Confidence intervals with imprecise data

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Abstract

The process of measuring a continuous variable is typically subject to imprecision due to causes as varied as measurement errors or rounding, and it is the duty of practitioners to take account of this imprecision when performing different statistical inference tasks. In this paper, we address the problem of accounting for this imprecision in the context of confidence intervals for parameters of probability distributions. For such purpose, we formalize the notions of inner and outer confidence interval, both of which generalize the classical notion of confidence interval in the presence of imprecision. Different properties of both mathematical constructs are here studied and their explicit expressions are provided in five prominent cases in the field of Statistics.

Keywords: confidence interval, imprecision, random set, rounding, measurement error

1 Introduction

Measurements of continuous variables are inherently imprecise. This imprecision is not only due to human limitations in the ability to perform accurate measurements and to annotate long decimal numbers, but also to computers being constrained by machine precision. For instance, the effects of rounding, which can be understood as a source of imprecision, have been known for decades to be catastrophic for some statistical tasks [33, 34]. Things become significantly more complicated when the source of imprecision is random, even though the problem still has attracted the attention of the statistical community [7].

Many proposals have aimed at establishing a framework for statistical analysis with imprecise data. Some examples are error-in-variables models [8, 20], Bayesian statistics [16], imprecise probabilities [25, 36], mixture models [26], robust statistics [1, 2, 23] and random sets [28, 30]. The last model is the one we shall consider in this paper. Formally, a random set is a generalization of the notion of random variable to the case in which more than one outcome from the random experiment is allowed. Random intervals [27], i.e., real-valued compact random sets, are the most natural type of random set for modelling imprecision since rounding, error measurements and other frequent sources of imprecision typically result in interval outcomes [19, 21].

A vast body of mathematical knowledge has been developed around the notion of random set [32], and two different schools of thought have been proposed when performing statistical inference with random sets; their differences lie in the ontic vs epistemic dichotomy ([9]) in their interpretation. On the one hand, some authors (see, e.g., [4]) proposed to directly extend statistical inference techniques as set-valued operations; on the other hand, other authors (see, e.g., [12]) have opted for abandoning the classical dichotomy (acceptance/rejection) in statistical inference and embracing a third option related to the indecision caused by the imprecision. In the present paper, we follow [11] and present a partition of the parameter space into three regions by means of two confidence intervals, that we shall call the inner and the outer confidence intervals. In particular, the values that belong to the inner confidence interval are those that are sure to belong to the precise confidence interval regardless of the information loss caused by imprecision, while those values that do not belong to the outer confidence interval are sure not to belong to the precise confidence interval regardless of the information loss caused by imprecision. One crucial problem here that arises is the computation of these intervals, that shall involve an optimization problem in the family of measurable selections of a random set. This is a problem whose difficulties have been partially addressed in some papers by searching the minimum and maximum variance of a sample for which only a lower and an upper bound for each observation are known ([18, 19, 31]); in this paper, we contribute to its solution by introducing the notion of quasivertices, that will allow us to significantly reduce the quest for the optimizers.

The remainder of the paper is structured as follows. Firstly, some basic concepts on random sets, random intervals and confidence intervals are introduced in Section 2. Our definition of the inner and outer confidence intervals is given in Section 3, and some properties related to their coverage and characterization are presented in Section 4. In Section 5, we discuss five relevant examples of confidence intervals and provide the explicit formulae of the inner and outer confidence intervals under imprecision. Finally, we end with some conclusions and future perspectives in Section 6. To facilitate the reading, we have gathered the proofs in an Appendix.

2 Preliminary concepts

In this section, we recall the basics on random sets [10, 30], random intervals [14] and confidence intervals [5] necessary for the development of the present paper.

As discussed in the introduction, in this paper we shall model the imprecision in the observational process by means of multi-valued mappings. We shall thus give them an *epistemic* interpretation, meaning that we assume that the 'true' outcome of the experiment is included within a specified set of possible outcomes. Formally, we shall work with random sets [13].

Definition 1 Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\tilde{\Omega}, \mathcal{A}')$ a measurable space and $\Gamma : \Omega \to \mathcal{P}(\tilde{\Omega})$ a multi-valued mapping. For any $A \in \mathcal{A}^{'}$, its *lower inverse* by Γ is given by $\Gamma_*(A) := \{\omega \in \Omega \mid \emptyset \neq \Gamma(\omega) \subseteq A\}$, and its *upper inverse* by Γ is given by $\Gamma^*(A) := \{\omega \in \Omega \mid \Gamma(\omega) \cap A \neq \emptyset\}$. Γ is a *random set* when $\Gamma^*(A) \in \mathcal{A}$ for any $A \in \mathcal{A}'$.

The above condition is called *strong measurability*, and it is equivalent to requiring that $\Gamma_*(A) \in \mathcal{A}$ for any $A \in \mathcal{A}^{'}$. It reduces to the standard measurability condition in the case of random variables. For other notions of measurability in the context of multi-valued mappings, we refer the reader to [22].

If we give a random set an epistemic interpretation, then it makes sense to assume that $\Gamma(\omega) \neq \emptyset$ for any $\omega \in \Omega$, since the set $\Gamma(\omega)$ should at the very least include the value $X(\omega)$ taken by the random variable X that is imprecisely observed. Thus, we shall assume throughout that the images of the random set are non-empty. Moreover, our information about the imprecisely observed variable is given by the set of measurable selections of the random set.

Definition 2 Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\tilde{\Omega}, \mathcal{A}')$ be a measurable space and $\Gamma : \Omega \to \mathcal{P}(\Omega^{'})$ a random set. A function $X : \Omega \to \tilde{\Omega}$ is called a *measurable selection* of Γ if X is measurable and $X(\omega) \in \Gamma(\omega)$ for any $\omega \in \Omega$. The set of all measurable selections of Γ is denoted by $S(\Gamma)$.

While many types of random sets can be used to model the imprecise observation of a random variable, in this paper, we shall focus on random intervals.

Definition 3 Consider a probability space $(\Omega, \mathcal{A}, P)$, the measurable space $(\mathbb{R}, \beta_{\mathbb{R}})$ and a random set $\Gamma : \Omega \to \mathcal{P}(\mathbb{R})$. The random set Γ is called a *random interval* when $\Gamma(\omega)$ is an interval for any $\omega \in \Omega$.

Here we shall mostly consider random *compact* intervals, i.e., random sets for which $\Gamma(\omega)$ is a compact interval for every $\omega \in \Omega$. In that case, Γ can be represented by the mappings $Y, Z : \Omega \to \mathbb{R}$ given by $Y(\omega) = \min \Gamma(\omega)$ and $Z(\omega) = \max \Gamma(\omega)$. Moreover [27, Theorem 3.1], a multi-valued mapping whose images are compact intervals of $\mathbb{R}$ is a random set iff the mappings Y, Z are random variables.

Random intervals appear quite naturally as a model for the imprecise measurement of a random variable: assume there exists a random variable X of interest that is not observable in a precise manner, and that instead another random variable $\widetilde{X}$ is observed. It is through $\widetilde{X}$ that two random variables Y and Z such that $Y \leq \widetilde{X} \leq Z$

are constructed. It is assumed that it also holds that $Y \leq X \leq Z$, thus resulting in a random interval $[Y, Z]$ of which X is a measurable selection. Two typical examples of such setting are the following:

- Measurement error: It holds that $\widetilde{X} = X + \varepsilon$, with ε a random and unknown absolute error. If there is a bound $K > 0$ such that $|X - \widetilde{X}| = |\varepsilon| \leq K$, it suffices to consider $Y = \widetilde{X} - K$ and $Z = \widetilde{X} + K$ for ensuring that $Y \leq X \leq Z$.
- Rounding: the observed variable is $\widetilde{X} = \varphi(X)$, with φ the rounding function at the d-decimal number. Then the functions $\varphi_1(x) = x - 5 \cdot 10^{-(d+1)}$ and $\varphi_2(x) = x - 5 \cdot 10^{-(d+1)}$, verifying that $\varphi_1(\varphi(X)) \leq X \leq \varphi_2(\varphi(X))$, give respectively the minimum and maximum number that could have resulted in a certain number x rounded at the d-decimal number. Therefore, since it follows that $\varphi_1(\widetilde{X}) \leq X \leq \varphi_2(\widetilde{X})$, it suffices to consider $Y = \varphi_1(\widetilde{X})$ and $Z = \varphi_2(\widetilde{X})$ for ensuring that $Y \leq X \leq Z$. Similar approaches may be used for data that has been subject to the floor or ceiling operations.

We finally introduce the main object of interest in this paper: confidence intervals [29]. Let $X : \Omega \to \mathbb{R}$ be a random variable, and let $X_1, \ldots, X_n$ be a *simple random sample* of X, i.e., a collection of independent random variables that are identically distributed to X. Assume also that the distribution of X is dependent on a parameter $\theta \in \mathbb{R}$.

Definition 4 Given $\alpha \in (0, 1)$, a random interval $CI[X_1, \ldots, X_n]$ constructed from a simple random sample $X_1, \ldots, X_n$ of a random variable X with distribution dependent on a parameter $\theta \in \mathbb{R}$ is called a *confidence interval* for θ at confidence level $1 - \alpha$ if, for any $\theta \in \mathbb{R}$:

$$P_\theta(\theta \in CI[X_1, \ldots, X_n]) \geq 1 - \alpha\,.$$

When the inequality above holds with equality, the confidence interval is said to be *tight*.

We will write $CI[X_1, \ldots, X_n]$ for referring to the confidence interval as a function of the simple random sample $X_1, \ldots, X_n$ and $CI[x_1, \ldots, x_n]$ to denote the interval that is obtained at a particular sample realization of $(X_1, \ldots, X_n)$.

3 Confidence intervals under imprecision

In this section, we shall extend the notion of confidence interval to the case of random variables that are imprecisely observed. For this aim, we shall first of all generalize the notion of simple random sample.

Definition 5 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. We say that X is observed with imprecision through the random variables $Y, Z : \Omega \to \mathbb{R}$ if X is a measurable selection of the random interval $[Y, Z]$. A collection $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ is called an *imprecise simple random sample* of $(X; Y, Z)$ if $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ are simple random samples of Y and Z and $Y_i(\omega) \leq Z_i(\omega)$ for all $i \in \{1, \ldots, n\}$ and all $\omega \in \Omega$.

The goal of this paper is to obtain confidence intervals for the parameter θ associated with a random variable $X \sim F_\theta$ that is observed with imprecision through Y, Z. We shall define two confidence intervals, that we shall call *inner* and *outer* confidence intervals. The first gathers the values of θ that belong to the confidence interval for all measurable selections of $[Y, Z]$, whereas the second considers those values of θ that belong to the confidence interval for at least one measurable selection.

Definition 6 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable with distribution F_θ dependent on a parameter $\theta \in \mathbb{R}$. Assume X is observed with imprecision through $Y, Z : \Omega \to \mathbb{R}$ and let $\alpha \in (0, 1)$. Given $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ an imprecise simple random sample of $(X; Y, Z)$, the *outer confidence interval* for θ at confidence level $1 - \alpha$ is defined as

$$CI^*[(Y_1, Z_1), \ldots, (Y_n, Z_n)] = \bigcup_{\tilde{X}_i \in S([Y_i, Z_i])} CI[\tilde{X}_1, \ldots, \tilde{X}_n]\,,$$

whereas the *inner confidence interval* for θ at confidence level $1 - \alpha$ is defined as

$$CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)] = \bigcap_{\tilde{X}_i \in S([Y_i, Z_i])} CI[\tilde{X}_1, \ldots, \tilde{X}_n].$$

These notions align with those of outer and inner confidence intervals considered in [11, Section 2.1][1]. In that paper, it is also considered the case where the imprecise observations are represented by fuzzy sets, and a comparison with the notion of fuzzy confidence interval by [24] is made; we will not consider fuzzy sets in this paper.

It is important to clarify that in the definitions above make use of the formula of the confidence interval $CI[X_1, \ldots, X_n]$ for the parameter θ given a confidence level $1 - \alpha$ and a simple random sample $X_1, \ldots, X_n$ of the random variable X that is imprecisely observed. However, this does not entail that $CI[\tilde{X}_1, \ldots, \tilde{X}_n]$ is necessarily a confidence interval for θ at the desired confidence level for all measurable selections $\tilde{X}_i \in S([Y_i, Z_i])$, since these may not have the same parametric distribution as X; only the random variable of interest is assumed to belong to that parametric family. We should also remark that an outer confidence interval is not necessarily an interval (i.e., it need not be convex). We shall look at this issue in the following section.

4 Coverage properties and characterizations

We begin by showing that the inner and outer confidence intervals indeed reduce to the classical notion when there is no imprecision in the observations.

Proposition 1 *Under the conditions of Definition 6, if $Y = Z = X$, it holds that:*

$$CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)] = CI^*[(Y_1, Z_1), \ldots, (Y_n, Z_n)] = CI(X_1, \ldots, X_n)\,.$$

Our next result gives bounds on the coverage given by the outer and inner confidence intervals.

[1] A different approach would be that of *confidence structures* considered by [3], which make use of auxiliary variables and were also considered in [17] as a basis for the theory of *c-boxes*. See also [15].

Proposition 2 *Under the conditions of Definition 6:*

(i) $P_\theta(\theta \in CI^*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]) \geq 1 - \alpha$.
(ii) If CI *is tight, then* $P_\theta(\theta \in CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]) \leq 1 - \alpha$.

Note that the inequality in item (ii) does not necessarily hold if the confidence interval is not tight. For instance, $CI[X_1, \ldots, X_n] = \mathbb{R}$ is a confidence interval for θ at any confidence level, and, in particular, at $1 - \alpha$. In such case we would obtain $P_\theta(\theta \in CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]) = 1$, and the inequality would not hold.

Next we establish more operative expressions for the lower and upper confidence intervals. For this aim, note that given a realization $x_1, \ldots, x_n$ of a simple random sample $X_1, \ldots, X_n$ of X, the confidence interval for θ at confidence level $1 - \alpha$ may be expressed as follows:

$$CI[x_1, \ldots, x_n] = [g_1(x_1, \ldots, x_n),\, g_2(x_1, \ldots, x_n)] = [g_1(\vec{x}), g_2(\vec{x})]\,,$$

where $g_1, g_2 : \mathbb{R}^n \to \mathbb{R}$ are functions such that $g_1(\vec{x}) \leq g_2(\vec{x})$ for any $\vec{x} \in \mathbb{R}^n$. We refer to g_1 and g_2 as the *lower* and *upper bound functions* of the confidence interval. Given a realization of an imprecise simple random sample of $(X; Y, Z)$, we may consider, in case they exist, the following values:

$$\vec{x}^1_{\min} = \arg\min\{g_1(\vec{x}) : x_i \in [y_i, z_i],\, \forall i \in \{1, \ldots, n\}\}\,, \tag{1}$$
$$\vec{x}^1_{\max} = \arg\max\{g_1(\vec{x}) : x_i \in [y_i, z_i],\, \forall i \in \{1, \ldots, n\}\}\,, \tag{2}$$
$$\vec{x}^2_{\min} = \arg\min\{g_2(\vec{x}) : x_i \in [y_i, z_i],\, \forall i \in \{1, \ldots, n\}\}\,, \tag{3}$$
$$\vec{x}^2_{\max} = \arg\max\{g_2(\vec{x}) : x_i \in [y_i, z_i],\, \forall i \in \{1, \ldots, n\}\}\,. \tag{4}$$

These values allow us to characterize the inner and outer confidence intervals.

Theorem 3 *Under the conditions of Definition 6, if the values* $\vec{x}^1_{\min}$, $\vec{x}^1_{\max}$, $\vec{x}^2_{\min}$ *and* $\vec{x}^2_{\max}$ *in Eqs. (1)–(4) exist, then it holds that:*

i) The convex hull of CI^* *is the interval* $[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]$.
ii) If at least one of g_1 *or* g_2 *is continuous, then* $CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]$.
iii) If $g_1(\vec{x}^1_{\max}) > g_2(\vec{x}^2_{\min})$*, then* $CI_* = \emptyset$.
iv) If $g_1(\vec{x}^1_{\max}) \leq g_2(\vec{x}^2_{\min})$*, then* $CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] \neq \emptyset$.

We may trivially unify items iii) and iv) in the above theorem and simply state $CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})]$; we will dispose of such distinctions in subsequent results.

When both bound functions are continuous, as is the case for the majority of the usual confidence intervals, the previous result can be simplified.

Corollary 1 Under the general conditions of Definition 6, if g_1 and g_2 are continuous, then the values $\vec{x}^1_{\min}$, $\vec{x}^1_{\max}$, $\vec{x}^2_{\min}$ and $\vec{x}^2_{\max}$ in Eqs. (1)–(4) exist, and $CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]$, $CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})]$.

On the other hand, it is not uncommon that the two bound functions of the confidence interval can be expressed as increasing transformations of the same function g; in those cases, we can simplify further the computation of the inner and outer confidence intervals.

Corollary 2 Consider the general conditions of Definition 6. If g_1 and g_2 are continuous and there exist $g : \mathbb{R}^n \to \mathbb{R}$ and $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ increasing such that $g_1 = h_1 \circ g$ and $g_2 = h_2 \circ g$, then the values in in Eqs. (1)–(4) exist and moreover $\vec{x}^1_{\min} = \vec{x}^2_{\min} := \vec{x}_{\min}$ and $\vec{x}^1_{\max} = \vec{x}^2_{\max} := \vec{x}_{\max}$. As a consequence, $CI^* = [g_1(\vec{x}_{\min}), g_2(\vec{x}_{\max})]$ and $CI_* = [g_1(\vec{x}_{\max}), g_2(\vec{x}_{\min})]$.

Another common scenario is that when both g_1, g_2 are continuous and monotone, either both increasing or both decreasing. The implications are depicted in the following corollary.

Corollary 3 Consider the general conditions of Definition 6.

(i) If g_1 and g_2 are continuous and monotone increasing, then $CI^* = [g_1(\vec{y}), g_2(\vec{z})]$ and $CI_* = [g_1(\vec{z}), g_2(\vec{y})]$.
(ii) If g_1 and g_2 are continuous and monotone decreasing, then $CI^* = [g_1(\vec{z}), g_2(\vec{y})]$ and $CI_* = [g_1(\vec{y}), g_2(\vec{z})]$.

5 Prominent confidence intervals

The results of the previous section provide the inner and outer confidence intervals, assuming we can solve the box-constrained optimization problem of identifying the values that minimize and maximize the bound functions of the confidence interval in the hyperrectangle $I_n := [y_1, z_1] \times \ldots \times [y_n, z_n]$. This problem can be solved numerically for any parametric distribution family and has been studied in some cases in the context of statistics for interval data (see, e.g., [19]). In this section, we investigate it for five relevant examples within statistical inference.

5.1 Rate of an exponential distribution

Given an exponential distribution $\mathcal{E}(\lambda)$, a confidence interval for λ at confidence level $1 - \alpha$ is given by:

$$CI[x_1, \ldots, x_n] = \left[\frac{e_{n,\alpha/2}}{\overline{\vec{x}}}, \frac{e_{n,1-\alpha/2}}{\overline{\vec{x}}} \right],$$

where $\overline{\vec{x}} = \frac{\sum_{i=1}^n x_i}{n}$ and $e_{n,\beta}$ is the β-quantile of the gamma distribution $\Gamma(n, n)$, i.e., the value that satisfies $P(\Gamma(n, n) \leq e_{n,\beta}) = \beta$.

Proposition 4 *Let $X \sim \mathcal{E}(\lambda)$ be a random variable observed with imprecision through the random variables Y, Z, and $(y_1, z_1), \ldots, (y_n, z_n)$ be the realization of an imprecise simple* *random sample of* $(X;Y,Z)$. *The outer and inner confidence intervals for* λ *at confidence level* $1-\alpha$ *are given by:*

$$CI^* = \left[\frac{e_{n,\alpha/2}}{\overline{\overline{z}}}, \frac{e_{n,1-\alpha/2}}{\overline{\overline{y}}}\right] \quad \textit{and} \quad CI_* = \begin{cases} \left[\frac{e_{n,\alpha/2}}{\overline{\overline{y}}}, \frac{e_{n,1-\alpha/2}}{\overline{\overline{z}}}\right], & \textit{if } \frac{\overline{\overline{z}}}{\overline{\overline{y}}} \leq \frac{e_{n,1-\alpha/2}}{e_{n,\alpha/2}}, \\ \emptyset, & \textit{if } \frac{\overline{\overline{z}}}{\overline{\overline{y}}} > \frac{e_{n,1-\alpha/2}}{e_{n,\alpha/2}}. \end{cases}$$

Let us illustrate this result.

Example 1 The following is a random sample of size $n = 24$ of an exponential distribution with parameter $\lambda = 0.1$, with the values rounded at the second decimal number. Table 1 presents the rounded sample ($\tilde{x}_i$) and the corresponding values of the bounding values ($y_i = \tilde{x}_i - 0.005$ and $z_i = \tilde{x}_i + 0.005$).

i	y_i	$\tilde{x}_i$	z_i	i	y_i	$\tilde{x}_i$	z_i	i	y_i	$\tilde{x}_i$	z_i
1	7.545	7.55	7.555	9	9.565	9.57	9.575	17	18.755	18.76	18.765
2	11.815	11.82	11.825	10	1.465	1.47	1.475	18	6.545	6.55	6.555
3	1.455	1.46	1.465	11	13.905	13.91	13.915	19	3.365	3.37	3.375
4	1.395	1.40	1.405	12	7.615	7.62	7.625	20	5.875	5.88	5.885
5	4.355	4.36	4.365	13	12.375	12.38	12.385	21	23.645	23.65	23.655
6	28.945	28.95	28.955	14	44.235	44.24	44.245	22	6.415	6.42	6.425
7	12.295	12.30	12.305	15	10.545	10.55	10.555	23	2.935	2.94	2.945
8	5.395	5.40	5.405	16	10.345	10.35	10.355	24	5.655	5.66	5.665

Table 1 Realization of an imprecise simple random sample of $(X;Y,Z)$ with $X \sim \mathcal{E}(0.1)$ and the smallest (Y) and greatest (Z) possible values of X, assuming X can only be observed rounded to the second decimal integer.

It holds that $\overline{\overline{y}} = 10.685$ and $\overline{\overline{z}} = 10.695$. In addition, considering $\alpha = 0.05$, we have that $e_{n,\alpha/2} = e_{24,0.025} \approx 0.6407189$ and $e_{n,1-\alpha/2} = e_{24,0.975} \approx 1.437971$. Therefore, it follows from Proposition 4 that

$$CI^* = [0.05990826\,, 0.13457843] \;\text{ and } CI_* = [0.05996433\,, 0.13445260].$$

5.2 Mean of a normal distribution (known variance)

Assume next that the random variable of interest X follows a normal distribution $\mathcal{N}(\mu,\sigma)$ with σ known. A confidence interval for μ at confidence level $1-\alpha$ is given by:

$$CI[x_1,\ldots,x_n] = \left[\overline{\overline{x}} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{\overline{x}} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right],$$

where z_β is the $(1-\beta)$-quantile of the standard normal distribution $\mathcal{N}(0,1)$.

Proposition 5 *Let* $X \sim \mathcal{N}(\mu,\sigma)$, *with* σ *known, be a random variable observed with imprecision through the random variables* Y, Z, *and let* $(y_1, z_1), \ldots, (y_n, z_n)$ *be the realization of an* *imprecise simple random sample of* $(X;Y,Z)$*. The outer and inner confidence intervals for* μ *at confidence level* $1-\alpha$ *are given by:*

$$CI^* = \left[\overline{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{z} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] \text{ and } CI_* = \left[\overline{z} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

Our next example illustrates the result.

Example 2 We randomly generate from a normal distribution of parameters $\mu = 0$ and $\sigma = 1$ a sample of size $n = 24$, presenting the values rounded at the second decimal number. Table 2 presents the rounded sample $(\tilde{x}_i)$ and the corresponding values of the bounding values ($y_i = \tilde{x}_i - 0.005$ and $z_i = \tilde{x}_i + 0.005$).

i	y_i	$\tilde{x}_i$	z_i	i	y_i	$\tilde{x}_i$	z_i	i	y_i	$\tilde{x}_i$	z_i
1	2.285	2.29	2.295	9	0.145	0.15	0.155	17	-0.895	-0.89	-0.885
2	-1.205	-1.20	-1.195	10	2.185	2.19	2.195	18	-0.315	-0.31	-0.305
3	-0.695	-0.69	-0.685	11	0.355	0.36	0.365	19	-0.005	-0.00	0.005
4	-0.415	-0.41	-0.405	12	2.715	2.72	2.725	20	0.985	0.99	0.995
5	-0.975	-0.97	-0.965	13	2.275	2.28	2.285	21	0.835	0.84	0.845
6	-0.955	-0.95	-0.945	14	0.315	0.32	0.325	22	0.705	0.71	0.715
7	0.745	0.75	0.755	15	1.895	1.90	1.905	23	1.305	1.31	1.315
8	-0.125	-0.12	-0.115	16	0.465	0.47	0.475	24	-1.395	-1.39	-1.385

Table 2 Realization of an imprecise simple random sample of $(X;Y,Z)$ with $X \sim \mathcal{N}(0,1)$ and Y, Z the smallest and greatest possible values of X, assuming X can only be observed rounded to the second decimal integer.

It holds that $\overline{y} = 0.42625$ and $\overline{z} = 0.43625$. In addition, considering $\alpha = 0.05$, we have that $z_{\alpha/2} = z_{0.025} \approx 1.959964$. Applying Proposition 5, we obtain

$$CI^* = [0.02617403\,, 0.83632597] \text{ and } CI_* = [0.03617403\,, 0.82632597].$$

5.3 Variance of a normal distribution (known mean)

Next, if $X \sim \mathcal{N}(\mu, \sigma)$ with μ known, a confidence interval for σ^2 at confidence level $1-\alpha$ is given by:

$$CI[x_1, \ldots, x_n] = \left[\frac{\sum_{i=1}^n (x_i - \mu)^2}{c_{n,1-\alpha/2}}, \frac{\sum_{i=1}^n (x_i - \mu)^2}{c_{n,\alpha/2}}\right],$$

where $c_{n,\beta}$ denotes the β-quantile of the chi-squared distribution χ^2_n.

Proposition 6 *Let* $X \sim \mathcal{N}(\mu, \sigma)$*, with* μ *known, be a random variable observed with imprecision through the random variables* Y, Z*, and* $(y_1, z_1), \ldots, (y_n, z_n)$ *be the realization of an imprecise simple random sample of* $(X;Y,Z)$*. Compute* $\vec{x}_{\max} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ *and* $\vec{x}_{\min} = (\hat{x}_1, \ldots, \hat{x}_n)$*, where for any* $i \in \{1, \ldots, n\}$*:*

$$\tilde{x}_i = \arg\max\left\{(x-\mu)^2 \mid x \in \{y_i, z_i\}\right\},$$

$$\hat{x}_i = \begin{cases} \mu & \textit{if } \mu \in [y_i, z_i]\,, \\ \arg\min\left\{(x-\mu)^2 \mid x \in \{y_i, z_i\}\right\} & \textit{if } \mu \notin [y_i, z_i]\,, \end{cases}$$

and consider $g(\vec{x}) = \sum_{i=1}^{n}(x_i - \mu)^2$.

i) If $\mu \notin \bigcap_{i=1}^{n}[y_i, z_i]$, *then the outer and inner confidence intervals for* σ^2 *at confidence level* $1-\alpha$ *are given by:*

$$CI^* = \left[\frac{g(\vec{x}_{\min})}{c_{n,1-\alpha/2}}, \frac{g(\vec{x}_{\max})}{c_{n,\alpha/2}}\right] \text{ and } CI_* = \left[\frac{g(\vec{x}_{\max})}{c_{n,1-\alpha/2}}, \frac{g(\vec{x}_{\min})}{c_{n,\alpha/2}}\right].$$

ii) If $\mu \in \bigcap_{i=1}^{n}[y_i, z_i]$, *then the outer and inner confidence intervals for* σ^2 *at confidence level* $1-\alpha$ *are given by:*

$$CI^* = \left[0, \frac{g(\vec{x}_{\max})}{c_{n,\alpha/2}}\right] \text{ and } CI_* = \emptyset\,.$$

The following example illustrates the result.

Example 3 Continue with the data from Table 2. It holds that $g(\vec{x}_{\min}) = 37.71158$ and $g(\vec{x}_{\max}) = 38.1958$. In addition, considering $\alpha = 0.05$, we have that $c_{n,1-\alpha/2} = c_{24,0.975} \approx 39.36408$ and $c_{n,\alpha/2} = c_{24,0.025} \approx 12.40115$. Applying Proposition 6, it follows that

$$CI^* = [0.9580200\,, 3.0800210] \text{ and } CI_* = [0.9703212\,, 3.0409740]\,.$$

5.4 Variance of a normal distribution (unknown mean)

Next, we consider the case where $X \sim \mathcal{N}(\mu, \sigma)$ and μ is unknown. A confidence interval for σ^2 at confidence level $1-\alpha$ is given by:

$$CI[x_1, \ldots, x_n] = \left[\frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{c_{n-1,1-\alpha/2}}, \frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{c_{n-1,\alpha/2}}\right],$$

where $c_{n-1,\beta}$ is the β-quantile of the chi-squared distribution χ^2_{n-1}.

In order to determine the confidence interval under imprecision, we shall introduce the notion of quasivertices.

Definition 7 Consider a hyper-rectangle $I_n := [y_1, z_1] \times \cdots \times [y_n, z_n]$ and let $J \subsetneq \{1, \ldots, n\}$ be a set of indices. A vector $\vec{x}$ is a *quasivertex* of I_n associated with J if (i) $x_i \in \{y_i, z_i\}$ for any $i \notin J$; and (ii) $x_i = \dfrac{\sum_{j \notin J} x_j}{n - |J|}$ and it belongs to $\bigcap_{j \in J}[y_j, z_j]$ for any $i \in J$. The set of quasivertices is defined as $QV := \bigcup_{J \subsetneq \{1,\dots,n\}} QV_J$. The particular case in which $J = \emptyset$ defines the set of vertices, denoted by $V = QV_{\emptyset}$.

Proposition 7 *Let $X \sim \mathcal{N}(\mu, \sigma)$, with μ unknown, be a random variable observed with imprecision through the random variables Y, Z, and $(y_1, z_1), \dots, (y_n, z_n)$ be the realization of an imprecise simple random sample of $(X; Y, Z)$. Let us define $h(\vec{x}) = \sum_{i=1}^{n} (x_i - \overline{\vec{x}})^2$ and $\vec{x}_{\max} = \underset{\substack{x_i \in \{y_i, z_i\} \\ i \in \{1,\dots,n\}}}{\arg\max} \{h(\vec{x})\}$.*

i) If $\bigcap_{i=1}^{n} [y_i, z_i] = \emptyset$, then the outer and inner confidence intervals for σ^2 at confidence level $1 - \alpha$ are given by:

$$CI^* = \left[\frac{h(\vec{x}_{\min})}{c_{n-1,1-\alpha/2}}, \frac{h(\vec{x}_{\max})}{c_{n-1,\alpha/2}} \right] \quad \text{and } CI_* = \left[\frac{h(\vec{x}_{\max})}{c_{n-1,1-\alpha/2}}, \frac{h(\vec{x}_{\min})}{c_{n-1,\alpha/2}} \right],$$

where $\vec{x}_{\min} = \underset{\vec{x} \in QV}{\arg\min} \{h(\vec{x})\}$, QV is the set of quasivertices of $I_n = [y_1, z_1] \times \cdots \times [y_n, z_n]$ and, additionally, $0 < h(\vec{x}_{\min}) \leq h(\vec{x}_{\max})$.

ii) If $\bigcap_{i=1}^{n} [y_i, z_i] \neq \emptyset$, then the outer and inner confidence intervals for σ^2 at confidence level $1 - \alpha$ are given by:

$$CI^* = \left[0, \frac{h(\vec{x}_{\max})}{c_{n-1,\alpha/2}} \right] \quad \text{and } CI_* = \emptyset.$$

Note that the set of quasivertices is finite, although its cardinality is greater than the set of vertices (which is 2^n). The number of quasivertices is bounded above by

$$\sum_{j=1}^{n} 2^j \binom{n}{j} = 3^n - 1,$$

where the equality can be established applying induction on n.

In addition, in case all intervals $[y_i, z_i]$ are pairwise disjoint, it suffices to study the quasivertices associated with sets of indices of cardinality zero or one, and there are $2^{n-1}(n+2)$ possible quasivertices. The interest of this scenario has also been raised in [19, Section 4.8].

In the following, we present an illustrative example.

Example 4 Continue with the data from Table 2. It holds that $h(\vec{x}_{\min}) = 33.25613$ and $h(\vec{x}_{\max}) = 33.72438$. Given $\alpha = 0.05$, we have that $c_{n-1,1-\alpha/2} = c_{23,0.975} \approx 38.07563$ and $c_{n-1,\alpha/2} = c_{23,0.025} \approx 11.68855$. Applying Proposition 7, we obtain

$$CI^* = [0.8734231\,, 2.8852491] \text{ and } CI_* = [0.885721\,, 2.845188]\,.$$

5.5 Mean of a normal distribution (unknown variance)

We conclude by considering the imprecise confidence intervals for the mean of a normal distribution $\mathcal{N}(\mu, \sigma)$ where the standard deviation σ is unknown. In the precise case, the confidence interval for μ at confidence level $1-\alpha$ is given by:

$$CI[x_1, \ldots, x_n] = \left[\overline{\vec{x}} - t_{n-1,\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{n(n-1)}},\, \overline{\vec{x}} + t_{n-1,\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{n(n-1)}}\right],$$

where $t_{n-1,\beta}$ is the $(1-\beta)$-quantile of the Student's t distribution t_{n-1}.

In order to establish the formulae for the lower and upper confidence intervals, we introduce next the notion of 2-quasivertices:

Definition 8 Let $J \subsetneq \{1, \ldots, n\}$ be a set of indices. A vector $\vec{x}$ is a *2-quasivertex* of $I_n := [y_1, z_1] \times \cdots \times [y_n, z_n]$ associated with J if (i) for any $i \notin J$ $x_i \in \{y_i, z_i\}$; and (ii) for any $i \in J$ $x_i = y$ with y a solution of the equation $\left(y - \overline{\vec{x}_k^*}\right)^2 = C\, \vec{x}_k^{*\prime} H_k \vec{x}_k^*$, with $k = n - |J|$, where

$$\vec{x}_k^* = (x_i)_{i \notin J}, A = \frac{k^2}{n^2} - \frac{(n-1)(n-k)k}{n^2\, t_{\alpha/2}^2}, B = \frac{n-1}{n}\frac{k}{t_{\alpha/2}^2} \text{ and } C = \frac{B}{A};$$

and moreover it belongs to $\bigcap_{j \in J} [y_j, z_j]$. The set of 2-quasivertices is denoted by $Q^2V := \bigcup_{J \subsetneq \{1,\ldots,n\}} QV_J$. The particular case in which $J = \emptyset$ defines again the set of vertices, denoted by $V = Q^2V_{\emptyset}$.

The matrix H_k^* is the *quasicentering matrix*, defined as $H_k^* := I_k - \frac{1}{n} J_k \in \mathcal{M}_k(\mathbb{R})$, where $n \in \mathbb{N}$, $k \in \{1, \ldots, n\}$, and I_k and J_k are respectively the identity and the all-ones matrices of order k. The quasicentering matrix is a positive definite matrix if $k \neq n$. When $k = n$, the quasicentering matrix is a positive semidefinite matix, since it coincides with the *centering matrix* of order n, denoted by H_n.

Proposition 8 *Let $X \sim \mathcal{N}(\mu, \sigma)$, with σ unknown, be a random variable observed with imprecision through the random variables Y, Z, and $(y_1, z_1), \ldots, (y_n, z_n)$ be the realization of an imprecise simple random sample of $(X; Y, Z)$. Compute*

$$\vec{x}_{\min}^1 = \arg\min_{\vec{x} \in Q^2V}\{g_1(\vec{x})\}, \quad \vec{x}_{\max}^1 = \arg\max_{\vec{x} \in Q^2V}\{g_1(\vec{x})\},$$
$$\vec{x}_{\min}^2 = \arg\min_{\vec{x} \in Q^2V}\{g_2(\vec{x})\}, \quad \vec{x}_{\max}^2 = \arg\max_{\vec{x} \in Q^2V}\{g_2(\vec{x})\},$$

where $g_1(\vec{x}) = \overline{\vec{x}} - t_{n-1,\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{n(n-1)}}$, $g_2(\vec{x}) = \overline{\vec{x}} + t_{n-1,\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{\vec{x}})^2}{n(n-1)}}$ and let Q^2V be the set of 2-quasivertices of $I_n = [y_1, z_1] \times \cdots \times [y_n, z_n]$.

The outer and inner confidence intervals for μ at confidence level $1-\alpha$ are given by:

$$CI^* = \left[g_1(\vec{x}_{\min}^1),\, g_2(\vec{x}_{\max}^2)\right] \quad \textit{and } CI_* = \left[g_1(\vec{x}_{\max}^1),\, g_2(\vec{x}_{\min}^2)\right].$$

Again, the set of 2-quasivertices is finite, and its cardinality is greater than the set of vertices (which is 2^n). The number of 2-quasivertices is bounded above by

$$\sum_{j=1}^{n} 2^{j+1}\binom{n}{j} = 2(3^n - 1),$$

where the equality can be established applying induction on n.

In addition, in case all intervals $[y_i, z_i]$ are pairwise disjoint, it suffices to study the 2-quasivertices associated with sets of indices of cardinality zero or one, and there are $2^n(n+2)$ possible quasivertices.

Example 5 Continue with the data from Table 2. It holds that $g_1(\vec{x}^1_{\min}) = -0.08328552$, $g_1(\vec{x}^1_{\max}) = -0.07335457$, $g_2(\vec{x}^2_{\min}) = 0.9357855$ and $g_2(\vec{x}^2_{\max}) = 0.9456466$. Applying Proposition 8,

$$CI^* = [-0.08328552\,, 0.94564660] \text{ and } CI_* = [-0.07335457\,, 0.93578552]\,.$$

6 Conclusions and future perspectives

In this paper, we extend the notion of confidence interval to the case where the random variable of interest is observed with imprecision. We have modelled this situation by means of random sets, and defined two random intervals, that determine a lower and an upper bound for the 'true' confidence interval. In particular, we have obtained the explicit formula for the inner and outer confidence intervals in five prominent examples in statistical inference. For this aim, we have introduced the notions of quasivertex and 2-quasivertex, that have allowed us to simplify the optimization problem associated with the computation of the inner and outer intervals.

Future work aims at extending the present results to other types of random sets different from random intervals, and to analyze the generalization to the multivariate case. Additionally, the well-known parallelism between confidence intervals and statistical tests makes the study of the connection between the inner/outer confidence interval and the acceptance/rejection region a fruitful topic of research. Finally, we may also generalise our work from random sets to fuzzy random variables, and this should allow to compare our work with the one carried out in [6, 35].

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Disclosure statement

The authors declare that they have no competing interests.

Appendix: Proofs

The following is an auxiliary lemma that shall be used in some of the subsequent proofs. Its proof is elementary and therefore omitted.

Auxiliary Lemma 1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, $k \in \{1, \ldots, n-1\}$, $U \subseteq \mathbb{R}^{n-k}$ an open set, $V \subseteq \mathbb{R}^k$, and $\vec{x}^* = (x_1^*, \ldots, x_n^*) \in U \times V$. Taking then

$$\vec{x}^*_{n-k} := (x_1^*, \ldots, x_{n-k}^*) \in U \subseteq \mathbb{R}^{n-k}, \quad \vec{x}^*_k := (x^*_{n-k+1}, \ldots, x_n^*) \in V \subseteq \mathbb{R}^k,$$

and the function

$$\begin{aligned} f^* : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R} \\ \vec{x}_{n-k} &\mapsto f\big((\vec{x}_{n-k}, \vec{x}^*_k)\big), \end{aligned}$$

it follows that:

ii) If $\vec{x}^*$ is a relative (respectively absolute) minimum of f in $U \times V$, then $\vec{x}^*_{n-k}$ is a relative (absolute) minimum of f^* in $\mathbb{R}^{n-k}$.
iii) If $\vec{x}^*$ is a relative (respectively absolute) maximum of f in $U \times V$, then $\vec{x}^*_{n-k}$ is a relative (absolute) maximum of f^* in $\mathbb{R}^{n-k}$.

Proof of Proposition 1. If $Y = Z = X$, any imprecise simple random sample $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ of $(X; Y, Z)$ is a simple random sample $X_1, \ldots, X_n$ of X, and, in particular $S([Y_i, Z_i]) = \{X_i\}$ for $i \in \{1, \ldots, n\}$. From this the result follows immediately. $\square$

Proof of Proposition 2. It suffices to take into account that

$$CI[X_1, \ldots, X_n] \subseteq \bigcup_{\tilde{X}_i \in S([Y_i, Z_i])} CI[\tilde{X}_1, \ldots, \tilde{X}_n] = CI^*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]\,,$$

and

$$CI[X_1, \ldots, X_n] \supseteq \bigcap_{\tilde{X}_i \in S([Y_i, Z_i])} CI[\tilde{X}_1, \ldots, \tilde{X}_n] = CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]\,,$$

from which it follows, by definition of confidence interval for θ at confidence level $1-\alpha$,

$$P_\theta(\theta \in CI^*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]) \geq P_\theta(\theta \in CI[X_1, \ldots, X_n]) \geq 1 - \alpha\,,$$

and

$$P_\theta(\theta \in CI_*[(Y_1, Z_1), \ldots, (Y_n, Z_n)]) \leq P_\theta(\theta \in CI[X_1, \ldots, X_n]) = 1 - \alpha\,,$$

where the last equality is necessarily true only if the confidence interval is tight. $\square$

Proof of Theorem 3. Under the conditions of Definition 6, since the values $\vec{x}^1_{\min}$, $\vec{x}^1_{\max}$, $\vec{x}^2_{\min}$ and $\vec{x}^2_{\max}$ in Eqs. (1–4) exist, then:

i) On the one hand, for all $\vec{x} \in \mathbb{R}^n$ with $x_i \in [y_i, z_i]$, it holds that $g_1(\vec{x}^1_{\min}) \leq g_1(\vec{x}) \leq g_2(\vec{x}) \leq g_2(\vec{x}^2_{\max})$, whence:

$$\bigcup_{x_i \in [y_i, z_i]} CI[x_1, \ldots, x_n] = \bigcup_{x_i \in [y_i, z_i]} [g_1(\vec{x}), g_2(\vec{x})] \subseteq [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})].$$

On the other hand, taking the samples $\vec{x}^1_{\min}$ y $\vec{x}^2_{\max}$, it is obtained that:

$$\begin{aligned}[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^1_{\min})] \cup [g_1(\vec{x}^2_{\max}), g_2(\vec{x}^2_{\max})] &= CI[\vec{x}^1_{\min}] \cup CI[\vec{x}^2_{\max}] \\ &\subseteq \bigcup_{x_i \in [y_i, z_i]} CI[x_1, \ldots, x_n] = CI^*.\end{aligned}$$

Thus,

$$[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^1_{\min})] \cup [g_1(\vec{x}^2_{\max}), g_2(\vec{x}^2_{\max})] \subseteq CI^* \subseteq [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]. \tag{5}$$

Since $[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]$ is the convex hull of the set $[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^1_{\min})] \cup [g_1(\vec{x}^2_{\max}), g_2(\vec{x}^2_{\max})]$, we deduce that it is also the convex hull of CI^*.

ii) Since by assumption g_1 is a continuous function, it follows that for all $\lambda \in [g_1(\vec{x}^1_{\min}), g_1(\vec{x}^2_{\max})]$ there exists $\vec{x}^\lambda \in \mathbb{R}^n$ with $x^\lambda_i \in [y_i, z_i]$ $\forall i \in \{1, \ldots, n\}$, such that $g_1(\vec{x}^\lambda) = \lambda$. Thus,

$$\lambda \in [g_1(\vec{x}^\lambda), g_2(\vec{x}^\lambda)] = CI[\vec{x}^\lambda] \subseteq \bigcup_{x_i \in [y_i, z_i]} CI[x_1, \ldots, x_n] = CI^*,$$

and $[g_1(\vec{x}^1_{\min}), g_1(\vec{x}^2_{\max})] \subseteq CI^*$. Since by Eq. (5) $[g_1(\vec{x}^2_{\max}), g_2(\vec{x}^2_{\max})] \subseteq CI^*$, we deduce that $[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = [g_1(\vec{x}^1_{\min}), g_1(\vec{x}^2_{\max})] \cup [g_1(\vec{x}^2_{\max}), g_2(\vec{x}^2_{\max})]$ is included in CI^*. Since the reverse inclusion follows from the first part, we conclude that

$$CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})].$$

When g_2 is a continuous function and g_1 is not, the result follows similarly using $[g_2(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] \subseteq CI^*$ and $[g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^1_{\min})] \cup [g_2(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})]$.

iii) Let us suppose that $g_1(\vec{x}^1_{\max}) > g_2(\vec{x}^2_{\min})$, so it holds that $[g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] = \emptyset$.
Taking the samples $\vec{x}^1_{\max}$ y $\vec{x}^2_{\min}$, since $g_1(\vec{x}^1_{\max}) > g_2(\vec{x}^2_{\min})$, it follows that:

$$\begin{aligned}\emptyset = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] &= [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^1_{\max})] \cap [g_1(\vec{x}^2_{\min}), g_2(\vec{x}^2_{\min})] \\ &= CI[\vec{x}^1_{\max}] \cap CI[\vec{x}^2_{\min}] \supseteq \bigcap_{x_i \in [y_i, z_i]} CI[x_1, \ldots, x_n] = CI_*,\end{aligned}$$

so $CI_* = \emptyset = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})]$.

iv) Consider now that $g_1(\vec{x}^1_{\max}) \le g_2(\vec{x}^2_{\min})$, so that the interval $[g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})]$ is not empty. For all $\vec{x} \in \mathbb{R}^n$ with $x_i \in [y_i, z_i]$ it holds that $g_1(\vec{x}) \le g_1(\vec{x}^1_{\max}) \le g_2(\vec{x}^2_{\min}) \le g_2(\vec{x})$, whence

$$\bigcap_{x_i \in [y_i, z_i]} CI[x_1, \dots, x_n] = \bigcap_{x_i \in [y_i, z_i]} [g_1(\vec{x}), g_2(\vec{x})] \supseteq [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] \neq \emptyset.$$

Taking the samples $\vec{x}^1_{\max}$ y $\vec{x}^2_{\min}$, since $g_1(\vec{x}^1_{\max}) \le g_2(\vec{x}^2_{\min})$,

$$\begin{aligned} \emptyset \neq [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] &= [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^1_{\max})] \cap [g_1(\vec{x}^2_{\min}), g_2(\vec{x}^2_{\min})] \\ &= CI[\vec{x}^1_{\max}] \cap CI[\vec{x}^2_{\min}] \supseteq \bigcap_{x_i \in [y_i, z_i]} CI[x_1, \dots, x_n] = CI_*, \end{aligned}$$

from which we conclude that $CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] \neq \emptyset$. □

Proof of Corollary 1. Since g_1 and g_2 are continuous functions, the values $\vec{x}^1_{\min}$, $\vec{x}^1_{\max}$, $\vec{x}^2_{\min}$ and $\vec{x}^2_{\max}$ defined in Eqs. (1)–(4) trivially exist, and the result follows immediately from Theorem 3. □

Proof of Corollary 2. The result follows immediately from Corollary 1. □

Proof of Corollary 3. Since g_1 and g_2 are continuous and increasing functions, the values $\vec{x}^1_{\min}$, $\vec{x}^1_{\max}$, $\vec{x}^2_{\min}$ and $\vec{x}^2_{\max}$ defined in Eqs. (1)–(4) trivially exist, and their values are $\vec{x}^1_{\min} = \vec{x}^2_{\min} = \vec{y}$ and $\vec{x}^1_{\max} = \vec{x}^2_{\max} = \vec{z}$. The result follows immediately from Corollary 1. □

Proof of Proposition 4. The proposition follows immediately from Corollary 3(ii). □

Proof of Proposition 5. The proposition follows immediately from Corollary 3(i). □

Proof of Proposition 6. Consider the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(\vec{x}) = \sum_{i=1}^n (x_i - \mu)^2$ and the increasing functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ defined by $h_1(x) = \frac{x}{c_{n,1-\alpha/2}}$ and $h_2(x) = \frac{x}{c_{n,\alpha/2}}$. Since $g_1 = h_1 \circ g$ and $g_2 = h_2 \circ g$, it follows from Corollary 2 that the inner confidence interval and the outer confidence interval are characterized by the minimum $\vec{x}_{\min}$ and the maximum $\vec{x}_{\max}$ of g on $I_n = [y_1, z_1] \times \ldots \times [y_n, z_n]$.

Given $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = (x - \mu)^2 \ge 0$, it follows that $g(\vec{x}) = \displaystyle\sum_{i=1}^n (x_i - \mu)^2 = \sum_{i=1}^n h(x_i)$, whence $\vec{x} = (x_1, \dots, x_n)$ is a maximum (resp. a minimum) of g on I_n if and only if x_i is a maximum (resp. a minimum) of h on $[y_i, z_i]$ for any $i \in \{1, \dots, n\}$. Since $h'(x) = 2(x - \mu)$ and $h''(x) = 2 > 0$, it immediately follows that h has a unique relative minimum at $x = \mu$ and has no relative maxima on $\mathbb{R}$. Studying h on the interval $[y_i, z_i]$, it is clear that the function reaches its maximum at the endpoints of the interval, i.e., at either y_i or z_i, and the minimum will be reached at $x = \mu$ if $\mu \in [y_i, z_i]$, or at the endpoints of the interval if $\mu \notin [y_i, z_i]$. Thus, if $\vec{x}_{\max} = (\tilde{x}_1, \dots, \tilde{x}_n)$ and $\vec{x}_{\min} = (\hat{x}_1, \dots, \hat{x}_n)$ respectively denote the maximum and the minimum of function g in I_n, it holds that:

$$\tilde{x}_i = \arg\max \left\{ (x - \mu)^2 \mid x \in \{y_i, z_i\} \right\},$$

$$\hat{x}_i = \begin{cases} \mu\,, & \text{if } \mu \in [y_i, z_i]\,, \\ \arg\min\left\{(x-\mu)^2 \mid x \in \{y_i, z_i\}\right\}\,, & \text{if } \mu \notin [y_i, z_i]\,, \end{cases}$$

for any $i \in \{1, \ldots, n\}$.

We now distinguish between two cases, depending on whether μ lies within $\bigcap_{i=1}^{n} [y_i, z_i]$ or not.

i) If $\mu \notin \bigcap_{i=1}^{n} [y_i, z_i]$, it follows that there exists $i \in \{1, \ldots, n\}$ such that $\mu \notin [y_i, z_i]$, so $\hat{x}_i \neq \mu$ and $h(\hat{x}_i) > 0$. Accordingly, it follows that $g(\vec{x}_{\min}) = \underbrace{h(\hat{x}_1)}_{\geq 0} + \ldots + \underbrace{h(\hat{x}_i)}_{>0} + \ldots + \underbrace{h(\hat{x}_n)}_{\geq 0} > 0$, and in particular $g_1(\vec{x}^1_{\min}) = g_2(\vec{x}^2_{\min}) > 0$. Since g_1 and g_2 are continuous, it follows from Corollary 2 that

$$CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = \left[\frac{g(\vec{x}_{\min})}{c_{1-\alpha/2}}, \frac{g(\vec{x}_{\max})}{c_{\alpha/2}}\right],$$
$$CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] = \left[\frac{g(\vec{x}_{\max})}{c_{1-\alpha/2}}, \frac{g(\vec{x}_{\min})}{c_{\alpha/2}}\right].$$

ii) If $\mu \in \bigcap_{i=1}^{n} [y_i, z_i]$, it follows that $\mu \in [y_i, z_i]$ for any $i \in \{1, \ldots, n\}$, so $\hat{x}_i = \mu$, $h(\hat{x}_i) = 0$ and $g(\vec{x}_{\min}) = \underbrace{h(\hat{x}_1)}_{0} + \ldots + \underbrace{h(\hat{x}_n)}_{0} = 0$. In particular, $g_1(\vec{x}^1_{\min}) = g_2(\vec{x}^2_{\min}) = 0$. Thus,

$$g_1(\vec{x}^1_{\max}) \leq g_2(\vec{x}^2_{\min}) \Leftrightarrow g(\vec{x}_{\max}) = 0 \Leftrightarrow y_i = z_i = \mu\,,\ \forall i \in \{1, \ldots, n\}\,.$$

In this case, the exact value of x_i would be perfectly known for any $i \in \{1, \ldots, n\}$. For this reason, we assume, without loss of generality, that this case will not occur, so it holds that $g_1(\vec{x}^1_{\max}) > g_2(\vec{x}^2_{\min})$. Since g_1 and g_2 are continuous functions, it follows from Corollary 2 that the outer and inner confidence intervals for σ^2 are given by:

$$CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = \left[0, \frac{g(\vec{x}_{\max})}{c_{\alpha/2}}\right] \text{ and } CI_* = \emptyset\,. \quad \square$$

Proof of Proposition 7. Consider the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(\vec{x}) = \sum_{i=1}^{n} (x_i - \overline{\vec{x}})^2 \tag{6}$$

and the increasing functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \frac{x}{c_{n-1,1-\alpha/2}}$ and $f_2(x) = \frac{x}{c_{n-1,\alpha/2}}$. Since $g_1 = f_1 \circ h$ and $g_2 = f_2 \circ h$, it follows from Corollary 2 that the inner confidence interval and the outer confidence interval are characterized by the minimum $\vec{x}_{\min}$ and the maximum $\vec{x}_{\max}$ of h on $I_n = [y_1, z_1] \times \ldots \times [y_n, z_n]$.

If we denote by H_n the centering matrix, then h can be expressed as $h(\vec{x}) = \sum_{i=1}^{n}(x_i - \overline{x})^2 = \vec{x}' H_n \vec{x}$. Thus, the gradient and Hessian of h are given by

$$\nabla h = \frac{\partial h}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}} (\vec{x}' H_n \vec{x}) = 2H_n\vec{x} \;\text{ and } \operatorname{Hess}(h) = \frac{\partial \nabla h}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}} (2H_n\vec{x}) = 2H_n. \quad (7)$$

Since H_n is a positive semidefinite matrix, so is $\operatorname{Hess}(h)$ and consequently the function h will not have relative maxima on $\mathbb{R}^n$, and only relative minima on $\mathbb{R}^n$ at those points where ∇h vanishes. Considering that:

$$\nabla h = 2H_n\vec{x} = \vec{0}_n \Leftrightarrow 2(\vec{x} - \overline{x}\,\vec{1}_n) = \vec{0}_n \Leftrightarrow \vec{x} = \overline{x}\,\vec{1}_n \Leftrightarrow x_i = x_j \,\forall i, j \in \{1, \ldots, n\},$$

the points where ∇h vanishes are of the form $\vec{x} = \gamma\,\vec{1}_n$ for any $\gamma \in \mathbb{R}$. Moreover, the function h also vanishes at those points, since:

$$\nabla h = 2H_n\vec{x} = \vec{0}_n \quad \Rightarrow \quad h(\vec{x}) = \vec{x}' \underbrace{H_n\vec{x}}_{0} = 0.$$

Having found the relative maxima and minima on $\mathbb{R}^n$ of function h, it must be taken into account that the extrema of the function h in I_n are the ones that must be found to build the inner and outer confidence intervals. For this purpose, the following family of functions is constructed. Given $k \in \{1, \ldots, n-1\}$ and $\vec{x}^*_k \in \mathbb{R}^k$ fixed, the auxiliary function h_{n-k} is defined as follows:

$$\begin{aligned} h_{n-k} : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R} \\ \vec{x}_{n-k} &\hookrightarrow h_{n-k}(\vec{x}_{n-k}) := h(\underbrace{(\vec{x}_{n-k}, \vec{x}^*_k)}_{\in \mathbb{R}^n}). \end{aligned}$$

The importance of the function above lies in the fact that, if there exists $k \in \{1, \ldots, n-1\}$ and $\vec{x}^* = (\vec{x}^*_{n-k}, \vec{x}^*_k) \in \mathbb{R}^n$ such that:

$$\vec{x}^*_{n-k} \in I^o_{n-k} := (y_1, z_1) \times \ldots \times (y_{n-k}, z_{n-k}),$$

then by Lemma 1, if $\vec{x}^*$ is a relative maximum (minimum) of h on I_n, $\vec{x}^*_{n-k}$ is a relative maximum (minimum) of h_{n-k} on $\mathbb{R}^{n-k}$. Therefore, the extrema of the auxiliary function h_{n-k} must be calculated. In order to do this, we shall give an explicit expression of this function in terms of $\vec{x}_{n-k}$.

$$h_{n-k}(\vec{x}_{n-k}) = h((\vec{x}_{n-k}, \vec{x}^*_k)) = (\vec{x}'_{n-k}, \vec{x}^{*\prime}_k) H_n (\vec{x}_{n-k}, \vec{x}^*_k)$$

$$= (\vec{x}'_{n-k}, \vec{x}^{*\prime}_k) \left(\begin{array}{c|c} H^*_{n-k} & -\frac{1}{n} J_{n-k,k} \\ \hline -\frac{1}{n} J_{k,n-k} & H^*_k \end{array} \right) (\vec{x}_{n-k}, \vec{x}^*_k)$$
$$= \vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} - \frac{1}{n} \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k - \frac{1}{n} \vec{x}^{*\prime}_k J_{k,n-k} \vec{x}_{n-k} + \vec{x}^{*\prime}_k H^*_k \vec{x}^*_k. \tag{8}$$

Noticing that $\vec{x}^{*\prime}_k J_{k,n-k} \vec{x}_{n-k} \in \mathbb{R}$, it follows that:

$$\vec{x}^{*\prime}_k J_{k,n-k} \vec{x}_{n-k} = (\vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k)' = \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k, \tag{9}$$

since a real number coincides with its transpose. Combining Eqs. (8) and (9) we obtain:

$$h_{n-k}(\vec{x}_{n-k}) = \vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k + \vec{x}^{*\prime}_k H^*_k \vec{x}^*_k. \tag{10}$$

Thus, the relative extrema of the function can now be determined by computing the gradient and Hessian of h_{n-k}. With respect to the gradient, notice that:

$$\nabla h_{n-k} = \frac{\partial h_{n-k}}{\partial \vec{x}_{n-k}} = \frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k + \vec{x}^{*\prime}_k H^*_k \vec{x}^*_k \right)$$
$$= \frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} \right) - \frac{2}{n} \frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k \right) + \frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}^{*\prime}_k H^*_k \vec{x}^*_k \right). \tag{11}$$

Given that:

$$\frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} \right) = 2 H^*_{n-k} \vec{x}_{n-k},$$
$$\frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k \right) = J_{n-k,k} \vec{x}^*_k,$$
$$\frac{\partial}{\partial \vec{x}_{n-k}} \left(\vec{x}^{*\prime}_k H^*_k \vec{x}^*_k \right) = \vec{0}_{n-k},$$

and combining this with Eq. (11), it follows that:

$$\nabla h_{n-k} = 2 H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} J_{n-k,k} \vec{x}^*_k. \tag{12}$$

The Hessian can now be computed using the expression of the gradient:

$$\mathrm{Hess}(h_{n-k}) = \frac{\partial \nabla h_{n-k}}{\partial \vec{x}_{n-k}} = \frac{\partial}{\partial \vec{x}_{n-k}} \left(2 H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} J_{n-k,k} \vec{x}^*_k \right)$$
$$= 2 \frac{\partial}{\partial \vec{x}_{n-k}} \left(H^*_{n-k} \vec{x}_{n-k} \right) - \frac{2}{n} \frac{\partial}{\partial \vec{x}_{n-k}} \left(J_{n-k,k} \vec{x}^*_k \right).$$

Since

$$\frac{\partial}{\partial \vec{x}_{n-k}}\left(H^*_{n-k}\vec{x}_{n-k}\right) = H^*_{n-k} \quad \text{and} \quad \frac{\partial}{\partial \vec{x}_{n-k}}\left(J_{n-k,k}\vec{x}^*_k\right) = 0_{n-k,n-k},$$

we deduce that $\mathrm{Hess}(h_{n-k}) = 2H^*_{n-k}$. As H^*_{n-k} is a positive definite matrix, so will be $\mathrm{Hess}(h_{n-k})$ and consequently the function h_{n-k} will not have relative maxima on $\mathbb{R}^{n-k}$, and only relative minima on $\mathbb{R}^{n-k}$ at those points where ∇h_{n-k} vanishes. Given that

$$\nabla h_{n-k} = 2H^*_{n-k}\vec{x}_{n-k} - \frac{2}{n}J_{n-k,k}\vec{x}^*_k = \vec{0}_{n-k} \Leftrightarrow H^*_{n-k}\vec{x}_{n-k} = \frac{1}{n}J_{n-k,k}\vec{x}^*_k$$

and expanding both sides of the last equality,

$$H^*_{n-k}\,\vec{x}_{n-k} = \vec{x}_{n-k} - \frac{n-k}{n}\,\overline{x}_{n-k}\,\vec{1}_{n-k} \quad \text{and} \quad \frac{1}{n}J_{n-k,k}\,\vec{x}^*_k = \frac{k}{n}\,\overline{x}^*_{n-k}\,\vec{1}_{n-k},$$

we obtain that:

$$\begin{aligned}
&\nabla h_{n-k} = \vec{0}_{n-k} \Leftrightarrow H^*_{n-k}\,\vec{x}_{n-k} = \frac{1}{n}\,J_{n-k,k}\,\vec{x}^*_k \\
&\Leftrightarrow \vec{x}_{n-k} - \frac{n-k}{n}\,\overline{x}_{n-k}\,\vec{1}_{n-k} = \frac{k}{n}\,\overline{x}^*_k\,\vec{1}_{n-k} \Leftrightarrow \;\vec{x}_{n-k} = \frac{n-k}{n}\,\overline{x}_{n-k}\,\vec{1}_{n-k} + \frac{k}{n}\,\overline{x}^*_k\,\vec{1}_{n-k}.
\end{aligned}$$

If we express

$$\frac{n-k}{n}\,\overline{x}_{n-k}\,\vec{1}_{n-k} + \frac{k}{n}\,\overline{x}^*_k\,\vec{1}_{n-k} = \frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^*_k}{n}\,\vec{1}_{n-k},$$

we obtain that it must be

$$x_i = \frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^*_k}{n}, \quad \forall i \in \{1,\ldots,n-k\}.$$

In particular, this means that $x_i = x_j$ for all $i, j \in \{1,\ldots,n-k\}$ and as a consequence that $x_i = \overline{x}_{n-k}$ for all $i \in \{1,\ldots,n-k\}$. As a consequence,

$$x_i = \frac{(n-k)x_i + k\,\overline{x}^*_k}{n} = x_i - \frac{k\,x_i - k\,\overline{x}^*_k}{n} \Leftrightarrow 0 = \frac{k\,x_i - k\,\overline{x}^*_k}{n} = \frac{k}{n}(x_i - \overline{x}^*_k) \Leftrightarrow x_i = \overline{x}^*_k.$$

This allows us to conclude that

$$\nabla h_{n-k} = \vec{0}_{n-k} \quad \Leftrightarrow \quad \vec{x}_{n-k} = \overline{x}^*_k\,\vec{1}_{n-k}.$$

Thus, the function h_{n-k} does not have any relative maxima on $\mathbb{R}^{n-k}$, and has a unique relative minimum on $\mathbb{R}^{n-k}$ at $\vec{x}_{n-k} = \overline{x}^*_k\,\vec{1}_{n-k}$.

We are now in a position to find the expression of $\vec{x}_{\max}$ and $\vec{x}_{\min}$. Let us begin with the maximum. Let $\vec{x}_{\max} = (\tilde{x}_1,\ldots,\tilde{x}_n)$, and let us prove that $\tilde{x}_i \in \{y_i, z_i\}$,

$\forall i \in \{1, \ldots, n\}$. Assume ex-absurdo that there exists a non empty set of indices $J \subseteq \{1, \ldots, n\}$ such that $\tilde{x}_i \in \{y_i, z_i\}$ for all $i \notin J$ and $\tilde{x}_i \notin \{y_i, z_i\} \, \forall i \in J$ (and as a consequence that $\tilde{x}_i \in (y_i, z_i) \, \forall i \in J$). Considering $k \in \{0, \ldots, n-1\}$ such that $|J| = n - k$, the invariance under permutations of h implies that it can be assumed w.l.o.g. that $J = \{1, \ldots, n-k\}$, obtaining that:

$$\tilde{x}_i \in \{y_i, z_i\} \quad \forall i \in \{n-k+1, \ldots, n\} \text{ and } \tilde{x}_i \in (y_i, z_i) \quad \forall i \in \{1, \ldots, n-k\}.$$

This allows to express the maximum as $\vec{x}_{\max} = (\vec{x}^*_{n-k}, \vec{x}^*_k)$, with

$$\begin{aligned}
\vec{x}^*_{n-k} &= (\tilde{x}_1, \ldots, \tilde{x}_{n-k}) \in (y_1, z_1) \times \ldots \times (y_{n-k}, z_{n-k}), \\
\vec{x}^*_k &= (\tilde{x}_{n-k+1}, \ldots, \tilde{x}_n) \in \{y_{n-k+1}, z_{n-k+1}\} \times \ldots \times \{y_n, z_n\}.
\end{aligned}$$

Since $\vec{x}_{\max}$ is a maximum of h in I_n, according to Lemma 1, $\vec{x}^*_{n-k}$ will be a relative maximum on $\mathbb{R}^{n-k}$ of

$$\begin{aligned}
h_{n-k} : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R} \\
\vec{x}_{n-k} &\hookrightarrow h_{n-k}(\vec{x}_{n-k}) := h(\underbrace{(\vec{x}_{n-k}, \vec{x}^*_k)}_{\in \mathbb{R}^n}).
\end{aligned} \tag{13}$$

But it has already been proven that the function h_{n-k} does not have any relative maximum on $\mathbb{R}^{n-k}$, which leads to a contradiction. Therefore,

$$\vec{x}_{\max} = \underset{\substack{x_i \in \{y_i, z_i\} \\ i \in \{1, \ldots, n\}}}{\arg\max} \{h(\vec{x})\} .$$

Next, we study the expression of the minimum, for which we will distinguish between two cases, depending on whether $\bigcap_{i=1}^{n} [y_i, z_i]$ is empty or not.

If $\bigcap_{i=1}^{n} [y_i, z_i]$ is not empty, there exists $\gamma \in \mathbb{R}$ such that $\gamma \in [y_i, z_i]$ for all $i \in \{1, \ldots, n\}$, and therefore $\vec{\gamma} := \gamma \vec{1}_n \in I_n$. Since it holds that $h(\vec{\gamma}) = 0$ and h is a non-negative function, it follows that $\min\{h(\vec{x}) : \vec{x} \in I_n\} = 0$.

If $\bigcap_{i=1}^{n} [y_i, z_i]$ is empty, considering $\vec{x}_{\min} = (\hat{x}_1, \ldots, \hat{x}_n)$, there exists $J \subseteq \{1, \ldots, n\}$ a subset of indices such that $\hat{x}_i \in \{y_i, z_i\}$ for all $i \notin J$ and $\hat{x}_i \notin \{y_i, z_i\}$ for all $i \in J$, whence $\hat{x}_i \in (y_i, z_i)$ for all $i \in J$.

The purpose now is to prove that $\vec{x}_{\min}$ belongs to the set of quasivertices. For that reason, we will distinguish into three separate cases:

Firstly, if $J = \{1, \ldots, n\}$, it follows that:

$$\hat{x}_i \in (y_i, z_i), \quad \forall i \in \{1, \ldots, n\} \Rightarrow \vec{x}_{\min} \in (I_n)^o = (y_1, z_1) \times \ldots \times (y_n, z_n).$$

Since $\vec{x}_{\min}$ is a minimum of h in I_n and $\vec{x}_{\min} \in (I_n)^o$, it follows that $\vec{x}_{\min}$ is a relative minimum of h on $\mathbb{R}^n$. Considering that the relative minima of h on $\mathbb{R}^n$ are the points of the form $\vec{x} = \gamma\,\vec{1}_n$ where $\gamma \in \mathbb{R}$, it follows that $\vec{x}_{\min} = \gamma\,\vec{1}_n$. Thus, $\gamma = \hat{x}_i \in [y_i, z_i]$ for all $i \in \{1, \ldots, n\}$, and consequently $\gamma \in \bigcap_{i=1}^{n} [y_i, z_i]$. Since $\bigcap_{i=1}^{n} [y_i, z_i] = \emptyset$, this has led to a contradiction, leading us to conclude that it must be $J \neq \{1, \ldots, n\}$.

Secondly, if $J = \emptyset$, it follows that:

$$\hat{x}_i \in \{y_i, z_i\}, \quad \forall i \in \{1, \ldots, n\} \Rightarrow \vec{x}_{\min} \in \{y_1, z_1\} \times \ldots \times \{y_n, z_n\} = V \subseteq QV.$$

Thirdly, if $\emptyset \neq J \subsetneq \{1, \ldots, n\}$, considering $k \in \{0, \ldots, n-1\}$ such that $|J| = n - k$, due to the invariance under permutations of the indices of h it can be assumed that $J = \{1, \ldots, n-k\}$, obtaining that:

$$\hat{x}_i \in \{y_i, z_i\}, \quad \forall i \in \{n-k+1, \ldots, n\}, \text{ and } \hat{x}_i \in (y_i, z_i), \quad \forall i \in \{1, \ldots, n-k\}.$$

Thus, the minimum can be written as $\vec{x}_{\min} = (\vec{x}^*_{n-k}, \vec{x}^*_k)$, with

$$\begin{aligned} \vec{x}^*_{n-k} &= (\tilde{x}_1, \ldots, \tilde{x}_{n-k}) \in (y_1, z_1) \times \ldots \times (y_{n-k}, z_{n-k}), \\ \vec{x}^*_k &= (\tilde{x}_{n-k+1}, \ldots, \tilde{x}_n) \in \{y_{n-k+1}, z_{n-k+1}\} \times \ldots \times \{y_n, z_n\}. \end{aligned}$$

Since $\vec{x}_{\min}$ is a minimum of h in I_n, according to Lemma 1, $\vec{x}^*_{n-k}$ will be a relative minimum on $\mathbb{R}^{n-k}$ of the function h_{n-k} defined in Eq. (13), bearing now in mind that $\vec{x}^*_k$ are now the components corresponding to the minimum rather than the maximum as in Eq. (13). But it has already been proven that the function h_{n-k} has a unique relative minimum on $\mathbb{R}^{n-k}$ at $\vec{x}_{n-k} = \overline{x}^*_k\,\vec{1}_{n-k}$, i.e.,

$$\hat{x}_i = \overline{x}^*_k = \frac{\sum_{j=n-k+1}^{n} \hat{x}_j}{k} = \frac{\sum_{j \notin J} \hat{x}_j}{n - |J|}, \quad \forall i \in \{1, \ldots, n-k\}.$$

Since $\hat{x}_i \in [y_i, z_i]$ for all $i \in \{1, \ldots, n-k\} = J$, we deduce that $\frac{\sum_{j \notin J} \hat{x}_j}{n-|J|} \in \bigcap_{i \in J} [y_i, z_i] \neq \emptyset$, and therefore that $\vec{x}_{\min} \in QV_J \subseteq QV$.

We conclude then that $\vec{x}_{\min} \in QV$ and as a consequence that if $\bigcap_{i=1}^{n} [y_i, z_i]$ is empty, $\vec{x}_{\min} = \underset{\vec{x} \in QV}{\arg\min}\{h(\vec{x})\}$. The proof ends now by direct application of Corollary 1, considering again the cases of $\bigcap_{i=1}^{n} [y_i, z_i]$ empty or not.

If $\bigcap_{i=1}^{n}[y_i, z_i] = \emptyset$, since g_1 and g_2 are continuous, it follows from Corollary 1 that the outer and inner confidence intervals for σ^2 at confidence level $1-\alpha$ are given by:

$$CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = \left[\frac{h(\vec{x}_{\min})}{c_{n-1,1-\alpha/2}}, \frac{h(\vec{x}_{\max})}{c_{n-1,\alpha/2}}\right],$$
$$CI_* = [g_1(\vec{x}^1_{\max}), g_2(\vec{x}^2_{\min})] = \left[\frac{h(\vec{x}_{\max})}{c_{n-1,1-\alpha/2}}, \frac{h(\vec{x}_{\min})}{c_{n-1,\alpha/2}}\right],$$

where it has been proven that the maximum and minimum of the function h in I_n are given by:

$$\vec{x}_{\max} = \mathop{\arg\max}_{\substack{x_i\in\{y_i,z_i\}\\ i\in\{1,\ldots,n\}}} \{h(\vec{x})\} \quad \text{and} \quad \vec{x}_{\min} = \mathop{\arg\min}_{\vec{x}\in QV}\{h(\vec{x})\}.$$

If $\bigcap_{i=1}^{n}[y_i, z_i]$ is not empty, it has been proven that $h(\vec{x}_{\min}) = \min\{h(\vec{x}) : \vec{x} \in I_n\} = 0$, whence $g_1(\vec{x}^1_{\min}) = g_2(\vec{x}^2_{\min}) = 0$. Noting that:

$$g_1(\vec{x}^1_{\max}) \le g_2(\vec{x}^2_{\min}) \Leftrightarrow \underbrace{\frac{h(\vec{x}_{\max})}{c^*_{1-\alpha/2}}}_{\ge 0} \le \underbrace{\frac{h(\vec{x}_{\min})}{c^*_{\alpha/2}}}_{0} \Leftrightarrow h(\vec{x}_{\max}) = 0 \Leftrightarrow y_i = z_i \ \forall i \in \{1,\ldots,n\}$$

we deduce that in this case, the exact value of x_i would be perfectly known for any $i \in \{1,\ldots,n\}$. Assume then that this case does not occur, so that it holds $g_1(\vec{x}^1_{\max}) > g_2(\vec{x}^2_{\min})$. Thus, since g_1 and g_2 are continuous functions, it follows from Corollary 1 that the outer and inner confidence intervals for σ^2 at confidence level $1-\alpha$ are given by:

$$CI^* = [g_1(\vec{x}^1_{\min}), g_2(\vec{x}^2_{\max})] = \left[0, \frac{h(\vec{x}_{\max})}{c^*_{\alpha/2}}\right] \text{ and } CI_* = \emptyset,$$

where it has been proven that the minimum of the function h in I_n is zero and the maximum is given by $\vec{x}_{\max} = \mathop{\arg\max}_{\substack{x_i\in\{y_i,z_i\}\\ i\in\{1,\ldots,n\}}} \{h(\vec{x})\}$. $\square$

Proof of Proposition 8. Since g_1 and g_2 are continuous functions, it follows from Corollary 1 that the inner confidence interval and the outer confidence interval are characterized by the minimum $\vec{x}^1_{\min}$ and $\vec{x}^2_{\min}$ and maximum $\vec{x}^1_{\max}$ and $\vec{x}^2_{\max}$ of g_1 and g_2 respectively.

Considering the centering matrix H_n, g_1 and g_2 can be expressed as:

$$g_1(\vec{x}) = \overline{x} - t_{\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n(n-1)}} = \frac{1}{n}\vec{x}'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{\vec{x}'H_n\vec{x}}}{\sqrt{n(n-1)}},$$
$$g_2(\vec{x}) = \overline{x} + t_{\alpha/2}\sqrt{\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n(n-1)}} = \frac{1}{n}\vec{x}'\,\vec{1}_n + t_{\alpha/2}\frac{\sqrt{\vec{x}'H_n\vec{x}}}{\sqrt{n(n-1)}}.$$

Moreover, considering the function h defined in Eq. (6), g_1 and g_2 can be expressed as

$$g_1(\vec{x}) = \frac{1}{n}\vec{x}'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{\vec{x}' H_n \vec{x}}}{\sqrt{n(n-1)}} = \frac{1}{n}\vec{x}'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{h(\vec{x})}}{\sqrt{n(n-1)}},$$
$$g_2(\vec{x}) = \frac{1}{n}\vec{x}'\,\vec{1}_n + t_{\alpha/2}\frac{\sqrt{\vec{x}' H_n \vec{x}}}{\sqrt{n(n-1)}} = \frac{1}{n}\vec{x}'\,\vec{1}_n + t_{\alpha/2}\frac{\sqrt{h(\vec{x})}}{\sqrt{n(n-1)}}.$$

Let us establish some properties of g_1 and g_2. First, we shall prove that they do not have relative extrema.

Regarding function g_1, its gradient is

$$\nabla g_1 = \frac{\partial g_1}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}\left(\frac{1}{n}\vec{x}'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{h(\vec{x})}}{\sqrt{n(n-1)}}\right)$$
$$= \frac{1}{n}\frac{\partial}{\partial \vec{x}}\left(\vec{x}'\,\vec{1}_n\right) - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\frac{\partial}{\partial \vec{x}}\left(\sqrt{h(\vec{x})}\right). \quad (14)$$

Thus, g_1 will be differentiable whenever $\sqrt{h(\vec{x})}$ is, i.e., in those points in which $h(\vec{x}) \neq 0$. It is already known from Proposition 7 that the function h vanishes at those points of the form $\vec{x} = \gamma\,\vec{1}_n$ for any $\gamma \in \mathbb{R}$. Let us distinguish between a number of cases, depending on whether $\vec{x}$ can be expressed in this way or not. Taking the closed set $\Delta_n := \{\vec{x} \in \mathbb{R}^n : \vec{x} = x\,\vec{1}_n, x \in \mathbb{R}\}$, it holds that g_1 is differentiable at $\vec{x}$ if and only if $\vec{x} \notin \Delta_n$.

Case 1: $\vec{x} \notin \Delta_n$.

In this case, g_1 is differentiable on the set $\mathbb{R}^n \setminus \Delta_n$, so it follows that if a point $\vec{x} \in \mathbb{R}^n \setminus \Delta_n$ is a relative extreme of g_1, the gradient must vanish at this point.

According to the expression of ∇h from (7), it follows that:

$$\frac{\partial}{\partial \vec{x}}\left(\vec{x}'\,\vec{1}_n\right) = \vec{1}_n \text{ and } \frac{\partial}{\partial \vec{x}}\left(\sqrt{h(\vec{x})}\right) = \frac{\nabla h(\vec{x})}{\sqrt{h(\vec{x})}} = \frac{2H_n\vec{x}}{\sqrt{h(\vec{x})}}.$$

By combining it with (14), we obtain

$$\nabla g_1 = \frac{1}{n}\,\vec{1}_n - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\,\frac{2H_n\vec{x}}{\sqrt{h(\vec{x})}} = \frac{1}{n}\,\vec{1}_n - A_{\vec{x}}\,H_n\vec{x},$$

using the notation $A_{\vec{x}} = \dfrac{t_{\alpha/2}}{\sqrt{n(n-1)}}\,\dfrac{2}{\sqrt{h(\vec{x})}} \in \mathbb{R}$. Since $A_{\vec{x}} = 0$ would imply that:

$$\nabla g_1 = \frac{1}{n}\,\vec{1}_n - \underbrace{A_{\vec{x}}\,H_n\vec{x}}_{0} = \frac{1}{n}\,\vec{1}_n \neq \vec{0}_n,$$

we deduce that it must be $A_{\vec{x}} \neq 0$, and then:

$$\nabla g_1 = \vec{0}_n \Leftrightarrow \frac{1}{n}\,\vec{1}_n - A_{\vec{x}}\,H_n\vec{x} = \vec{0}_n \Leftrightarrow H_n\vec{x} = \frac{1}{A_{\vec{x}}\,n}\,\vec{1}_n$$
$$\Leftrightarrow \vec{x} - \overline{x}\,\vec{1}_n = \frac{1}{A_{\vec{x}}\,n}\,\vec{1}_n \Leftrightarrow \vec{x} = \left(\overline{x} + \frac{1}{A_{\vec{x}}}\right)\vec{1}_n = B_{\vec{x}}\,\vec{1}_n,$$

where the notation $B_{\vec{x}} = \overline{x} + \dfrac{1}{A_{\vec{x}}} \in \mathbb{R}$ has been considered. However, this leads to a contradiction, since it has been proved that the gradient of g_1 vanishes at those points of the form of $\vec{x} = B_{\vec{x}}\,\vec{1}_n$ and for these $\vec{x} \in \Delta_n$, a contradiction with the assumption $\vec{x} \notin \Delta_n$.
Case 2: $\vec{x} \in \Delta_n$.
In this case $h(\vec{x}) = 0$ and consequently the expression of g_1 is:

$$g_1(\vec{x}) = \frac{1}{n}\vec{x}'\,\vec{1}_n - \underbrace{t_{\alpha/2}\frac{\sqrt{h(\vec{x})}}{\sqrt{n(n-1)}}}_{0} = \frac{1}{n}\vec{x}'\,\vec{1}_n.$$

Since $\vec{x} = x\,\vec{1}_n,\ x \in \mathbb{R}$, we are dealing with a real function:

$$\tilde{g}_1(x) := g_1(x\,\vec{1}_n) = \frac{1}{n}(x\,\vec{1}_n)'\,\vec{1}_n = x\,\frac{\vec{1}_n'\,\vec{1}_n}{n} = x\,\frac{n}{n} = x.$$

It is clear that the function g_1 will not have extrema on $\mathbb{R}^n$. A similar reasoning allows us to conclude that g_2 has no extrema on $\mathbb{R}^n$.

Having studied the relative maxima and minima on $\mathbb{R}^n$ of functions g_1 and g_2, it must be taken into account that the extrema of those functions on I_n are the ones that must be found to construct the inner and outer confidence intervals. For this purpose, the following families of functions are constructed. Given $k \in \{1,\dots,n-1\}$ and $\vec{x}_k^* \in \mathbb{R}^k$ fixed, the auxiliary functions h_{n-k}^- and h_{n-k}^+ are defined as follows:

$$\begin{aligned}
h_{n-k}^- : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R}\\
\vec{x}_{n-k} &\hookrightarrow h_{n-k}^-(\vec{x}_{n-k}) := g_1(\underbrace{(\vec{x}_{n-k},\vec{x}_k^*)}_{\in\mathbb{R}^n}),\\
h_{n-k}^+ : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R}\\
\vec{x}_{n-k} &\hookrightarrow h_{n-k}^+(\vec{x}_{n-k}) := g_2(\underbrace{(\vec{x}_{n-k},\vec{x}_k^*)}_{\in\mathbb{R}^n}).
\end{aligned}$$

The importance of those functions lies in the fact that, if there exists $k \in \{1,\dots,n-1\}$ and $\vec{x}^* = (\vec{x}_{n-k}^*, \vec{x}_k^*) \in \mathbb{R}^n$ such that: $\vec{x}_{n-k}^* \in I_{n-k}^o := (y_1,z_1)\times\ldots\times(y_{n-k},z_{n-k})$, according to Lemma 1 it follows that if $\vec{x}^*$ is a relative maximum (minimum) of g_1 in I_n, then $\vec{x}_{n-k}^*$ is a relative maximum (minimum) of h_{n-k}^- on $\mathbb{R}^{n-k}$. An analogous relationship exists between the extrema of the functions g_2 and h^+_{n-k}. Therefore, the extrema on $\mathbb{R}^{n-k}$ of the auxiliary function functions h^-_{n-k} and h^+_{n-k} must be computed.

In order to calculate the extrema of functions h^-_{n-k} and h^+_{n-k}, an explicit expression of those function in terms of $\vec{x}_{n-k}$ must be provided. These more practical expressions will be related with the function h_{n-k} given by Eq. (13) and will allow us to compute its gradient by relating it to the centering and quasicentering matrices. Firstly, we have that:

$$h^-_{n-k}(\vec{x}_{n-k}) = g_1((\vec{x}_{n-k}, \vec{x}^*_k)) = \frac{1}{n}(\vec{x}_{n-k}, \vec{x}^*_k)'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{h((\vec{x}_{n-k}, \vec{x}^*_k))}}{\sqrt{n(n-1)}}.$$

Since $h((\vec{x}_{n-k}, \vec{x}^*_k)) = h_{n-k}(\vec{x}_{n-k})$ and expanding the first term of the sum we obtain

$$\frac{1}{n}(\vec{x}_{n-k}, \vec{x}^*_k)'\,\vec{1}_n = \frac{1}{n}\vec{x}'_{n-k}\,\vec{1}_{n-k} + \frac{1}{n}\vec{x}^{*\prime}_k\,\vec{1}_k.$$

It follows that:

$$\begin{aligned} h^-_{n-k}(\vec{x}_{n-k}) &= \frac{1}{n}(\vec{x}_{n-k}, \vec{x}^*_k)'\,\vec{1}_n - t_{\alpha/2}\frac{\sqrt{h((\vec{x}_{n-k}, \vec{x}^*_k))}}{\sqrt{n(n-1)}} \\ &= \frac{1}{n}\vec{x}'_{n-k}\,\vec{1}_{n-k} + \frac{1}{n}\vec{x}^{*\prime}_k\,\vec{1}_k - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\sqrt{h_{n-k}(\vec{x}_{n-k})}. \end{aligned}$$

Exchanging the roles of h^-_{n-k} and h^+_{n-k}, as well as those of g_1 and g_2, it is shown that:

$$\begin{aligned} h^+_{n-k}(\vec{x}_{n-k}) &= \frac{1}{n}(\vec{x}_{n-k}, \vec{x}^*_k)'\,\vec{1}_n + t_{\alpha/2}\frac{\sqrt{h((\vec{x}_{n-k}, \vec{x}^*_k))}}{\sqrt{n(n-1)}} \\ &= \frac{1}{n}\vec{x}'_{n-k}\,\vec{1}_{n-k} + \frac{1}{n}\vec{x}^{*\prime}_k\,\vec{1}_k + \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\sqrt{h_{n-k}(\vec{x}_{n-k})}. \end{aligned}$$

Thus, the relative extrema on $\mathbb{R}^{n-k}$ of the functions can now be calculated by computing their gradients. Firstly, to calculate the gradient of function h^-_{n-k} it holds that:

$$\begin{aligned} \nabla h^-_{n-k} &= \frac{\partial h^-_{n-k}}{\partial \vec{x}_{n-k}} = \frac{\partial}{\partial \vec{x}_{n-k}}\left(\frac{1}{n}\vec{x}'_{n-k}\,\vec{1}_{n-k} + \frac{1}{n}\vec{x}^{*\prime}_k\,\vec{1}_k - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\sqrt{h_{n-k}(\vec{x}_{n-k})}\right) \\ &= \frac{1}{n}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}'_{n-k}\,\vec{1}_{n-k}\right) + \frac{1}{n}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}^{*\prime}_k\,\vec{1}_k\right) - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\sqrt{h_{n-k}(\vec{x}_{n-k})}\right). \end{aligned} \tag{15}$$

With respect to the gradient of h^+_{n-k}, it holds that:

$$\nabla h^+_{n-k}$$

$$= \frac{1}{n}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}'_{n-k}\,\vec{1}_{n-k}\right)+\frac{1}{n}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}^{*\prime}_{k}\,\vec{1}_{k}\right)+\frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\frac{\partial}{\partial \vec{x}_{n-k}}\left(\sqrt{h_{n-k}(\vec{x}_{n-k})}\right).$$

Thus, h^{-}_{n-k} y h^{+}_{n-k} will be differentiable whenever $\sqrt{h_{n-k}(\vec{x}_{n-k})}$ is, i.e., in those points in which $h_{n-k}(\vec{x}_{n-k}) = h((\vec{x}_{n-k}, \vec{x}^*_k)) \neq 0$. It is already known from Proposition 7 that the function h vanishes at those points of the form $\vec{x} = \gamma\,\vec{1}_n$ for any $\gamma \in \mathbb{R}$, so it follows that the points in which those functions will be differentiable are of the form of $\vec{x}_{n-k} = \gamma\,\vec{1}_{n-k}$, if the fixed components are also of the form of $\vec{x}^*_k = y\,\vec{1}_k$. Let us distinguish between a number of cases, depending on whether $\vec{x}_{n-k}$ can be expressed in this way or not.

Case 1: $\vec{x}^*_k \neq \gamma\,\vec{1}_k$ or $\vec{x}^*_k = \gamma\,\vec{1}_k$ and $\vec{x}_{n-k} \neq \gamma\,\vec{1}_{n-k}$.

In this case the functions h^{-}_{n-k} and h^{+}_{n-k} will be differentiable at $\vec{x}_{n-k}$, that will belong to an open set in which the functions will be differentable, either $\mathbb{R}^{n-k}$ if $\vec{x}^*_k \neq \gamma\,\vec{1}_k$ or $\mathbb{R}^{n-k} \setminus \{\gamma\,\vec{1}_{n-k}\}$ if $\vec{x}^*_k = \gamma\,\vec{1}_k$, so it follows that if $\vec{x}_{n-k}$ is a relative extreme of h^{-}_{n-k} or h^{+}_{n-k} , the respective gradient must vanish at this point.
Starting with the function h^{-}_{n-k}, and according to the expression of ∇h_{n-k} from (12), it follows that:

$$\begin{aligned}
&\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}'_{n-k}\,\vec{1}_{n-k}\right) = \vec{1}_{n-k},\\
&\frac{\partial}{\partial \vec{x}_{n-k}}\left(\vec{x}^{*\prime}_{k}\,\vec{1}_{k}\right) = \vec{0}_{n-k},\\
&\frac{\partial}{\partial \vec{x}_{n-k}}\left(\sqrt{h_{n-k}(\vec{x}_{n-k})}\right) = \frac{\nabla h_{n-k}(\vec{x}_{n-k})}{2\sqrt{h_{n-k}(\vec{x}_{n-k})}} = \frac{2H^*_{n-k}\vec{x}_{n-k} - \frac{2}{n}J_{n-k,k}\vec{x}^*_k}{2\sqrt{h_{n-k}(\vec{x}_{n-k})}}.
\end{aligned}$$

By combining it with (15), it is obtained that:

$$\nabla h^{-}_{n-k} = \frac{1}{n}\vec{1}_{n-k} - \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\frac{H^*_{n-k}\vec{x}_{n-k} - \frac{1}{n}J_{n-k,k}\vec{x}^*_k}{\sqrt{h_{n-k}(\vec{x}_{n-k})}}.$$

Now the points in which ∇h^{-}_{n-k} vanishes must be found. For this, notice that

$$\begin{aligned}
\nabla h^{-}_{n-k} = \vec{0}_{n-k} &\Leftrightarrow \frac{1}{n}\vec{1}_{n-k} = \frac{t_{\alpha/2}}{\sqrt{n(n-1)}}\frac{H^*_{n-k}\vec{x}_{n-k} - \frac{1}{n}J_{n-k,k}\vec{x}^*_k}{\sqrt{h_{n-k}(\vec{x}_{n-k})}}\\
&\Leftrightarrow \sqrt{\frac{n-1}{n}}\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\,\vec{1}_{n-k} = H^*_{n-k}\vec{x}_{n-k} - \frac{1}{n}J_{n-k,k}\vec{x}^*_k. \qquad (16)
\end{aligned}$$

Expanding the right-hand side of the equation,

$$H^*_{n-k}\vec{x}_{n-k} - \frac{1}{n}J_{n-k,k}\vec{x}^*_k = \vec{x}_{n-k} - \frac{n-k}{n}\overline{x}_{n-k}\vec{1}_{n-k} - \frac{k}{n}\overline{x}^*_{n-k}\vec{1}_{n-k}. \qquad (17)$$

By introducing (17) into (16) we deduce that:

$$\nabla h^{-}_{n-k} = \vec{0}_{n-k} \Leftrightarrow \vec{x}_{n-k} = \frac{n-k}{n}\overline{x}_{n-k}\vec{1}_{n-k} + \frac{k}{n}\overline{x}^{*}_{k}\,\vec{1}_{n-k} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\,\vec{1}_{n-k}. \tag{18}$$

Noting that:

$$\begin{aligned}&\frac{n-k}{n}\overline{x}_{n-k}\vec{1}_{n-k} + \frac{k}{n}\overline{x}^{*}_{k}\,\vec{1}_{n-k} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\,\vec{1}_{n-k}\\ &= \left(\frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^{*}_{k}}{n} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)\vec{1}_{n-k},\end{aligned} \tag{19}$$

it follows that:

$$\vec{x}_{n-k} = \left(\frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^{*}_{k}}{n} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)\vec{1}_{n-k}, \tag{20}$$

which implies that:

$$x_i = \frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^{*}_{k}}{n} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}, \quad \forall i \in \{1,\dots,n-k\}. \tag{21}$$

In particular,

$$x_i = x_j, \quad \forall i,j \in \{1,\dots,n-k\} \quad \Rightarrow \quad x_i = \overline{x}_{n-k}. \tag{22}$$

By combining the reasoning made from Equations (18) to (22) it is obtained that:

$$\begin{aligned}&\nabla h^{-}_{n-k} = \vec{0}_{n-k}\\ &\qquad\Leftrightarrow x_i = \frac{(n-k)\overline{x}_{n-k} + k\,\overline{x}^{*}_{k}}{n} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\ \forall i \in \{1,\dots,n-k\}.\end{aligned}$$

Taking then $\vec{x}_{n-k}$ of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$ with $y \in \mathbb{R}$, it holds that $\overline{x}_{n-k} = y$, so it follows that:

$$\begin{aligned}\nabla h^{-}_{n-k} = \vec{0}_{n-k} \quad &\Leftrightarrow \quad y = \frac{(n-k)y + k\,\overline{x}^{*}_{k}}{n} + \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}} \quad \Leftrightarrow\\ &\frac{y-\overline{x}^{*}_{k}}{n}\,k = \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}} \quad \Rightarrow\\ &\left(\frac{y-\overline{x}^{*}_{k}}{n}\,k\right)^2 = \left(\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)^2.\end{aligned} \tag{23}$$

Following the same process for the function h^+_{n-k}, it is obtained that the points in which its gradient vanishes are of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$ with $y \in \mathbb{R}$ and it also holds that:

$$\begin{aligned}
\nabla h^+_{n-k} = \vec{0}_{n-k} \quad \Leftrightarrow \quad y = \frac{(n-k)y + k\,\overline{x}^*_k}{n} - \sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}} \quad \Leftrightarrow \\
\frac{y - \overline{x}^*_k}{n}\;k = -\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}} \quad \Rightarrow \\
\left(\frac{y - \overline{x}^*_k}{n}\;k\right)^2 = \left(\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)^2 .
\end{aligned} \tag{24}$$

Combining Equations (23) and (24), it holds that if $\vec{x}_{n-k}$ is a point in which the gradient of h^-_{n-k} or h^+_{n-k} vanishes, $\vec{x}_{n-k}$ is of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$, with $y \in \mathbb{R}$ verifying:

$$\left(\frac{y - \overline{x}^*_k}{n}\;k\right)^2 = \left(\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)^2 . \tag{25}$$

To expand the previous expression, we shall given an explicit expression of $h_{n-k}(\vec{x}_{n-k})$ when $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$. The expression of h_{n-k} given in Eq. (10) was:

$$h_{n-k}(\vec{x}_{n-k}) = \vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k + \vec{x}^{*\prime}_k H^*_k \vec{x}^*_k$$

Since

$$\begin{aligned}
\vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} &= (n-k)\left(\overline{x^2}_{n-k} - \frac{n-k}{n}\overline{x}^2_{n-k}\right) \\
&= (n-k)\left(y^2 - \frac{n-k}{n} y^2\right) = y^2\,\frac{k(n-k)}{n}, \\
\vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k &= (n-k)\overline{x}_{n-k}\,k\,\overline{x}^*_k = y\,(n-k)k\,\overline{x}^*_k, \\
\vec{x}^{*\prime}_k H^*_k \vec{x}^*_k &= k\left(\overline{x^{*2}_k} - \frac{k}{n}\,\overline{x^*_k}^2\right) = k\left(\overline{x^{*2}_k} - \overline{x^*_k}^2 + \frac{n-k}{n}\,\overline{x^*_k}^2\right) \\
&= k\left(\overline{x^{*2}_k} - \overline{x^*_k}^2\right) + \frac{k(n-k)}{n}\,\overline{x^*_k}^2 \\
&= k\;(\vec{x}^{*\prime}_k H_k \vec{x}^*_k) + \frac{k(n-k)}{n}\,\overline{x^*_k}^2,
\end{aligned}$$

it follows that:

$$\begin{aligned}
h_{n-k}(\vec{x}_{n-k}) &= \vec{x}'_{n-k} H^*_{n-k} \vec{x}_{n-k} - \frac{2}{n} \vec{x}'_{n-k} J_{n-k,k} \vec{x}^*_k + \vec{x}^{*\prime}_k H^*_k \vec{x}^*_k \\
&= y^2\,\frac{k(n-k)}{n} - 2y\,\frac{k(n-k)}{n}\overline{x}^*_k + \frac{k(n-k)}{n}\,\overline{x^*_k}^2 + k\;(\vec{x}^{*\prime}_k H_k \vec{x}^*_k)
\end{aligned}$$

$$= A_1\, y^2 - 2\, A_1\, y\, \overline{x_k^*} + A_1\, \overline{x_k^*}^2 + k\, (\vec{x}_k^{*\prime} H_k \vec{x}_k^*)$$
$$= A_1\, \left(y - \overline{x_k^*}\right)^2 + k\, (\vec{x}_k^{*\prime} H_k \vec{x}_k^*)\,,$$

where the notation $A_1 := \dfrac{k(n-k)}{n}$ has been considered. Therefore:

$$\begin{aligned}
\left(\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)^2 &= \frac{n-1}{n}\,\frac{h_{n-k}(\vec{x}_{n-k})}{t_{\alpha/2}^2}\\
&= \frac{n-1}{n}\,\frac{A_1\, \left(y - \overline{x_k^*}\right)^2 + k\, (\vec{x}_k^{*\prime} H_k \vec{x}_k^*)}{t_{\alpha/2}^2}\\
&= A_2\, \left(y - \overline{x_k^*}\right)^2 + B\, \vec{x}_k^{*\prime} H_k \vec{x}_k^*,
\end{aligned}$$

using the notation $A_2 := \dfrac{n-1}{n}\,\dfrac{A_1}{t_{\alpha/2}^2} = \dfrac{(n-1)(n-k)k}{n^2\, t_{\alpha/2}^2}$ and $B := \dfrac{n-1}{n}\,\dfrac{k}{t_{\alpha/2}^2}$. Thus,

$$\begin{aligned}
\left(\frac{y - \overline{x_k^*}}{n}\, k\right)^2 = \left(\sqrt{\frac{n-1}{n}}\,\frac{\sqrt{h_{n-k}(\vec{x}_{n-k})}}{t_{\alpha/2}}\right)^2&\\
\Leftrightarrow \frac{k^2}{n^2}\left(y - \overline{x_k^*}\right)^2 = A_2\, \left(y - \overline{x_k^*}\right)^2 + B\, \vec{x}_k^{*\prime} H_k \vec{x}_k^*&\\
\Leftrightarrow A\left(y - \overline{x_k^*}\right)^2 = B\, \vec{x}_k^{*\prime} H_k \vec{x}_k^* \Leftrightarrow \left(y - \overline{x_k^*}\right)^2 = C\, \vec{x}_k^{*\prime} H_k \vec{x}_k^*,&
\end{aligned} \tag{26}$$

where $A := \frac{k^2}{n^2} - A_2 = \frac{k^2}{n^2} - \dfrac{(n-1)(n-k)k}{n^2\, t_{\alpha/2}^2}$ and $C := \dfrac{B}{A}$.

Finally, combining Equations (25) and (26), it is obtained that if $\vec{x}_{n-k}$ is a point in which the gradient of h_{n-k}^- or h_{n-k}^+ vanishes, $\vec{x}_{n-k}$ is of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$, with $y \in \mathbb{R}$ verifying:

$$\left(y - \overline{x_k^*}\right)^2 = C\, \vec{x}_k^{*\prime} H_k \vec{x}_k^*.$$

Case 2: $\vec{x}_k^* = y\,\vec{1}_k$ and $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$.
In this case, $\vec{x}_{n-k}$ is of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$ so it holds that:

$$\vec{x}_k^* = y\,\vec{1}_k \quad \Rightarrow \quad \overline{x_k^*} = y, \quad \overline{x_k^{*2}} = y^2, \tag{27}$$

Thus,

$$\vec{x}_k^{*\prime} H_k \vec{x}_k^* = k\, \left(\overline{x_k^{*2}} - \overline{x_k^*}^2\right) = k\,(y^2 - y^2) = 0. \tag{28}$$

Combining Equations (27) and (28), it holds that $\vec{x}_{n-k}$ is of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$ with $y \in \mathbb{R}$ verifying $\left(y - \overline{x_k^*}\right)^2 = 0 = C\,\vec{x}_k^{*\prime} H_k \vec{x}_k^*$, so the relative extrema of functions h_{n-k}^- or h_{n-k}^+ on $\mathbb{R}^{n-k}$ will be points of that form.

Thus, it has been proved that in both cases, if a point $\vec{x}_{n-k}$ is a relative extreme of h_{n-k}^- or h_{n-k}^+ on $\mathbb{R}^{n-k}$, $\vec{x}_{n-k}$ is of the form of $\vec{x}_{n-k} = y\,\vec{1}_{n-k}$, with $y \in \mathbb{R}$ verifying: $\left(y - \overline{x_k^*}\right)^2 = C\,\vec{x}_k^{*\prime} H_k \vec{x}_k^*$. We are now in a position to find the set in which the extrema of functions g_1 and g_2 in I_n are reached.

Starting with the function g_1, considering $\vec{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ an extreme point of g_1 in I_n, there exists $J \subseteq \{1, \ldots, n\}$ a set of indices such that $\hat{x}_i \in \{y_i, z_i\}$ for all $i \notin J$ and $\hat{x}_i \notin \{y_i, z_i\}$ for all $i \in J$ (whence $\hat{x}_i \in (y_i, z_i) \quad \forall i \in J$). The purpose now is to prove that $\vec{x}$ belongs to the set of 2-quasivertices. We shall consider three separate cases.

Firstly, if $J = \{1, \ldots, n\}$, it follows that:

$$\hat{x}_i \in (y_i, z_i) \quad \forall i \in \{1, \ldots, n\} \Rightarrow \vec{x} \in (I_n)^o = (y_1, z_1) \times \ldots \times (y_n, z_n).$$

Since $\vec{x}$ is an extreme point of g_1 in I_n and $\vec{x} \in (I_n)^o$, it follows that $\vec{x}$ is a relative extreme of g_1 on $\mathbb{R}^n$, which is known to be a function with no relative extrema on $\mathbb{R}^n$. This is a contradiction, from which we conclude that $J \neq \{1, \ldots, n\}$.

Secondly, if $J = \emptyset$ it follows that:

$$\hat{x}_i \in \{y_i, z_i\} \quad \forall i \in \{1, \ldots, n\} \Rightarrow \vec{x} \in \{y_1, z_1\} \times \ldots \times \{y_n, z_n\} = V \subseteq Q^2 V.$$

Thirdly, if $\emptyset \neq J \subsetneq \{1, \ldots, n\}$, considering $k \in \{0, \ldots, n-1\}$ such that $|J| = n-k$, due to the invariance under permutations of the indices of g_1 it can be assumed that $J = \{1, \ldots, n-k\}$, whence

$$\begin{aligned} \hat{x}_i &\in \{y_i, z_i\} \quad \forall i \in \{n-k+1, \ldots, n\}, \\ \hat{x}_i &\in (y_i, z_i) \quad \forall i \in \{1, \ldots, n-k\}. \end{aligned}$$

Thus, $\vec{x}$ can be written as $\vec{x} = (\vec{x}_{n-k}^*, \vec{x}_k^*)$, taking

$$\vec{x}_{n-k}^* \in (y_1, z_1) \times \ldots \times (y_{n-k}, z_{n-k}), \quad \vec{x}_k^* \in \{y_{n-k+1}, z_{n-k+1}\} \times \ldots \times \{y_n, z_n\}.$$

Since $\vec{x}$ is an extreme point of g_1 in I_n, according to Lemma 1, $\vec{x}_{n-k}^*$ will be a relative extreme on $\mathbb{R}^{n-k}$ of

$$\begin{aligned} h_{n-k}^- : \mathbb{R}^{n-k} &\longrightarrow \mathbb{R} \\ \vec{x}_{n-k} &\hookrightarrow h_{n-k}^-(\vec{x}_{n-k}) := g_1(\underbrace{(\vec{x}_{n-k}, \vec{x}_k^*)}_{\in \mathbb{R}^n}). \end{aligned}$$

But it has already been proven that the extrema on $\mathbb{R}^{n-k}$ of the function h_{n-k}^- are of the form of $y\,\vec{1}_{n-k}$, with $y \in \mathbb{R}$ verifying $\left(y - \overline{x_k^*}\right)^2 = 0 = C\,\vec{x}_k^{*\prime} H_k \vec{x}_k^*$. Thus, $\vec{x} \in Q^2 V_J \subseteq Q^2 V$.

We conclude then that both the maximum and the minimum of g_1 on I_n belong to the set of 2-quasivertices. A similar reasoning shows that the maximum and the minimum of g_2 on I_n belong to the set of 2-quasivertices, and as a consequence

$$\begin{aligned}
\vec{x}^1_{\min} &= \underset{\vec{x}\in Q^2V}{\arg\min}\{g_1(\vec{x})\}\,, \qquad \vec{x}^1_{\max} = \underset{\vec{x}\in Q^2V}{\arg\max}\{g_1(\vec{x})\}\,,\\
\vec{x}^2_{\min} &= \underset{\vec{x}\in Q^2V}{\arg\min}\{g_2(\vec{x})\}\,, \qquad \vec{x}^2_{\max} = \underset{\vec{x}\in Q^2V}{\arg\max}\{g_2(\vec{x})\}\,.
\end{aligned}$$

The proof ends by direct application of Corollary 1. Since g_1 and g_2 are continuous functions, the outer and inner confidence intervals for μ at confidence level $1-\alpha$ are given by:

$$CI^* = \left[g_1(\vec{x}^1_{\min}),\, g_2(\vec{x}^2_{\max})\,\right] \quad \text{and } CI_* = \left[g_1(\vec{x}^1_{\max}),\, g_2(\vec{x}^2_{\min})\right]. \quad \square$$

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