance that a system will fail?
- Decision making: Given uncertain models, what is the best possible design choice to “ensure” system functionality? (Or how to properly balance reliability and expenses?)

Aging models

For many systems of interest, failures are often delayed!

Assume that:

- System state X is binary - either functional or failed ($X \in \{1, 0\}$).
- System is functional at time 0.
- Once system fails, it remains in failed state.

Then, the system state process $X(t)$ can be equivalently described by a RV $T > 0$, representing the **time to failure**, and
$Pr(X(t) = 1) = Pr(T > t)$.

Given some continuity assumptions, T can be uniquely described by either of $F, R, f, r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the distribution, survival, probability density, and hazard rate functions, respectively.

Distribution of T can be inferred from observations of failures.

Classification of failure laws [Barlow and Proschan 1967]

Scientists love their zoos!

The hazard rate function $r(t) := \lim_{\Delta \to 0} \frac{Pr(T \in [t;t+\Delta] | T > t)}{\Delta}$ is effectively the transition rate from functional to failed state.

Many useful properties may be derived for failure laws belonging to specific classes.

T is said to be of:

- Increasing Failure Rate if $r(t)$ is increasing.
- Increasing Failure Rate in Average if $\int_0^t r(x)dx/t$ is increasing.
- New Better than Used if $R(x|t) \leq R(x); \forall x \geq t \geq 0$.
- ...

Imprecise reliability [Lai and Xie 2006]

Engineers have been doing IP since at least the 60s!

Qualitative assessments may be made, which helps us to construct bounds on the survival function ($Pr(T \geq t)$).

If T is IFR and μ_r its r-th general moment, then:

$$R(t) \geq \exp(-\alpha t) \quad \forall t < \mu_r^{1/r}, \qquad \alpha = [\Gamma(r+1)/\mu_r]^{1/r}.$$

If T is IFRA and ξ_p its $p-$th quantile and α the rate of exponential distribution with the same quantile, then:

$$R(t) \begin{cases} \geq \exp(-\alpha t) & 0 \leq t \leq \xi_p, \\ \leq \exp(-\alpha t) & t > \xi_p. \end{cases}$$

If T us NBU s.t. $R(x) = \alpha$, then:

$$R(t) \begin{cases} \geq \alpha^{1/k} & \frac{x}{k+1} \leq t \leq \frac{x}{k}, k \in \mathbb{N}, \\ \leq \alpha^k & kx \leq t \leq (k+1)x, k \in \mathbb{N}. \end{cases}$$

Complex systems

Systems are systems of systems!

- Systems, we use, are composed of sub-systems, states of which influence the state of the super-system.
- This dependency is modelled by a reliability function $h : \{p_1, \ldots, p_n\} \to [0, 1]$, which is monotone (for reasonable systems).

$\Rightarrow$ We can infer the distribution laws for the sub-systems (cheaper) and the dependency model to assess reliability of the super-system (also cheaper than breaking the system over and over again).

Maintenance

If a system fails, we can repair it, or buy a new one!

There exists many maintenance scenarios.

- Corrective maintenance (replacement upon discovery of failure)
- Preventive maintenance (replacement upon failure and at specified time)
- Partial maintenance (system is just made operational, not as good as new)

Both for:

- Systems with immediate repair - renewal process
- Systems with positive repair time - alternating process
- Systems with latent failures (e.g. we can observe state of the system only at "inspection times")
- Systems with finite amount of service personal and/or limited resources
- ...

Renewal process - definition

Let us assume that a system replaced by a new one immediately upon its failure. We can model the replacement times as a point process:

- Be $\{T_i; i \in \mathbb{N}\}$ positive RVs with corresponding laws F_i, R_i, f_i, r_i.
- Define $S_0 = 0$ and $S_n := \sum_{k=1}^{n} T_k$ with corresponding laws F^n, R^n, f^n, r^n,

$\Rightarrow$ $$F^n = F * F^{n-1} \text{ and } f^n = f * f^{n-1}$$

- The Point Process of interest is $\sum_{n=1}^{\infty} \delta_{S_n}$.

Figure: A part of a point process.

The renewal process $\{N(t) \in \mathbb{Z}_{\geq 0}; t \geq 0\}$ models the number of renewals in interval $(0, t]$, i.e.:

$$Pr(N(t) \geq n) = Pr(S_n \leq t)$$

Renewal process - properties

Over the years, the interest has been put on evaluating the mean of renewal process (for expected utility purposes).
Denote $M(t) := \mathbb{E}\{N(t)\}$ (aka the **renewal function**).
Then:

- $N(t)$ is semi-Markov process (Markov for constant failure rate)
- $(N(t), S_{N(t)})$ is Markov process on $\mathbb{Z} \otimes \mathbb{R}$
- $M(t) = \sum_{n=1}^{\infty} F^n(t)$ (by definition).
- $M(t) = F(t) + (F * M)(t)$ (renewal equation)
- $M(t) = M(S_n) + M(t - S_n)$ (if T_i are i.i.d, restarting property)
- $\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mathbb{E}\{T_1\}}$ (if T_i are i.i.d, **Renewal theorem**)

Computation of the renewal function [Osaki 2002]

For special classes of T_i i.i.d:

- Exponential with rate λ: $M(t) = \lambda t$.
- Phase-type distributed: $M(t) = \frac{t - v(t)T^{-1}e - 1}{\mathbb{E}[T]}$, $\quad \frac{dv}{dt}(t) = v(t)Q^*$.

For special classes of T_i, $M(t)$ may be bounded, e.g. for T_i i.i.d.:

- NBUE: $\frac{t}{\mathbb{E}[T]} - 1 \leq M(t) \leq \frac{t}{\mathbb{E}[T]}$.
- IFR: $\frac{t}{\mathbb{E}[T]} \leq M(t) \leq \frac{tF(t)}{\mathbb{E}[T]}$.

Otherwise, there is no closed solution and numerical methods have to be employed:

- Laplace inversion,
- Spline approximation,
- Rational function approximation.

Renewal reward process

Take, again, the point process $\{S_n; n \in \mathbb{Z}_{\geq 0}\}$.
Assign with each jump time S_n a random variable R_n, the reward/cost.
(Observe that for renewal process $R_n = 1$).
Define $R(t) = \sum_{i=1}^{N(t)} R_n$.
Then:

- $R(t)$ is semi-Markov.
- $(R(t), S_{N(t)})$ is Markov.
- $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}\{R_1\}}{\mathbb{E}\{T_1\}}$ (if T_i, R_i are i.i.d, **Renewal reward theorem**)

Example - optimal preventive maintenance

We replace component upon failure (corrective maintenance) OR after predefined time T^* (preventive replacement).
Preventive replacement costs c_1 and corrective replacement costs $c_2 > c_1$.
Then mean cost per cycle (interval between two replacements) is:

$$\frac{c_2 Pr(T \leq T^*) + c_1 Pr(T > T^*)}{\int_0^{T^*} R(t)dt}.$$

What is the optimal T^*?

For T with decreasing failure rate (New Worse than Used), $T^* = \infty$.
Generally, the solution satisfies:

$$r(T^*) \int_0^{T^*} R(t)dt - F(T^*) = \frac{c_1}{c_2 - c_1}.$$

We can do it easier with IP!!!

Alternating process

Be:

- $T_n, G_n; n \in \mathbb{N}$ respectively i.i.d. RVs. (time to failure, time to repair, respectively)
- $Z_n := T_n + G_n$, the length of the n-th cycle.
- $S_n := Z_{n-1} + T_n$, the time to n-th failure.
- $\{X(t); t \in \mathbb{R}_{\geq 0}\}$ a process of system state s.t.

$$X(t) := \begin{cases} 0 & ; \exists n : S_n \leq t < Z_n \\ 1 & ; else \end{cases}$$

- $A(t) := \mathbb{E}\{X(t)\}$, the **availability**.

Then:

$$\lim_{t \to \infty} A(t) = \frac{\mathbb{E}\{T_1\}}{\mathbb{E}\{Z_1\}}.$$

Alternating process computation

- If T, G are exponential with rates λ, μ respectively, then $A(t) = \frac{\lambda}{\lambda+\mu}(1 + \exp(-(\lambda+\mu)t))$.
- Generally, $U(t) := 1 - A(t)$ satisfies recursive formula:

$$U(t) = \int_0^t f_T(x)[1 - F_G(t-x)]dx + \int_0^t (f_T * f_G)(x)U(t-x)dx.$$

- Even more generally: Simulate.

Multi-state systems

Sometimes it is desirable to distinguish states of partial failure.
Be X in $\{0, 1, \ldots, K\}$, s.t. 0 represents failure state, K fully functional one and the rest partial failures.
We then have to model more general stochastic process $X(t)$.

Dynamic reliability

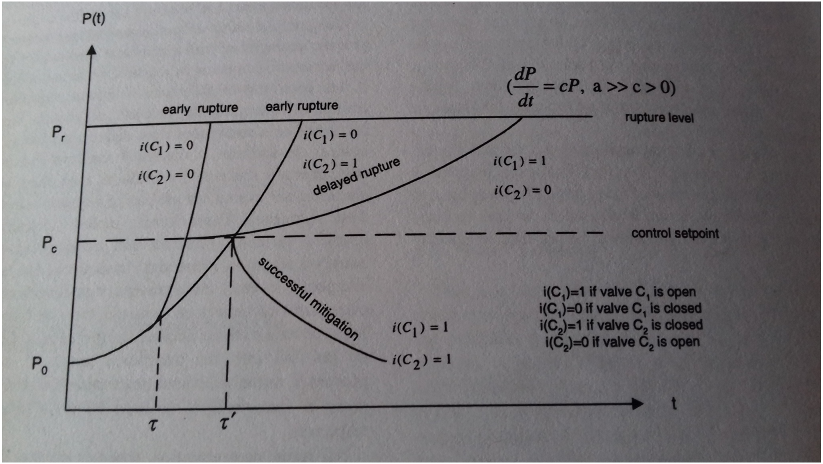

Figure: Piecewise deterministic Markov process; img. from [Labeau, Smidts, and Swaminathan 2000]

Conclusions

I should have stick with being an electrician - life would have been much easier now...
But much less interesting on the other hand!

- A lot of (coherent?) bounds has been discovered - possibly easily extended for imprecise assessments.
- Restarting property looks like worth exploiting.
- How does it look in Laplace's world?
- Can we find dominating processes, which are tight enough to make reasonable assessments?
- Can we simulate with imprecisely specified laws??!?!?!??!??!?!

Thank You for Your attention!

By the way...

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(you can follow us on @ResearchGate:project UTOPIAE)

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