

International Journal of General Systems

ISSN: 0308-1079 (Print) 1563-5104 (Online) Journal homepage: www.tandfonline.com/journals/ggen20

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To cite this article: Enrique Miranda & Ignacio Montes (15 Jun 2026): On the closedness of imprecise probability models under aggregation, International Journal of General Systems, DOI: 10.1080/03081079.2026.2686085

To link to this article: https://doi.org/10.1080/03081079.2026.2686085

Published online: 15 Jun 2026.

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INTERNATIONAL JOURNAL OF GENERAL SYSTEMS
https://doi.org/10.1080/03081079.2026.2686085

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On the closedness of imprecise probability models under aggregation

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ABSTRACT

Given a number of imprecise probability models, we aim at aggregating them into a joint one using an aggregation rule such as the conjunction, disjunction, convex mixture, Pareto, conjunction-disjunction or maximal consistent subsets rules. We focus on the problem of analysing if these operators are closed, in the sense that the output belongs to the same family as the inputs. Specifically, we analyse this problem for the family of comparative probabilities, 2-monotone capacities, probability intervals, belief functions, *p*-boxes and minitive measures.

ARTICLE HISTORY
Received 1 October 2025
Accepted 1 June 2026

KEYWORDS
Belief pooling; 2-monotone capacities; *p*-boxes; comparative probabilities; probability intervals

1. Introduction

The problem of belief aggregation appears naturally in contexts where the information is coming from different sources, or when different experts provide their input on the uncertainty. The second of these scenarios is often linked to subjective assessments, and in those cases it may be more reliable to model the uncertainty using an imprecise model. This leads to the problem of aggregating imprecise probabilities, that adds some layers of difficulty to that of belief pooling.

Building on the work carried out in Bradley (2019), Destercke, Dubois, and Chojnacki (2009), Dubois et al. (2016), Moral and del Sagrado (1998), Stewart and Quintana (2018) and Walley (1982), in Miranda, Salamanca, and Montes (2025) we tackled the problem of aggregating coherent lower previsions. Specifically, we compared six different rules: the conjunction, disjunction, convex mixture, Pareto, conjunction-disjunction and the maximal consistent subsets rules, and our comparison was made in terms of twenty different rationality criteria, based on the earlier works in the aforementioned references.

While coherent lower previsions are undoubtedly interesting, they are not the only mathematical model that has been proposed to deal with imprecise information; we may for instance consider Choquet capacities (Choquet 1953–1954), belief functions (Shafer 1976), probability intervals (de Campos, Huete, and Moral 1994) or possibility and necessity measures (Dubois and Prade 1988). These models differ from each other in their mathematical properties and, in some cases, in the underlying interpretation

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attached. Their different features make them intrinsically interesting, to the point that we may focus on any of these families as our uncertainty model.

Should we decide to model imprecision by means of one of these models, the problem of aggregation must be reconsidered, and some of the rationality axioms we discussed in Miranda, Salamanca, and Montes (2025) may have to be adapted. For instance, in Miranda, Salamanca, and Montes (2025) we considered as a basic requirement that the output of the aggregation of a finite number of coherent lower previsions should again be a coherent lower prevision; but if we model our uncertainty in terms of belief functions, it would make sense to impose that their aggregation should be a belief function too. There are two main reasons for this: on the one hand, there are some specific mathematical advantages of some of the models, that we may want to preserve through the aggregation; but also from a conceptual point of view, if for instance we work in the realm of belief functions and give an evidential interpretation to our uncertainty, this interpretation should carry on to the aggregated model.

In this paper, we consider a finite possibility space and analyse the aforementioned aggregation rules for a number of imprecise probability models: comparative probabilities (de Finetti 1931; Koopman 1940), 2-monotone capacities (Choquet 1953–1954), probability intervals (de Campos, Huete, and Moral 1994), belief functions (Shafer 1976), minitive measures (Zadeh 1978) and probability boxes (p-boxes, for short) (Ferson et al. 2003). We shall investigate which of the aggregation procedures is a closed operator within each of these families, and in some cases provide sufficient conditions for this to be the case. This should assist the practitioner in the choice of an aggregation rule or of an uncertainty model.

The paper is organised as follows: after recalling in Section 2 the imprecise probability models of interest and in Section 3 the aggregation rules that shall be analysed in this paper, we proceed to study whether they are closed for comparative probabilities Section 4, 2-monotone capacities Section 5, probability intervals Section 6, belief functions Section 7, minitive measures Section 8 and p-boxes Section 9. Finally, a comparison of the results obtained and some additional discussion are given in Section 10.

A preliminary version of this paper was presented at the ISIPTA 2025 conference (Miranda, Salamanca, and Montes 2025). This extended version differs in a number of ways: on the one hand, we analyse the problem of aggregation for any finite number of models, and not necessarily two; secondly, we incorporate into our study the conjunction-disjunction and maximal consistent subsets rules; thirdly, we discuss in detail the preservation of the properties from events to gambles; fourthly, we give detailed proofs and not only sketch of proofs; and finally, we expand some of the comments and results presented in the conference version.

2. Imprecise probability models

Let $\mathcal{X}$ be a finite possibility space of cardinality n, and denote by $\mathbb{P}(\mathcal{X})$ the set of probability measures on $\mathcal{P}(\mathcal{X})$. Following Walley (1991), a real-valued function $f : \mathcal{X} \to \mathbb{R}$ is called *gamble* and the set of all gambles is denoted by $\mathcal{L}(\mathcal{X})$. We will use $P(f)$ to denote the expectation of the gamble f with respect to the probability measure $P \in \mathbb{P}(\mathcal{X})$. As particular cases of gambles, we have the indicator functions I_A of subsets A of $\mathcal{X}$, that take the value 1 on A and 0 on A^c.

A *lower prevision* is map $\underline{P} : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X})$. It determines a (possibly empty) closed and convex set of probability measures, called *credal set*, by

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\mathcal{X}) : P(f) \geq \underline{P}(f)\ \forall f \in \mathcal{K}\}.$$

The lower prevision is said to *avoid sure loss* when $\mathcal{M}(\underline{P}) \neq \emptyset$ and it is called *coherent* when $\underline{P}(f) = \min\{P(f) : P \in \mathcal{M}(\underline{P})\}$ for every $f \in \mathcal{K}$. A lower prevision has a conjugate *upper* prevision $\overline{P}$ given by $\overline{P}(f) = -\underline{P}(-f)$ for any f such that $-f \in \mathcal{K}$.

Conversely, any closed and convex non-empty set of probability measures $\mathcal{M}$ determines a coherent lower prevision $\underline{P} : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$ by

$$\underline{P}(f) := \inf\{P(f) : P \in \mathcal{M}\} \quad \forall f \in \mathcal{L}(\mathcal{X}).$$

Any coherent lower prevision $\underline{P}$ defined on a set of gambles $\mathcal{K}$ can always be extended to $\mathcal{L}(\mathcal{X})$ using the *natural extension*:

$$\underline{E}(f) = \min\{P(f) : P \in \mathcal{M}(\underline{P})\} \quad \forall f \in \mathcal{L}(\mathcal{X}), \tag{1}$$

which satisfies $\underline{P}(f) = \underline{E}(f)$ for any $f \in \mathcal{K}$ and $\mathcal{M}(\underline{P}) = \mathcal{M}(\underline{E})$.

In the remaining of this section we present the particular families of lower previsions that will be considered in Sections 4–8.

2.1. Comparative probabilities

The first model we consider in this paper is that of *comparative probabilities* (de Finetti 1931; Walley and Fine 1979) made on singletons[1], a model thoroughly analysed in Miranda and Destercke (2015).

Consider a set $\mathcal{C} \subseteq \mathcal{X} \times \mathcal{X}$, where each pair $(x_i, x_j) \in \mathcal{C}$ is interpreted as "the probability of x_i is at least as large as that of x_j". These assessments can be represented by the lower prevision $\underline{P}$ defined on $\mathcal{K} = \{I_{x_i} - I_{x_j} : (x_i, x_j) \in \mathcal{C}\}$ by:

$$\underline{P}(I_{x_i} - I_{x_j}) = 0 \quad \forall\, (x_i, x_j) \in \mathcal{C}. \tag{2}$$

Therefore, $\underline{P}$ is constant and equal to 0 on $\mathcal{K}$. $\underline{P}$ is called the *comparative probability* determined by $\mathcal{C}$ and its credal set can be expressed as:

$$\begin{aligned}\mathcal{M}(\underline{P}) &= \{P \in \mathbb{P}(\mathcal{X}) : P(I_{x_i} - I_{x_j}) \geq \underline{P}(I_{x_i} - I_{x_j}) = 0\ \forall\, (x_i, x_j) \in \mathcal{C}\} \\ &= \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_i\}) \geq P(\{x_j\})\ \forall\, (x_i, x_j) \in \mathcal{C}\},\end{aligned} \tag{3}$$

noting that $P(I_{x_i} - I_{x_j}) = P(I_{x_i}) - P(I_{x_j}) = P(\{x_i\}) - P(\{x_j\})$.

The credal set $\mathcal{M}(\underline{P})$ above is non-empty because the uniform probability distribution is compatible with any set $\mathcal{C}$ of pairwise comparisons. As a consequence, any comparative probability determines a coherent lower prevision. Moreover, when $\mathcal{C} = \emptyset$ it holds that $\mathcal{M}(\underline{P}) = \mathbb{P}(\mathcal{X})$.

The natural extension of the comparative probability $\underline{P}$ determined by $\mathcal{C}$ is the coherent lower prevision that is the lower envelope of the credal set $\mathcal{M}(\underline{P})$ given in Equation (3), as explained in Equation (1). When dealing with comparative probabilities, and for the sake of simplicity, we keep the same notation $\underline{P}$ for the natural extension.

2.2. 2-monotone capacities

A *capacity*, also called fuzzy measure or lower probability in the literature (see for instance Grabisch 2016), is a function $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ that is normalised ($\underline{P}(\emptyset) = 0$ and $\underline{P}(\mathcal{X}) = 1$) and monotone increasing ($A \subseteq B$ implies $\underline{P}(A) \leq \underline{P}(B)$). Its conjugate capacity, denoted by $\overline{P}$, is defined by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for any $A \subseteq \mathcal{X}$. The reason for using the notation $\underline{P}$ and $\overline{P}$ is that a capacity and its conjugate can be interpreted respectively as a lower and an upper prevision defined on indicators of events $\mathcal{K} = \{I_A : A \subseteq \mathcal{X}\}$, identifying $\underline{P}(A) = \underline{P}(I_A)$ and $\overline{P}(A) = \overline{P}(I_A)$. Hence, the notions of avoiding sure loss and coherence apply as well to capacities. In particular, any coherent capacity satisfies superadditivity, meaning that $\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$ for any pairwise disjoint $A, B \subseteq \mathcal{X}$.

A capacity $\underline{P}$ is called *2-monotone*, or supermodular, when it satisfies

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall\, A, B \subseteq \mathcal{X}. \tag{4}$$

A 2-monotone capacity is always coherent. Moreover, while not every coherent capacity is 2-monotone, both concepts are equivalent when $\mathcal{X}$ is of cardinality three (de Cooman, Troffaes, and Miranda 2008).

2.3. Probability intervals

Probability intervals are one particular case of 2-monotone capacities; they are determined by lower and upper bounds on the probability of singletons. Specifically, we consider a set of intervals $\mathcal{I} = \{[l_i, u_i] : i = 1, \ldots, n\}$ where $l_i \leq u_i$ and $[l_i, u_i] \subseteq [0,1]$ for any $i = 1, \ldots, n$, and the credal set

$$\{P \in \mathbb{P}(\mathcal{X}) : l_i \leq P(\{x_i\}) \leq u_i \,\forall\, i = 1, \ldots, n\}.$$

Whenever this set is non-empty, which holds if and only if $\sum_{i=1}^n l_i \leq 1 \leq \sum_{i=1}^n u_i$ (see e.g. de Campos, Huete, and Moral 1994, Equation (2)), taking lower and upper envelopes on events we obtain a coherent capacity $\underline{P}$ and its conjugate $\overline{P}$ given, for any $A \subseteq \mathcal{X}$, by

$$\underline{P}(A) = \max\left\{\sum_{x_i \in A} l_i, 1 - \sum_{x_i \notin A} u_i\right\}, \quad \overline{P}(A) = \min\left\{\sum_{x_i \in A} u_i, 1 - \sum_{x_i \notin A} l_i\right\}. \tag{5}$$

We refer to $\underline{P}$ as the *probability interval* determined by $\mathcal{I}$. Moreover, the probability interval is called *reachable* when the capacity $\underline{P}$ and its conjugate $\overline{P}$ satisfy $\underline{P}(\{x_i\}) = l_i$ and $\overline{P}(\{x_i\}) = u_i$ for any $i = 1, \ldots, n$. Probability intervals were quite thoroughly studied in de Campos, Huete, and Moral (1994), where it was shown that reachable probability intervals are also 2-monotone.

2.4. Belief functions

We focus next on another particular case of 2-monotone capacities: belief functions. A capacity $\underline{P}$ is *k-monotone* if it satisfies:

$$\underline{P}\left(\cup_{i=1}^k A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \underline{P}(\cap_{i \in I} A_i) \quad \forall\, A_1, \ldots, A_k \subseteq \mathcal{X}, \tag{6}$$

and it is called *belief function* when it is k-monotone for any $k \in \mathbb{N}$. We refer to Shafer (1976) for a thorough study of belief functions, and to Dempster (1967) for their connection to random sets. They can be characterised by their Möbius inverse m, given by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B) \quad \forall\, A \subseteq \mathcal{X}$$

through the formula

$$\underline{P}(A) = \sum_{B \subseteq A} m(B) \quad \forall\, A \subseteq \mathcal{X}. \tag{7}$$

While the Möbius inverse and the relationship in Equation (7) hold for any capacity $\underline{P}$, belief functions stand out because for them $m(A) \geq 0$ for every $A \subseteq \mathcal{X}$ and $\sum_{A \subseteq \mathcal{X}} m(A) = 1$, so m can be regarded as a probability mass function on the subsets of $\mathcal{X}$; this is the basis of the *evidential* interpretation (Shafer 1976). The *focal elements* are those events A for which $m(A) > 0$, and the set of focal elements is denoted by $\mathcal{F}$.

2.5. Minitive measures

Minitive measures are a particular case of belief functions. They are those capacities $\underline{P}$ satisfying

$$\underline{P}(A \cap B) = \min\{\underline{P}(A), \underline{P}(B)\} \quad \forall\, A, B \subseteq \mathcal{X}.$$

They correspond to belief functions whose focal elements are nested. Indeed, it is also possible to characterise the focal elements as those sets A such that $\underline{P}(A) > \max\{\underline{P}(B) : B \subset A\}$.

The conjugate upper probability of a minitive measure is called *maxitive*, and satisfies $\overline{P}(A \cup B) = \max\{\overline{P}(A), \overline{P}(B)\}$ for every $A, B \subseteq \mathcal{X}$.

An example of minitive measure is the so-called *vacuous model* on an event $V \subseteq \mathcal{X}$, given by:

$$\underline{P}_V(A) = \begin{cases} 1, & \text{if } V \subseteq A, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

It corresponds with a belief function with only one focal event: $m(V) = 1$ and $m(A) = 0$ for any other $A \neq V$, and its credal set $\mathcal{M}(\underline{P})$ is formed by the probability measures P satisfying $P(V) = 1$.

2.6. p-boxes

The final family we consider in this paper are probability boxes, or p-boxes for short. These correspond to models that are determined by the lower and upper probabilities of the cumulative events. Assume then that the possibility space $\mathcal{X} = \{x_1, \ldots .x_n\}$ is ordered, $x_1 < x_2 < \ldots < x_n$. Any pair $(\underline{F}, \overline{F})$ of ordered $(\underline{F} \leq \overline{F})$ cumulative distribution functions $\underline{F}, \overline{F} : \mathcal{X} \to [0, 1]$ $(\underline{F}(x_n) = \overline{F}(x_n) = 1)$ determine a credal set by

$$\mathcal{M}(\underline{F}, \overline{F}) = \{P \in \mathbb{P}(\mathcal{X}) : \underline{F}(x_i) \leq F_P(x_i) \leq \overline{F}(x_i)\ \forall\, i = 1, \ldots, n\},$$

where F_P denotes the cumulative distribution function associated with P. A capacity $\underline{P}$ is called a *p-box* if there exists $(\underline{F}, \overline{F})$ such that $\mathcal{M}(\underline{P}) = \mathcal{M}(\underline{F}, \overline{F})$. In such case, $\underline{P}$ is a

belief function whose focal elements are ordered intervals, in the sense that (Troffaes and Destercke 2011)

$$[a,b] \preceq [c,d] \Leftrightarrow a \leq c \text{ and } b \leq d. \tag{9}$$

Note that any coherent capacity $\underline{P}$, and its conjugate $\overline{P}$, determines a pair $(\underline{F},\overline{F})$ by

$$\underline{F}(x_i) = \underline{P}(\{x_1,\ldots,x_i\}), \quad \overline{F}(x_i) = \overline{P}(\{x_1,\ldots,x_i\}),\ \forall\, i = 1,\ldots,n.$$

However, in general $\mathcal{M}(\underline{P}) \subseteq \mathcal{M}(\underline{F},\overline{F})$, and the equality will hold if and only if $\underline{P}$ is a belief function whose focal elements are ordered intervals. We refer to Ferson et al. (2003); Troffaes and Destercke (2011) for some seminal studies on p-boxes.

3. Aggregation rules

After presenting the uncertainty models of interest, we are now ready to introduce the aggregation problem. Consider a group of m experts, each of them modelling the uncertainty about the experiment using a coherent lower prevision on $\mathcal{L}(\mathcal{X})$. An aggregation procedure aims at determining another lower prevision on $\mathcal{L}(\mathcal{X})$ summarising the global opinion of the group. In our formal definition of an aggregation rule, we shall denote by $\underline{\mathbb{P}}(\mathcal{X})$ the set of lower previsions on $\mathcal{X}$ and by $\underline{\mathbb{P}}'(\mathcal{X})$ the subset of coherent lower previsions.

Definition 3.1: An *aggregation rule on coherent lower previsions* is a map $\mathcal{A} : (\underline{\mathbb{P}}'(\mathcal{X}))^m \to \underline{\mathbb{P}}(\mathcal{X})$.

In what follows, and to ease the notation, we will simply write $\mathcal{A}$ instead of $\mathcal{A}(\underline{P}_1,\ldots,\underline{P}_m)$ when no confusion is possible.

The aggregation rules we shall analyse in this paper are the following[2]:

Conjunction ($\mathcal{A}_C$) It corresponds (Destercke, Dubois, and Chojnacki 2009; Walley 1982) to the *natural extension* of the lower prevision $\max_{i=1,\ldots,m} \underline{P}_i$, given, for any $f \in \mathcal{L}(\mathcal{X})$, by

$$\mathcal{A}_C(f) = \sup_{f_1,\ldots,f_k,k} \inf_{x\in\mathcal{X}} \left\{ f(x) - \sum_{i=1}^{k} \left(f_i(x) - \left(\max_{j=1,\ldots,m} \underline{P}_j(f_i) \right) \right) : f_i \in \mathcal{L}(\mathcal{X}), i = 1,\ldots,k \right\},$$

where the supremum is taken over all $f_1,\ldots,f_k \in \mathcal{L}(\mathcal{X})$ and $k \in \mathbb{N}$. Its credal set can be expressed as $\mathcal{M}(\mathcal{A}_C) = \mathcal{M}(\underline{P}_1) \cap \ldots \cap \mathcal{M}(\underline{P}_m)$. Therefore, the conjunction rule is a coherent lower prevision if, and only if, $\mathcal{M}(\underline{P}_1) \cap \ldots \cap \mathcal{M}(\underline{P}_m)$ is non-empty or, equivalently, if $\max_{i=1,\ldots,m} \underline{P}_i$ avoids sure loss.

Disjunction ($\mathcal{A}_D$) It gives (Destercke, Dubois, and Chojnacki 2009; Walley 1982) $\mathcal{A}_D(f) = \min\{\underline{P}_1(f),\ldots,\underline{P}_m(f)\}$ for any gamble $f \in \mathcal{L}(\mathcal{X})$. This rule is called *convex pooling* in Stewart and Quintana (2018). Its credal set can be expressed by $\mathcal{M}(\mathcal{A}_D) = \mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m))$, where $\mathcal{CH}$ denotes the *convex hull*.

Convex mixture ($\mathcal{A}_M$) Given fixed $\alpha_1,\ldots,\alpha_m \geq 0$ satisfying $\alpha_1 + \ldots + \alpha_m = 1$, the convex mixture aggregation rule gives (Destercke, Dubois, and Chojnacki 2009;

Walley 1982)

$$\mathcal{A}_M(f) = \sum_{i=1}^{m} \alpha_i \underline{P}_i(f), \quad \forall f \in \mathcal{L}(\mathcal{X}).$$

This rule is called *linear opinion pooling* in Stewart and Quintana (2018). Its credal set can be expressed by:

$$\begin{aligned}&\alpha_1 \mathcal{M}(\underline{P}_1) \oplus \ldots \oplus \alpha_m \mathcal{M}(\underline{P}_m)\\ &\quad := \left\{\alpha_1 P_1 + \ldots + \alpha_m P_m : P_i \in \mathcal{M}(\underline{P}_i)\ \forall\, i = 1, \ldots, m\right\},\end{aligned}$$

where $\oplus$ represents the Minkowski sum. In what follows, we shall assume w.l.o.g. that all the weights α_i are strictly positive to include in the aggregation all the m sources.

Pareto ($\mathcal{A}_P$) The Pareto rule (Walley 1982) aggregates the models in the following lower prevision:

$$\mathcal{A}_P(f) = \min\left\{\max_{j=1,\ldots,m} \underline{P}_j(f), \min_{j=1,\ldots,m} \overline{P}_j(f)\right\}, \quad \forall f \in \mathcal{L}(\mathcal{X}).$$

Conjunction-Disjunction (CD) rule ($\mathcal{A}_{CD}$) This rule applies the conjunction rule when it leads to a coherent lower prevision (that is, when the lower prevision $\max_{i=1,\ldots,m} \underline{P}_i$ avoids sure loss) and the disjunction rule otherwise (Dubois et al. 2016).

Maximal consistent subsets (MCS) rule ($\mathcal{A}_{MCS}$) A set of indices $I \subseteq \{1, \ldots, m\}$ is a *maximal consistent subset* if $\cap_{i\in I} \mathcal{M}(\underline{P}_i) \neq \emptyset$, or equivalently if $\max_{i\in I} \underline{P}_i$ avoids sure loss, and it is maximal with this property. Given $I_1, \ldots, I_k$ the maximal consistent subsets, this rule gives rise to the coherent lower prevision that is the lower envelope of the credal set:

$$\mathcal{M} = \mathcal{CH}\left(\bigcup_{j=1}^{k} \bigcap_{i\in I_j} \mathcal{M}(\underline{P}_i)\right).$$

Equivalently, this rule applies the conjunction to the maximal consistent subsets and then aggregates them using the disjunction rule.

We refer to Bradley (2019), Dubois et al. (2016), Moral and del Sagrado (1998), Stewart and Quintana (2018) and Walley (1982) for a deeper discussion of the rationality underlying these rules. Some of them are more appropriate from the perspective of probabilistic opinion pooling, while others are more interesting from the point of view of consensus seeking. See also (Miranda, Salamanca, and Montes 2025) for a thorough discussion of these rules in the context of coherent lower previsions. In particular, in this reference it was shown that the Pareto rule need not give a coherent lower prevision as an output, that the conjunction rule does so if and only if the maximum of the inputs avoids sure loss and that the other rules always provide a coherent lower prevision. That the conjunction rule does not always give a coherent lower prevision is the motivation for the conjunction-disjunction and the MCS rules, that try to follow the ideas underlying the conjunction rule while taking into account the presence of conflict.

It is worth mentioning that convex mixtures are strongly related to the approach based on eventwise aggregations of capacities. Given m capacities $\mu_1, \ldots, \mu_m$, this approach considers *consensus* functions $C_A : [0,1]^m \to [0,1]$ for any $A \subset \mathcal{X}$, which ideally satisfy some sort of idempotence property, and defines the aggregate capacity as $C(\mu_1, \ldots, \mu_m)(A) = C_A(\mu_1(A), \ldots, \mu_m(A))$ for any $A \subset \mathcal{X}$. The consensus approach has been widely investigated for necessity and possibility measures (Dubois and Prade 1990), belief functions (Wagner 1989), capacities (Bronevich 2005, 2007) or the general class of decomposable measures (Dubois et al. 1996). When the function C_A is given by $C_A(t_1, \ldots, t_m) = \alpha_1 t_1 + \ldots + \alpha_m t_m$ for every $A \subset \mathcal{X}$ for some $\alpha_1, \ldots, \alpha_m \in [0,1]$ such that $\alpha_1 + \ldots + \alpha_m = 1$, we recover the convex mixture rule. In this work, we focus on convex mixtures rather than eventwise aggregations; this has the advantage of being applicable also to coherent lower previsions or credal sets, and as a consequence to models that are not necessarily defined by their values on events.

Also, the aim of the CD and MCS rules is to find an alternative to the conjunction rule when the sources of information are incompatible. On the one hand, the CD rule applies the conjunction whenever all the sources are compatible, while it applies the disjunction otherwise. On the other hand, the MCS rule applies the conjunction to all the maximal consistent subsets, and later applies the disjunction to the resulting credal sets. In the literature, other approaches to the conjunction rule for conflicting credal sets have been proposed; an interesting summary can be found in Bronevich (2025). One of the main alternatives is the use of generalised credal sets, as suggested by Bronevich and Rozenberg (2015b, 2018), or distortion-based methods using the total variation distance (Nieto-Barba, Miranda, and Montes 2026). Moreover, this problem has also been addressed in the particular case of belief functions, where a distance-based approach has been proposed (Klein, Destercke, and Colot 2018), and where the approach by Bronevich and Rozenberg (2019) is also applicable.

We are now ready to tackle the problem that is our focus in this paper: given a number of coherent lower previsions that belong to one of the families in Section 2, does the aggregate model determined by the rules in this section belong to the same family?

4. Comparative probabilities

The first model we consider is that of comparative probabilities on singletons. Let us denote by $\underline{P}_1, \ldots, \underline{P}_m$ the comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. As explained in Section 2.1, each comparative probability $\underline{P}_k$, for $k \in \{1, \ldots, m\}$, is defined on the domain $\mathcal{K}_k = \{I_{x_i} - I_{x_j} : (x_i, x_j) \in \mathcal{C}_k\}$, meaning that the domains of $\underline{P}_1, \ldots, \underline{P}_m$ may be different. However, as explained in Section 2.1, we may take their natural extensions, that for the sake of simplicity will also be denoted by $\underline{P}_1, \ldots, \underline{P}_m$, and that are defined on $\mathcal{L}(\mathcal{X})$. In this manner, all the coherent lower prevision $\underline{P}_1, \ldots, \underline{P}_m$ are defined on the same domain.

The aggregation of comparative probabilities has partly been considered in Miranda and Destercke (2015, Section 4.5); in this section, we complete the analysis.

4.1. Conjunction

As we mentioned in Section 3, the conjunction rule gives a non-coherent lower prevision when the intersection $\mathcal{M}(\underline{P}_1) \cap \cdots \cap \mathcal{M}(\underline{P}_m)$ is empty. However, this shall not be the case in the context of comparative probabilities, given that the uniform probability distribution is always compatible with a comparative probability on the singletons.

As a consequence, the conjunction $\mathcal{A}_C$ is always coherent. It was established in Miranda and Destercke (2015) that it is moreover associated with a comparative model.

Proposition 4.1 (Miranda and Destercke 2015**, Proposition 9):** *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. Then, their conjunction $\mathcal{A}_C$ is the comparative probability determined by $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$.*

4.2. Disjunction

As shown in Miranda and Destercke (2015, Example 7), the disjunction of a finite number of comparative probabilities need not be a comparative probability. Nevertheless, it is possible to give a necessary and sufficient condition for the disjunction to be within this family, inspired by the work in Miranda and Destercke (2015).

Theorem 4.2: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively, and let $\mathcal{A}_D$ denote their disjunction.*

(i) *If $\underline{P}$ denotes the comparative probability determined by*[3] *$\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$, then:*

$$\mathcal{M}(\mathcal{A}_D) = \mathcal{CH}\left(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m)\right) \subseteq \mathcal{M}(\underline{P}).$$

(ii) *As a consequence, the disjunction $\mathcal{A}_D$ is a comparative probability if and only if:*

$$\mathcal{CH}\left(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m)\right) = \mathcal{M}(\underline{P}).$$

Proof: (i) This has been established in Miranda and Destercke (2015, Proposition 10).

(ii) Since trivially the equality $\mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m)) = \mathcal{M}(\underline{P})$ implies that the disjunction is a comparative probability, it suffices to establish the converse.

Assume then that $\mathcal{A}_D$ is a comparative probability induced by some set of comparisons $\mathcal{C}^{\cup} \subseteq \mathcal{X} \times \mathcal{X}$, and let us prove that $\mathcal{C}^{\cup} = \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$.

On the one hand, if $(x_i, x_j) \in \mathcal{C}^{\cup}$ then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m))$. From (i), it follows that $P \in \mathcal{M}(\underline{P})$ and as a consequence that $(x_i, x_j) \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$.

Conversely, if $(x_i, x_j) \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$, then:

$$P(\{x_i\}) \geq P(\{x_j\}) \quad \forall\, P \in \mathcal{M}(\underline{P}_k),\ k = 1, \ldots, m$$

implying that $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m)$. Now, take $P \in \mathcal{M}(\mathcal{A}_D) = \mathcal{CH}(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m))$. Then, there exist $\alpha_k \in [0, 1]$ and $P_k \in \mathcal{M}(\underline{P}_k)$

for $k = 1, \ldots, m$ such that $\sum_{k=1}^{m} \alpha_k = 1$ and $P = \sum_{k=1}^{m} \alpha_k P_k$. Thus,

$$P(\{x_i\}) = \sum_{k=1}^{m} \alpha_k P_k(\{x_i\}) \geq \sum_{k=1}^{m} \alpha_k P_k(\{x_j\}) = P(\{x_j\}),$$

meaning that $(x_i, x_j) \in \mathcal{C}^{\cup}$. ■

The first item of this result tells us that the comparative probability determined by $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$ is an *outer approximation* of the disjunction $\mathcal{A}_D$ in the family of comparative probabilities[4]. In fact, it is not difficult to show that it is the only *undominated*[5] outer approximation:

Corollary 4.3: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. Then, the comparative probability $\underline{P}$ determined by $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$ is the unique undominated outer approximation of the disjunction $\mathcal{A}_D$ in the family of comparative probabilities.*

Proof: Let $\underline{P}^*$ be another comparative probability, determined by $\mathcal{C}^*$, that outer approximates the disjunction $\mathcal{A}_D$. Consider $(x_i, x_j) \in \mathcal{C}^*$, then:

$$P(\{x_i\}) \geq P(\{x_j\}) \quad \forall\, P \in \mathcal{M}(\underline{P}^*),$$

meaning that in particular $P(\{x_i\}) \geq P(\{x_j\})$ holds for any $P \in \mathcal{M}(\underline{P}_k)$ and any $k = 1, \ldots, m$. Hence, $(x_i, x_j) \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$. We conclude that $\mathcal{C}^* \subseteq \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$, and consequently $\underline{P} \geq \underline{P}^*$. ■

4.3. Convex mixtures

Let us start showing a simple but useful property of comparative probabilities.

Lemma 4.4: *Let $\underline{P}$ be a comparative probability determined by $\mathcal{C}$. If $(x_i, x_j) \notin \mathcal{C}$, then $\underline{P}(I_{x_i} - I_{x_j}) < 0$.*

Proof: Any comparative probability can be represented using a graph $\mathcal{G}$ where the nodes are the elements in $\mathcal{X}$ and we add an edge between x_i and x_j when $(x_i, x_j) \in \mathcal{C}$; see (Miranda and Destercke 2015) for more details. Let $H(x_j)$ be the set of ancestors of x_j, which satisfies $x_i \notin H(x_j)$ and $H(H(x_j)) = H(x_j)$. According to Miranda and Destercke (2015), the probability measure defined by:

$$P(\{x\}) = \begin{cases} \dfrac{1}{|H(x_j)|}, & \text{if } x \in H(x_j), \\ 0, & \text{otherwise,} \end{cases}$$

belongs to $\mathcal{M}(\underline{P})$, hence:

$$\underline{P}(I_{x_i} - I_{x_j}) \leq P(I_{x_i} - I_{x_j}) = P(\{x_i\}) - P(\{x_j\}) = 0 - \frac{1}{|H(x_j)|} < 0.$$

■

Using this property we can give the form of the set of comparisons when the convex mixture is a comparative probability.

Proposition 4.5: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0,1)$. If $\mathcal{A}_M$ is a comparative probability, then it is determined by $\mathcal{C}^M = \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$.*

Proof: On the one hand, if $(x_i, x_j) \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$, then $P(\{x_i\}) \geq P(\{x_j\})$ for any $P \in \mathcal{M}(\underline{P}_k)$ and any $k = 1, \ldots, m$. Given a probability measure $P \in \alpha_1 \mathcal{M}(\underline{P}_1) \oplus \ldots \oplus \alpha_m \mathcal{M}(\underline{P}_m)$, there are $P_k \in \mathcal{M}(\underline{P}_k)$ such that $P = \sum_{k=1}^m \alpha_k P_k$. This implies that

$$P(\{x_i\}) = \sum_{k=1}^m \alpha_k P_k(\{x_i\}) \geq \sum_{k=1}^m \alpha_k P_k(\{x_j\}) = P(\{x_j\}),$$

meaning that $(x_i, x_j) \in \mathcal{C}^M$. This implies $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m \subseteq \mathcal{C}^M$.

On the other hand, if $(x_i, x_j) \in \mathcal{C}^M$, this implies that

$$0 = \underline{P}(I_{x_i} - I_{x_j}) = \sum_{k=1}^m \alpha_k \underline{P}_k(I_{x_i} - I_{x_j}). \tag{10}$$

If $(x_i, x_j) \notin \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$, then there is some k such that $(x_i, x_j) \notin \mathcal{C}_k$, whence $\underline{P}_k(I_{x_i} - I_{x_j}) < 0$ from Lemma 4.4; since for any $k' \neq k$ it is $\underline{P}_{k'}(I_{x_i} - I_{x_j}) \leq 0$, we conclude that Equation (10) is violated. Therefore, $\mathcal{C}^M = \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$. ■

From Proposition 4.5 and the results in the previous subsection we deduce that the convex mixture is a comparative probability just in very particular scenarios.

Corollary 4.6: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0,1)$. $\mathcal{A}_M$ is a comparative probability if and only if $\mathcal{C}_1 = \ldots = \mathcal{C}_m$.*

Proof: On the one hand, if $\mathcal{C}_1 = \ldots = \mathcal{C}_m$, we trivially obtain $\mathcal{M}(\underline{P}_1) = \ldots = \mathcal{M}(\underline{P}_m)$, hence $\underline{P}_1 = \ldots = \underline{P}_m = \mathcal{A}_M$.

On the other hand, if the convex mixture is a comparative probability, then by the previous proposition it is determined by the set of comparisons $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m$. Also, we have the following inclusions:

$$\begin{aligned}\mathcal{M}(\mathcal{A}_M) &\supseteq \mathcal{CH}\left(\mathcal{M}(\underline{P}_1) \cup \ldots \cup \mathcal{M}(\underline{P}_m)\right) \\ &\supseteq \alpha_1 \mathcal{M}(\underline{P}_1) \oplus \ldots \oplus \alpha_m \mathcal{M}(\underline{P}_m) = \mathcal{M}(\mathcal{A}_M)\end{aligned}$$

where the first inclusion comes from Theorem 4.2. We deduce that the disjunction coincides with the convex mixture, but this is only possible if $\mathcal{M}(\underline{P}_1) = \ldots = \mathcal{M}(\underline{P}_m)$ or, equivalently, if $\mathcal{C}_1 = \ldots = \mathcal{C}_m$. ■

We therefore conclude that, whenever the comparative probabilities to be aggregated are different, the convex mixture will not be a comparative probability. On the contrary, when all the comparative probabilities coincide, the convex mixture coincides as well with the initial models and also with the disjunction and conjunction.

4.4. Pareto

As discussed before, the intersection of the credal sets of comparative probabilities is non-empty. It follows then from Miranda, Salamanca, and Montes (2025, Proposition 7.1) that the Pareto rule is given by:

$$\mathcal{A}_P(f) = \max_{i=1,\ldots,m} \underline{P}_i(f) \quad \forall f \in \mathcal{L}(\mathcal{X}). \tag{11}$$

From this condition we obtain the set of comparisons determining its credal set when the Pareto rule gives rise to a comparative probability.

Proposition 4.7: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. If the Pareto rule $\mathcal{A}_P$ is a comparative probability, then it is determined by $\mathcal{C}^P = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$.*

Proof: If $(x_i, x_j) \in \mathcal{C}^P$, then $0 = \mathcal{A}_P(I_{x_i} - I_{x_j}) = \max_{k=1,\ldots,m} \underline{P}_k(I_{x_i} - I_{x_j})$, which implies that $(x_i, x_j) \in \mathcal{C}_k$ for some $k = 1, \ldots, m$, because according to Lemma 4.4 $\underline{P}_l(I_{x_i} - I_{x_j}) < 0$ when $(x_i, x_j) \notin \mathcal{C}_l$. Thus, $\mathcal{C}^P \subseteq \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$.

Conversely, if $(x_i, x_j) \in \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$, we get that $\underline{P}_k(I_{x_i} - I_{x_j}) = 0$ whenever $(x_i, x_j) \in \mathcal{C}_k$ and it is smaller than zero otherwise. Then:

$$\mathcal{A}_P(I_{x_i} - I_{x_j}) = \max_{k=1,\ldots,m} \underline{P}_k(I_{x_i} - I_{x_j}) = 0,$$

meaning that $(x_i, x_j) \in \mathcal{C}^P$. ■

From this result we easily deduce that the Pareto rule gives rise to a comparative probability if and only if it coincides with the conjunction.

Corollary 4.8: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be comparative probabilities determined by $\mathcal{C}_1, \ldots, \mathcal{C}_m$, respectively. Then, the Pareto rule $\mathcal{A}_P$ is a comparative probability if and only if it coincides with the conjunction rule $\mathcal{A}_C$.*

Proof: If $\mathcal{A}_P$ coincides with the conjunction $\mathcal{A}_C$, it is a comparative probability from Proposition 4.1. Conversely, if $\mathcal{A}_P$ is a comparative probability, then from Proposition 4.7 $\mathcal{A}_P$ is determined by the set of comparisons $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$, which is the set of comparisons that corresponds to the conjunction $\mathcal{A}_C$. ■

We also deduce that if $\mathcal{A}_P$ is a comparative probability, the expression of $\mathcal{A}_P$ given in Equation (11) is a coherent lower prevision.

4.5. Conjunction-disjunction and MSC

As already mentioned, the intersection of the credal sets of comparative probabilities is never empty because the uniform distribution is compatible with any comparative probability assessment on the singletons. This implies in particular that the conjunction-disjunction rule becomes the conjunction rule that, as shown in Proposition 4.1, is again a comparative probability.

The same argument leads to the equality between the MCS and the conjunction rule, since there will always be a unique maximal consistent subset, given by $\{1, \ldots, m\}$. Therefore, when working with comparative probabilities, the maximal consistent subsets, conjunction-disjunction and conjunction rules all give the same output.

5. 2-monotone capacities

In the next sections we centre our attention on specific families of capacities. Since capacities are evaluated on (indicators of) events, and not on the set $\mathcal{L}(\mathcal{X})$ of all gambles, we shall focus also on the restriction of the aggregated model $\mathcal{A}_\cdot$ to (indicators of) events, and use the notation $\mathcal{A}_\cdot(B)$ instead of $\mathcal{A}_\cdot(I_B)$ for simplicity.

We start considering the case of 2-monotone capacities $\underline{P}_1, \ldots, \underline{P}_m$.

5.1. Conjunction

In general, the conjunction of two 2-monotone capacities need not be 2-monotone: it suffices to observe that any probability measure is 2-monotone and take two different probability measures $P_1 \neq P_2$, given that if the maximum of a number of coherent lower previsions does not avoid sure loss then the procedure of natural extension gives $\underline{E}(f) = +\infty$ for any gamble $f \in \mathcal{L}(\mathcal{X})$ (Walley 1991, Section 3.1.1).

For a less trivial example where $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss and therefore the conjunction produces a coherent lower probability, consider the following:

Example 5.1: Consider $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be the coherent capacities given in the following table, that also gives the values of $\max\{\underline{P}_1, \underline{P}_2\}$:

A	$\underline{P}_1(A)$	$\underline{P}_2(A)$	$\max\{\underline{P}_1(A), \underline{P}_2(A)\}$
$\{x_1\}$	0.25	0.25	0.25
$\{x_2\}$	0.25	0.25	0.25
$\{x_3\}$	0	0	0
$\{x_4\}$	0	0	0
$\{x_1, x_2\}$	0.5	0.5	0.5
$\{x_1, x_3\}$	0.5	0.25	0.5
$\{x_1, x_4\}$	0.25	0.5	0.5
$\{x_2, x_3\}$	0.25	0.5	0.5
$\{x_2, x_4\}$	0.5	0.25	0.5
$\{x_3, x_4\}$	0	0	0
$\{x_1, x_2, x_3\}$	0.75	0.75	0.75
$\{x_1, x_2, x_4\}$	0.75	0.75	0.75
$\{x_1, x_3, x_4\}$	0.5	0.5	0.5
$\{x_2, x_3, x_4\}$	0.5	0.5	0.5
$\{x_1, x_2, x_3, x_4\}$	1	1	1

It can be verified that both $\underline{P}_1, \underline{P}_2$ are 2-monotone. On the other hand, $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent, since it is the lower envelope of the probability measures with mass functions $(0.25, 0.25, 0.25, 0.25)$ and $(0.5, 0.5, 0, 0)$, and thus coincides with the conjunction $\mathcal{A}_C$. It follows that $\mathcal{A}_C(\{x_1, x_2, x_3\}) + \mathcal{A}_C(\{x_3\}) = 0.75 < 1 = \mathcal{A}_C(\{x_1, x_3\}) + \mathcal{A}_C(\{x_2, x_3\})$, meaning that $\mathcal{A}_C$ is not 2-monotone.

In spite of this, and inspired by Miranda, Troffaes, and Destercke (2015, Proposition 7), it is possible to obtain a sufficient condition for the conjunction to preserve 2-monotonicity:

Proposition 5.1: *Let $\underline{P}_1, \underline{P}_2$ be 2-monotone capacities. If $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex, then $\mathcal{A}_C$ is a 2-monotone capacity and $\mathcal{A}_C = \max\{\underline{P}_1, \underline{P}_2\}$.*

Proof: Taking into account the characterisation of 2-monotonicity from Walley (1981, Corollaries 6.4, 6.5), it suffices to establish that for any pair of events $A \subseteq B$ there exists some $P \geq \mathcal{A}_C$ such that $P(A) = \mathcal{A}_C(A)$ and $P(B) = \mathcal{A}_C(B)$. Consider then such events A, B. Since $\underline{P}_1$ is 2-monotone, there exists some $P_1 \in \mathcal{M}(\underline{P}_1)$ such that $P_1(A) = \underline{P}_1(A)$ and $P_1(B) = \underline{P}_1(B)$; similarly, there is some $P_2 \in \mathcal{M}(\underline{P}_2)$ such that $P_2(A) = \underline{P}_2(A)$ and $P_2(B) = \underline{P}_2(B)$. Since $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is convex, it follows from Zaffalon and Miranda (2013, Theorem 6) that there exists some $\alpha \in [0,1]$ such that $P := \alpha P_1 + (1-\alpha)P_2$ belongs to $\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$ and therefore dominates $\mathcal{A}_C$. On the other hand,

$$P(A) \leq \max\{P_1(A), P_2(A)\} = \max\{\underline{P}_1(A), \underline{P}_2(A)\} \leq \mathcal{A}_C(A), \quad \text{and}$$
$$P(B) \leq \max\{P_1(B), P_2(B)\} = \max\{\underline{P}_1(B), \underline{P}_2(B)\} \leq \mathcal{A}_C(B),$$

whence $P(A) = \mathcal{A}_C(A)$ and $P(A) = \mathcal{A}_C(B)$. Thus, $\mathcal{A}_C$ is 2-monotone and moreover $\mathcal{A}_C(A) = \max\{\underline{P}_1(A), \underline{P}_2(A)\}$ for any event $A \subseteq \mathcal{X}$. ■

Let us show that this sufficient condition is not necessary:

Example 5.2: On the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ consider the following coherent capacities:

B	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}_1(B)$	0.16	0	0	0.4	0.4	0
$\underline{P}_2(B)$	0.15	0	0	0.5	0.3	0
$\mathcal{A}_C(B)$	0.16	0	0	0.5	0.4	0

and with 0 and 1 on $\emptyset$ and $\mathcal{X}$, respectively. Since $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent, by Miranda, Salamanca, and Montes (2025, Proposition 7) it coincides with the conjunction of $\underline{P}_1$ and $\underline{P}_2$. Moreover, since $\mathcal{X}$ has cardinality three, it follows that $\underline{P}_1$, $\underline{P}_2$ and $\max\{\underline{P}_1, \underline{P}_2\}$ are all 2-monotone.

To see however that $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is not convex, note that

$$P_1 = (0.16, 0.24, 0.6) \in \mathcal{M}(\underline{P}_1) \quad \text{and} \quad P_2 = (0.15, 0.35, 0.5) \in \mathcal{M}(\underline{P}_2)$$

but $P = 0.5P_1 + 0.5P_2 \notin \mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$: it holds that $P(\{x_1\}) = 0.155 < \underline{P}_1(\{x_1\})$ and $P(\{x_1, x_2\}) = 0.45 < \underline{P}_2(\{x_1, x_2\})$.

The condition in Proposition 5.1 does not generalise to more than two models, in the sense that, given 2-monotone capacities $\underline{P}_1, \ldots, \underline{P}_m$, it may be that $\cup_{i=1}^m \mathcal{M}(\underline{P}_i)$ is convex while $\mathcal{A}_C$ is not 2-monotone. For a counterexample, it suffices to consider $\underline{P}_1$ and $\underline{P}_2$ as in Example 5.1 and $\underline{P}_3$ the vacuous model on $\mathcal{X}$, given by $\underline{P}_3(A) = 0$ if $A \neq \mathcal{X}$ and $\underline{P}_3(\mathcal{X}) = 1$. In that case, $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2) \cup \mathcal{M}(\underline{P}_3) = \mathcal{M}(\underline{P}_3) = \mathbb{P}(\mathcal{X})$, which is trivially convex; however, $\mathcal{A}_C(\underline{P}_1, \underline{P}_2, \underline{P}_3) = \mathcal{A}_C(\underline{P}_1, \underline{P}_2)$, which is not 2-monotone.

5.2. Disjunction

That 2-monotonicity is not preserved by disjunction follows from the fact that coherent capacities, that are the lower envelopes of a set of probability measures (which are in particular 2-monotone) need not be 2-monotone. Next, we show an explicit example.

Example 5.3: Consider Walley's (1981) classical example where P_1 and P_2 are the probability measures on $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ associated with the mass functions $P_1 = (0.5, 0.5, 0, 0)$ and $P_2 = (0.25, 0.25, 0.25, 0.25)$. Their disjunction $\mathcal{A}_D = \min\{P_1, P_2\}$ satisfies:

$$\mathcal{A}_D(\{x_1, x_3, x_4\}) + \mathcal{A}_D(\{x_1\}) = 0.75 < 1 = \mathcal{A}_D(\{x_1, x_3\}) + \mathcal{A}_D(\{x_1, x_4\})$$

meaning that $\mathcal{A}_D$ is not 2-monotone.

Nevertheless, it is still possible to establish the following sufficient condition for $\mathcal{A}_D$ to preserve 2-monotonicity:

Proposition 5.2: *Let $\underline{P}_1$ be a 2-monotone capacity and $\underline{P}_V$ be the vacuous model on a fixed event $V \subseteq \mathcal{X}$ given by Equation (8). Then, $\mathcal{A}_D$ is a 2-monotone capacity.*

Proof: It follows from the definition of the disjunction rule that

$$\mathcal{A}_D(A) = \begin{cases} \underline{P}_1(A), & \text{if } V \subseteq A, \\ 0, & \text{otherwise.} \end{cases}$$

Let us prove that it satisfies Equation (4). Given events $A, B \subseteq \mathcal{X}$, we have the following possibilities:

- If $V \not\subseteq A$, then

  $$\mathcal{A}_D(A \cup B) + \mathcal{A}_D(A \cap B) - \mathcal{A}_D(A) - \mathcal{A}_D(B) = \mathcal{A}_D(A \cup B) \geq 0$$

  where the inequality follows from the monotonicity of $\mathcal{A}_D$. A similar result can be established if $V \not\subseteq B$.
- If on the other hand $V \subseteq A$ and $V \subseteq B$, then also $V \subseteq A \cap B$, whence

  $$\begin{aligned} &\mathcal{A}_D(A \cup B) + \mathcal{A}_D(A \cap B) - \mathcal{A}_D(A) - \mathcal{A}_D(B) \\ &= \underline{P}_1(A \cup B) + \underline{P}_1(A \cap B) - \underline{P}_1(A) - \underline{P}_1(B) \geq 0, \end{aligned}$$

  taking into account that $\underline{P}_1$ is 2-monotone.

■

A number of comments are in order. First of all, while the vacuous model on a fixed event is a minitive measure, this result does not generalise to arbitrary minitive measures, in the sense that the disjunction of a 2-monotone capacity and a minitive measure need not be 2-monotone; a counterexample is our Example 7.2 later on. Secondly, while Proposition 5.2 only considers two models, the result can be easily extended to a finite number of them, in the following manner:

Corollary 5.3: *Let $\underline{P}_1$ be a 2-monotone capacity and let $\underline{P}_{V_2}, \ldots, \underline{P}_{V_m}$ be the vacuous models on the respective events $V_2, \ldots, V_m$ determined via Equation (8). Then, their disjunction $\mathcal{A}_D$ is a 2-monotone capacity.*

Proof: Reasoning as in Proposition 5.2, the disjunction of $\underline{P}_1, \underline{P}_{V_2} \ldots, \underline{P}_{V_m}$ satisfies

$$\mathcal{A}_D(A) = \begin{cases} \underline{P}_1(A), & \text{if } V_2 \cup \ldots \cup V_m \subseteq A \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, $\mathcal{A}_D$ coincides with the disjunction of $\underline{P}_1$ and $\underline{P}_{V_2 \cup \ldots \cup V_m}$, where $\underline{P}_{V_2 \cup \ldots \cup V_m}$ is the vacuous model on $V_2 \cup \ldots \cup V_m$. Applying Proposition 5.2, we deduce that $\mathcal{A}_D$ is a 2-monotone capacity. ■

5.3. Convex mixtures

Like coherence (Walley 1991, Theorem 2.6.4), 2-monotonicity is preserved by convex mixtures[6]:

Proposition 5.4: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be 2-monotone capacities. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0, 1)$. Then, $\mathcal{A}_M$ is also a 2-monotone capacity.*

Proof: Let $A, B \subseteq \mathcal{X}$. It holds that:

$$\begin{aligned} \mathcal{A}_M(A \cup B) + \mathcal{A}_M(A \cap B) &= \sum_{k=1}^{m} \alpha_k \underline{P}_k(A \cup B) + \sum_{k=1}^{m} \alpha_k \underline{P}_k(A \cap B) \\ &= \sum_{k=1}^{m} \alpha_k \left(\underline{P}_k(A \cup B) + \underline{P}_k(A \cap B) \right) \geq \sum_{k=1}^{m} \alpha_k \left(\underline{P}_k(A) + \underline{P}_k(B) \right) \\ &= \sum_{k=1}^{m} \alpha_k \underline{P}_k(A) + \sum_{k=1}^{m} \alpha_k \underline{P}_k(B) = \mathcal{A}_M(A) + \mathcal{A}_M(B), \end{aligned}$$

from which we conclude that $\mathcal{A}_M$ is 2-monotone. ■

5.4. Pareto

From Miranda, Salamanca, and Montes (2025, Proposition 7), when $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss it coincides with the output of the Pareto rule; however, this capacity need not be coherent (that is, it need not coincide with the result of the conjunction rule), and as a consequence it need not be 2-monotone either. For an explicit example on a space of cardinality three, we refer to Miranda, Salamanca, and Montes (2025, Example 3).

5.5. Conjunction-disjunction and MCS

Considering that neither the conjunction nor the disjunction procedures preserve 2-monotonicity, we deduce immediately that the conjunction-disjunction procedure does

not do it either; given that, as established in Miranda, Salamanca, and Montes (2025), it preserves coherence, we can trivially establish that it will also preserve 2-monotonicity when the possibility space has cardinality three, simply because in that case coherence is equivalent to 2-monotonicity. However, this property does not hold for higher cardinalities, as can be shown in Example 5.3, where the $\mathcal{A}_{CD}$ rule coincides with $\mathcal{A}_D$ since $\max\{P_1, P_2\}$ does not avoid sure loss, and none of them is 2-monotone.

In addition, since the MCS rule agrees with the CD rule when we only aggregate two models, the comments made about the previous rule allow us to immediately establish that 2-monotonicity is not preserved by the MCS rule in general. We also deduce that it will do so if the possibility space has cardinality three, given that in that case coherence is equivalent to 2-monotonicity. The counterexample mentioned in the previous paragraph shows that this does not extend to higher cardinalities.

6. Probability intervals

In this section we consider reachable probability intervals $\underline{P}_1, \ldots, \underline{P}_m$, determined by $\mathcal{I}_1, \ldots, \mathcal{I}_m$, respectively, where $\mathcal{I}_j = \{[l_i^j, u_i^j] : i = 1, \ldots, n\}$.

6.1. Conjunction

The same comments made for the conjunction of 2-monotone capacities allow us to deduce that the family of probability intervals is not closed under conjunction, simply because the conjunction may not be coherent (i.e. the intersection of their respective credal sets may be empty), and therefore it would not be reachable; for this, it suffices to take into account that probability measures are particular cases of reachable probability intervals, and take two different probability measures $P_1 \neq P_2$. It was proven in de Campos, Huete, and Moral (1994, Proposition 7) that given two reachable probability intervals determined by $\mathcal{I}_1 = \{[l_i^1, u_i^1] : i = 1, \ldots, n\}$ and $\mathcal{I}_2 = \{[l_i^2, u_i^2] : i = 1, \ldots, n\}$, their credal sets are compatible, i.e. they have a non-empty intersection, if and only if:

$$l_i^1 \leq u_i^2 \text{ and } l_i^2 \leq u_i^1 \quad \forall\, i = 1, \ldots, n, \text{ and}$$
$$\sum_{i=1}^{n} \max\left\{l_i^1, l_i^2\right\} \leq 1 \leq \sum_{i=1}^{n} \min\left\{u_i^1, u_i^2\right\}.$$

When two reachable probability intervals are compatible, de Campos et al. proved that the conjunction is again a reachable probability interval[7]:

Proposition 6.1 (de Campos, Huete, and Moral 1994, **Proposition 8**): *Let $\underline{P}_1$ and $\underline{P}_2$ be two compatible reachable probability intervals determined by $\mathcal{I}_1 = \{[l_i^1, u_i^1] : i = 1, \ldots, n\}$ and $\mathcal{I}_2 = \{[l_i^2, u_i^2] : i = 1, \ldots, n\}$, respectively. Then, their conjunction $\mathcal{A}_C$ is again a reachable probability interval determined by $\mathcal{I}_C = \{[l_i^C, u_i^C] : i = 1, \ldots, n\}$, where:*

$$l_i^C = \max\left\{l_i^1, l_i^2, 1 - \sum_{j \neq i} \min\left\{u_i^1, u_i^2\right\}\right\} \quad \text{and}$$

$$u_i^C = \min\left\{u_i^1, u_i^2, 1 - \sum_{j\neq i} \max\left\{l_i^1, l_i^2\right\}\right\} \quad \forall\, i = 1, \ldots, n.$$

Iterating this result, and using the symmetry and associativity of the conjunction rule (Miranda, Salamanca, and Montes 2025), it is not difficult to show that, whenever a finite number m of reachable probability intervals are compatible, their conjunction is also a reachable probability interval.

6.2. Disjunction

De Campos et al. also analysed the disjunction of two reachable probability intervals. In this case, they showed that the disjunction is not a reachable probability interval in general (de Campos, Huete, and Moral 1994, Example 1). They also proved that there is a unique undominated outer approximation of the disjunction using a reachable probability interval, and that it is determined by de Campos, Huete, and Moral (1994, Proposition 9):

$$\mathcal{I} = \left\{\left[\min\left\{l_i^1, l_i^2\right\}, \max\left\{u_i^1, u_i^2\right\}\right] : i = 1, \ldots, n\right\}.$$

This result goes in line with Montes, Miranda, and Vicig (2018, Proposition 5), where it was established that any coherent lower probability has a unique undominated outer approximation using reachable probability intervals, which is determined by the restriction of the capacity and its conjugate to singletons. The result can be easily generalised to the case where we take the disjunction of more than two probability intervals.

6.3. Convex mixtures

Given m reachable probability intervals $\underline{P}_1$, …, $\underline{P}_m$ determined by $\mathcal{I}_1$, …, $\mathcal{I}_m$, and weights $\alpha_1, \ldots, \alpha_m \in (0, 1)$, we can define another probability interval determined by $\mathcal{I}^\alpha = \{[l_i, u_i] : i = 1, \ldots, n\}$, where:

$$[l_i, u_i] = \left[\alpha_1 l_i^1 + \ldots + \alpha_m l_i^m, \alpha_1 u_i^1 + \ldots + \alpha_m u_i^m\right]$$

for $i = 1, \ldots, n$. The following result is a trivial consequence of the convexity of coherent capacities.

Lemma 6.2: *The probability interval $\mathcal{I}^\alpha$ is reachable.*

Next, we find a connection between the convex mixture and the probability interval determined by $\mathcal{I}^\alpha$.

Proposition 6.3: *Let $\underline{P}_1$, …, $\underline{P}_m$ be reachable probability intervals determined by $\mathcal{I}_1$, …, $\mathcal{I}_m$, respectively. If $\underline{P}^\alpha$ denotes the probability interval determined by $\mathcal{I}^\alpha$, then it holds that:*

$$\alpha_1 \mathcal{M}(\underline{P}_1) \oplus \ldots \oplus \alpha_m \mathcal{M}(\underline{P}_m) \subseteq \mathcal{M}(\underline{P}^\alpha). \tag{12}$$

Proof: Given $P_1 \in \mathcal{M}(\underline{P}_1), \ldots, P_m \in \mathcal{M}(\underline{P}_m)$, the prevision $P = \alpha_1 P_1 + \ldots + \alpha_m P_m$ satisfies:

$$P(\{x_i\}) = \alpha_1 P_1(\{x_i\}) + \ldots + \alpha_m P_m(\{x_i\}) \geq \alpha_1 l_i^1 + \ldots + \alpha_m l_i^m = l_i$$
$$P(\{x_i\}) = \alpha_1 P_1(\{x_i\}) + \ldots + \alpha_m P_m(\{x_i\}) \leq \alpha_1 u_i^1 + \ldots + \alpha_m u_i^m = u_i,$$

for any $i = 1, \ldots, n$, implying that $P \in \mathcal{M}(\underline{P}^\alpha)$. ■

Nevertheless, the convex mixture of two reachable probability intervals does not coincide in general with $\mathcal{I}^\alpha$, as we show next. In other words, the convex mixture need not be determined by the capacity and its conjugate of the singletons, and the inclusion in Equation (12) may be strict. This is illustrated in the following example:

Example 6.1: Consider the reachable probability intervals $\underline{P}_1$ and $\underline{P}_2$ determined by $\mathcal{I}_1$ and $\mathcal{I}_2$ given by:

	x_1	x_2	x_3	x_4
$[l_i^1, u_i^1]$	$[0.1, 0.2]$	$[0.2, 0.3]$	$[0.2, 0.3]$	$[0.2, 0.3]$
$[l_i^2, u_i^2]$	$[0.2, 0.4]$	$[0.2, 0.4]$	$[0.1, 0.3]$	$[0.1, 0.35]$
$[l_i, u_i]$	$[0.15, 0.3]$	$[0.2, 0.35]$	$[0.15, 0.3]$	$[0.15, 0.325]$

The table above also shows the values of $\mathcal{I}^\alpha$ for $\alpha_1 = \alpha_2 = 0.5$. Applying Equation (5), we get

$$\underline{P}_1(\{x_1, x_2\}) = \max\left\{l_1^1 + l_2^1, 1 - u_3^1 - u_4^1\right\} = \max\{0.3, 1 - 0.6\} = 0.4.$$
$$\underline{P}_2(\{x_1, x_2\}) = \max\left\{l_1^2 + l_2^2, 1 - u_3^2 - u_4^2\right\} = \max\{0.4, 1 - 0.65\} = 0.4.$$
$$\underline{P}^\alpha(\{x_1, x_2\}) = \max\{l_1 + l_2, 1 - u_3 - u_4\} = \max\{0.35, 1 - 0.625\} = 0.375.$$

Hence, $\underline{P}^\alpha(\{x_1, x_2\}) \neq \alpha_1 \underline{P}_1(\{x_1, x_2\}) + \alpha_2 \underline{P}_2(\{x_1, x_2\})$, whence $\alpha_1 \underline{P}_1 + \alpha_2 \underline{P}_2$ is not a probability interval.

The previous example, together with Proposition 6.3, allows us to deduce that $\mathcal{I}^\alpha$ is an undominated outer approximation of the convex mixture.

Corollary 6.4: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be reachable probability intervals determined by $\mathcal{I}_1, \ldots, \mathcal{I}_m$, respectively. Then, the probability interval $\underline{P}^\alpha$ determined by $\mathcal{I}^\alpha$ is an undominated outer approximation of their convex mixture $\mathcal{A}_M$ in the family of reachable probability intervals.*

Proof: From Proposition 6.3, $\mathcal{I}^\alpha$ is an outer approximation of the convex mixture $\mathcal{A}_M$. Since moreover the reachable probability interval $\underline{P}^\alpha$ determined by $\mathcal{I}^\alpha$ coincides with the convex mixture $\mathcal{A}_M$ on singletons, we deduce that it is undominated. ■

6.4. Pareto rule

It follows from Equation (11) that, when $\max_{i=1,\ldots,m} \underline{P}_i$ avoids sure loss, the Pareto rule evaluated on events can be expressed as $\mathcal{A}_P(A) = \max_{i=1,\ldots,m} \underline{P}_i(A)$ for any $A \subseteq \mathcal{X}$. In

particular, this produces the following lower and upper bounds on the singletons:

$$\mathcal{A}_P(\{x_j\}) = \max_{i=1,\dots,m} l_j^i, \quad \overline{\mathcal{A}_P}(\{x_j\}) = \min_{i=1,\dots,m} u_j^i \quad \forall\, j = 1,\dots,n.$$

Our next example shows that, even in that case, the Pareto rule need not give rise to a probability interval, because it need not be coherent.

Example 6.2: Consider a four-element possibility space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the reachable probability intervals $\underline{P}_1$ and $\underline{P}_2$ determined by $\mathcal{I}_1$ and $\mathcal{I}_2$ given by:

	x_1	x_2	x_3	x_4
$[l_i^1, u_i^1]$	$[0.05, 0.2]$	$[0.15, 0.4]$	$[0.1, 0.2]$	$[0.3, 0.5]$
$[l_i^2, u_i^2]$	$[0.1, 0.3]$	$[0.1, 0.3]$	$[0.1, 0.3]$	$[0.1, 0.3]$

It holds that $\max\{\underline{P}_1, \underline{P}_2\}$ avoids sure loss because taking P^* determined by the mass function $(0.2, 0.3, 0.2, 0.3)$ we get $P^*(A) \geq \max\{\underline{P}_1(A), \underline{P}_2(A)\}$ for any $A \subseteq \mathcal{X}$. However, it is not coherent, because $\mathcal{A}_P$ does not agree with the conjunction $\mathcal{A}_C$: indeed, $\mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2) = \{P^*\}$ because any probability measure Q in the intersection of these credal sets should satisfy

$$Q(\{x_j\}) \leq \min\left\{u_i^1(x_j), u_i^2(x_j)\right\} \quad \forall\, j = 1,\dots,4,$$

and $\sum_{j=1}^4 \min\{u_i^1(x_j), u_i^2(x_j)\} = 1$. From here we deduce that

$$\max\left\{\underline{P}_1(\{x_3\}), \underline{P}_2(\{x_3\})\right\} = 0.1 < 0.2 = P^*(\{x_3\}),$$

Therefore, $\max\{\underline{P}_1, \underline{P}_2\}$ is not coherent, and as a consequence it is not a probability interval.

If on the other hand $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent, then the Pareto rule coincides with the conjunction, and then it produces a reachable probability interval, as shown in Proposition 6.1.

Finally, when $\max\{\underline{P}_1, \underline{P}_2\}$ does not avoid sure loss, the Pareto rule on two reachable probability intervals need not produce another probability interval. For a counterexample, take (Miranda, Salamanca, and Montes 2025, Example 5): there, $\max\{P_1, P_2\}$ incur a sure loss and $\mathcal{A}_P(\underline{P}_1, \underline{P}_2)$ is not coherent, and as a consequence not a probability interval either.

6.5. Conjunction-disjunction and MCS

Taking into account the comments in Subsections 6.1–6.2, it is straightforward to see that the conjunction-disjunction is a reachable probability interval if $\underline{P}_1, \dots, \underline{P}_m$ are compatible coherent probability intervals. When they are not, it cannot be guaranteed that the CD rule gives rise to a probability interval, since in that case it would agree with the disjunction.

Similarly, since the disjunction of two reachable probability intervals is not a probability interval in general, we conclude that the maximal consistent subsets rule gives rise to a probability interval when all the reachable probability intervals are compatible (because it would coincide with the conjunction); otherwise, we cannot guarantee that this rule gives rise to a probability interval.

7. Belief functions

We focus next on another particular case of 2-monotone capacities: belief functions. We denote by $\underline{P}_1, \ldots, \underline{P}_m$ the belief functions to be aggregated, by $m_1, \ldots, m_m$ their Möbius inverses and by $\mathcal{F}_1, \ldots, \mathcal{F}_m$ their sets of focal elements, respectively.

7.1. Conjunction

It is quite straightforward to observe that the conjunction of belief functions is not necessarily a belief function, simply by considering m different probability measures. More interestingly, even if their associated credal sets are compatible (i.e. if $\mathcal{M}(\underline{P}_1) \cap \ldots \cap \mathcal{M}(\underline{P}_m) \neq \emptyset$, or equivalently if $\max_{i=1,\ldots,m} \underline{P}_i$ avoids sure loss), the conjunction rule need not produce a belief function. For a counterexample, consider the capacities $\underline{P}_1$ and $\underline{P}_2$ in Example 5.1, that are belief functions in addition to being 2-monotone.

In Proposition 5.1, we established a sufficient condition for the conjunction of 2-monotone capacities to be 2-monotone. The result does not generalise to belief functions, as we show next:

Example 7.1: Consider the possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be the belief functions, with Möbius inverses m_1 and m_2, given by:

A	$\underline{P}_1(A)$	$m_1(A)$	$\underline{P}_2(A)$	$m_2(A)$	$\mathcal{A}_C(A)$	$m_C(A)$
$\{x_1\}$	0	0	0	0	0	0
$\{x_2\}$	0	0	0	0	0	0
$\{x_3\}$	0	0	0	0	0	0
$\{x_1, x_2\}$	0.5	0.5	0.5	0.5	0.5	0.5
$\{x_1, x_3\}$	0	0	0.5	0.5	0.5	0.5
$\{x_2, x_3\}$	0.5	0.5	0	0	0.5	0.5
$\{x_1, x_2, x_3\}$	1	0	1	0	1	−0.5

$\underline{P}_1$ and $\underline{P}_2$ are belief functions because m_1 and m_2 are non-negative, and $\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2)$ is a convex set taking into account that:

$$\mathcal{M}(\underline{P}_1) = \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_1\}), P(\{x_3\}) \in [0, 0.5]\},$$
$$\mathcal{M}(\underline{P}_2) = \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_1\}), P(\{x_2\}) \in [0, 0.5]\},$$

hence:

$$\begin{aligned}\mathcal{M}(\underline{P}_1) \cup \mathcal{M}(\underline{P}_2) &= \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_1\}) \in [0, 0.5]\} \\ &= \mathcal{CH}\left(\{(0,1,0), (0.5,0,0.5), (1,0,0), (0,0.5,0.5)\}\right).\end{aligned}$$

Both credal sets $\mathcal{M}(\underline{P}_1), \mathcal{M}(\underline{P}_2)$ and their union are represented in Figure 1. Hence, $\mathcal{A}_C = \max\{\underline{P}_1, \underline{P}_2\}$ (its values are reported in the table above) and, applying Proposition 5.1, it is 2-monotone. However, $\mathcal{A}_C$ is not a belief function because its Möbius inverse m_C satisfies $m_C(\mathcal{X}) < 0$.

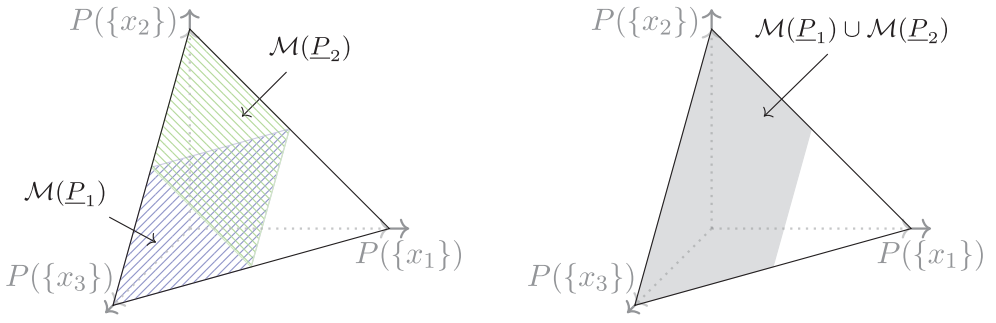


Figure 1. Credal sets of $\underline{P}_1$ and $\underline{P}_2$ (left hand side) and their union (right hand side).

7.2. Disjunction

Clearly, the disjunction of two belief functions does not produce a belief function in general: if it did, any coherent capacity would be a belief function, and we know that this is not the case.

On the other hand, we shall see in Proposition 8.2 that the minimum of two minitive measures is again minitive, and as a consequence also a belief function. We may then wonder if the minimum of a belief function and a minitive measure is again a belief function. Our next example shows that this is not the case:

Example 7.2: Consider a four-element possibility space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be two belief functions with Möbius inverses m_1 and m_2, respectively, reported in the following table:

A	$\underline{P}_1(A)$	$m_1(A)$	$\underline{P}_2(A)$	$m_2(A)$	$\mathcal{A}_D(A)$	$m_D(A)$
$\{x_1\}$	0.1	0.1	0.25	0.25	0.1	0.1
$\{x_2\}$	0.1	0.1	0	0	0	0
$\{x_3\}$	0.1	0.1	0	0	0	0
$\{x_4\}$	0.1	0.1	0	0	0	0
$\{x_1, x_2\}$	0.25	0.05	0.5	0.25	0.25	0.15
$\{x_1, x_3\}$	0.25	0.05	0.25	0	0.25	0.15
$\{x_1, x_4\}$	0.25	0.05	0.25	0	0.25	0.15
$\{x_2, x_3\}$	0.25	0.05	0	0	0	0
$\{x_2, x_4\}$	0.25	0.05	0	0	0	0
$\{x_3, x_4\}$	0.25	0.05	0	0	0	0
$\{x_1, x_2, x_3\}$	0.5	0.05	0.75	0.25	0.5	0.1
$\{x_1, x_2, x_4\}$	0.5	0.05	0.5	0	0.5	0.1
$\{x_1, x_3, x_4\}$	0.5	0.05	0.25	0	0.25	−0.15
$\{x_2, x_3, x_4\}$	0.5	0.05	0	0	0	0
$\{x_1, x_2, x_3, x_4\}$	1	0.1	1	0.25	1	0.4

Note that $\underline{P}_2$ is not only a belief function, but also minitive; its set of (nested) focal elements is $\mathcal{F}_2 = \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}\}$. The table above also gives their disjunction $\mathcal{A}_D = \min\{\underline{P}_1, \underline{P}_2\}$ and its associated Möbius inverse m_D, from which we deduce that $\mathcal{A}_D$ is not a belief function because $m_D(\{x_1, x_3, x_4\}) < 0$.

In fact, the disjunction $\mathcal{A}_D$ in the above example is not even 2-monotone:

$$\mathcal{A}_D(\{x_1, x_3, x_4\}) + \mathcal{A}_D(\{x_1\}) = 0.35 < 0.5 = \mathcal{A}_D(\{x_1, x_3\}) + \mathcal{A}_D(\{x_1, x_4\}).$$

On the other hand, the counterexample cannot be established for smaller cardinalities:

Proposition 7.1: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be belief functions defined on a space of cardinality three $\mathcal{X} = \{x_1, x_2, x_3\}$. If there exists some $i \in \{1, \ldots, m\}$ such that $\underline{P}_i$ is minitive, then the disjunction $\mathcal{A}_D$ is a belief function.*

Proof: If $\underline{P}_i$ is minitive, then its focal elements $\mathcal{F}_i$ are nested. We may assume without loss of generality that $\mathcal{F}_i \subseteq \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$. This implies that the disjunction satisfies

$$\mathcal{A}_D(\{x_2\}) = \mathcal{A}_D(\{x_3\}) = \mathcal{A}_D(\{x_2, x_3\}) = 0,$$

and that the Möbius inverse m_D of $\mathcal{A}_D$ satisfies

$$\begin{aligned} m_D(\{x_1\}) &= \mathcal{A}_D(\{x_1\}), \\ m_D(\{x_1, x_2\}) &= \mathcal{A}_D(\{x_1, x_2\}) - \mathcal{A}_D(\{x_1\}), \\ m_D(\{x_1, x_3\}) &= \mathcal{A}_D(\{x_1, x_3\}) - \mathcal{A}_D(\{x_1\}), \\ m_D(\mathcal{X}) &= 1 - m_D(\{x_1\}) - m_D(\{x_1, x_2\}) - m_D(\{x_1, x_3\}), \end{aligned}$$

and that it is zero otherwise. The monotonicity due to coherence of $\mathcal{A}_D$ allows us to deduce that $m_D(\{x_1\})$, $m_D(\{x_1, x_2\})$ and $m_D(\{x_1, x_3\})$ are non-negative. On the other hand,

$$m_D(\mathcal{X}) = 1 + \mathcal{A}_D(\{x_1\}) - \mathcal{A}_D(\{x_1, x_2\}) - \mathcal{A}_D(\{x_1, x_3\}) \geq 0,$$

taking into account that any coherent capacity on a space of cardinality three is 2-monotone (Walley 1981) and applying the 2-monotonicity condition to $A = \{x_1, x_2\}$ and $B = \{x_1, x_3\}$. We deduce that m_D is non-negative and therefore $\mathcal{A}_D$ is a belief function. ■

In spite of Example 7.2, it is possible to obtain some (albeit restrictive) condition for the disjunction of a belief function and a minitive measure to be a belief function on higher cardinalities.

Proposition 7.2: *Let $\underline{P}_1$ be a belief function and $\underline{P}_V$ be the vacuous model on a fixed event $V \subseteq \mathcal{X}$ given by Equation (8). Then, their disjunction $\mathcal{A}_D$ is a belief function.*

Proof: Reasoning as in the proof of Proposition 5.2, the disjunction of $\underline{P}_1$ and $\underline{P}_V$ satisfies

$$\mathcal{A}_D(A) = \min\{\underline{P}_1(A), \underline{P}_V(A)\} = \begin{cases} \underline{P}_1(A), & \text{if } V \subseteq A \\ 0, & \text{otherwise.} \end{cases}$$

Now, for a fixed $k \in \mathbb{N}$, let us see that $\mathcal{A}_D$ is k-monotone, i.e, that Equation (6) holds for any $A_1, \ldots, A_k \subseteq \mathcal{X}$. We consider the following cases:

Case 1 Assume that $A_1, \ldots, A_k \supseteq V$. This implies that $\cup_{i=1}^k A_i \supseteq V$ and $\cap_{i\in I} A_i \supseteq V$ for any non-empty $I \subseteq \{1, \ldots, k\}$, hence:

$$\mathcal{A}_D\left(\cup_{i=1}^k A_i\right) = \underline{P}_1\left(\cup_{i=1}^k A_i\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}} (-1)^{|I|+1}\underline{P}_1(\cap_{i\in I}A_i)$$
$$= \sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}} (-1)^{|I|+1}\mathcal{A}_D(\cap_{i\in I}A_i),$$

where the inequality follows from the k-monotonicity of the belief function $\underline{P}_1$.

Case 2 Assume that $A_1, \ldots, A_k \not\supseteq V$. This implies that $\cap_{i\in I} A_i \not\supseteq V$ for any non-empty set $I \subseteq \{1, \ldots, k\}$, meaning that $\mathcal{A}_D(\cap_{i\in I}A_i) = 0$ and consequently:

$$\mathcal{A}_D\left(\cup_{i=1}^k A_i\right) \geq 0 = \sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}} (-1)^{|I|+1}\mathcal{A}_D(\cap_{i\in I}A_i).$$

Case 3 Otherwise, assume $A_i \supseteq V$ and $A_j \not\supseteq V$ for some $i, j \in \{1, \ldots, k\}$. We may assume that $A_1, \ldots, A_l \supseteq V$ and $A_{l+1}, \ldots, A_k \not\supseteq V$. Hence, taking a non-empty $I \subseteq \{1, \ldots, k\}$ such that $I \cap \{l+1, \ldots, k\} \neq \emptyset$, it holds that $\cap_{i\in I} A_i \not\supseteq V$, meaning that $\mathcal{A}_D(\cap_{i\in I}A_i) = 0$. Hence:

$$\sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}} (-1)^{|I|+1}\mathcal{A}_D(\cap_{i\in I}A_i) = \sum_{\emptyset\neq I\subseteq\{1,\ldots,l\}} (-1)^{|I|+1}\mathcal{A}_D(\cap_{i\in I}A_i), \tag{13}$$

Also, for any non-empty $I \subseteq \{1, \ldots, l\}$ it holds that $\cap_{i\in I} A_i \supseteq V$, and consequently $\mathcal{A}_D(\cap_{i\in I}A_i) = \underline{P}_1(\cap_{i\in I}A_i)$. This implies that:

$$\sum_{\emptyset\neq I\subseteq\{1,\ldots,l\}} (-1)^{|I|+1}\mathcal{A}_D(\cap_{i\in I}A_i) = \sum_{\emptyset\neq I\subseteq\{1,\ldots,l\}} (-1)^{|I|+1}\underline{P}_1(\cap_{i\in I}A_i)$$
$$\leq \underline{P}_1\left(\cup_{i=1}^l A_i\right) \leq \underline{P}_1\left(\cup_{i=1}^k A_i\right) = \mathcal{A}_D\left(\cup_{i=1}^k A_i\right) \tag{14}$$

where the first inequality follows from the k-monotonicity of the belief function $\underline{P}_1$. Putting Equations (13) and (14) together, we conclude that $\mathcal{A}_D$ satisfies Equation (6).

■

The result extends immediately to the case where we aggregate more than two models:

Corollary 7.3: *Let $\underline{P}_1$ be a belief function and let $\underline{P}_{V_2}, \ldots, \underline{P}_{V_m}$ be the vacuous models on the fixed events $V_2, \ldots, V_m$, respectively. Then, their disjunction $\mathcal{A}_D$ is a belief function.*

Proof: Reasoning as in Proposition 5.2, the disjunction of $\underline{P}_1, \underline{P}_{V_2} \ldots, \underline{P}_{V_m}$ satisfies

$$\mathcal{A}_D(A) = \begin{cases} \underline{P}_1(A), & \text{if } V_2 \cup \ldots \cup V_m \subseteq A \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, $\mathcal{A}_D(\underline{P}_1, \underline{P}_{V_2} \ldots, \underline{P}_{V_m})$ coincides with $\mathcal{A}_D(\underline{P}_1, \underline{P}_{V_2\cup\ldots\cup V_m})$, where $\underline{P}_{V_2\cup\ldots\cup V_m}$ is the vacuous model associated with $V_2 \cup \ldots \cup V_m$. Applying Proposition 7.2, we deduce that $\mathcal{A}_D(\underline{P}_1, \underline{P}_{V_2} \ldots, \underline{P}_{V_m})$ is a belief function. ■

7.3. Convex mixtures

It is almost immediate to establish that belief functions are preserved by convex mixtures (see again (Bronevich 2005) for a more general approach), and moreover to establish a connection between their focal elements:

Proposition 7.4: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be belief functions with respective Möbius inverses $m_1, \ldots, m_m$ and families focal elements $\mathcal{F}_1, \ldots, \mathcal{F}_m$. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0, 1)$. Then, $\mathcal{A}_M$ is a belief function and its set of focal elements is $\mathcal{F}_M = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_m$.*

Proof: Denoting by m_M the Möbius inverse of $\mathcal{A}_M$, it follows that:

$$m_M(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mathcal{A}_M(B) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \left(\sum_{k=1}^{m} \alpha_k \underline{P}_k(B) \right)$$
$$= \sum_{k=1}^{m} \alpha_k \left(\sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}_k(B) \right) = \sum_{k=1}^{m} \alpha_k m_k(A), \quad \forall A \subseteq \mathcal{X}.$$

Since $\underline{P}_1, \ldots, \underline{P}_m$ are belief functions, this allows us to establish that $m_M(A) \geq 0$ for every A, whence $\mathcal{A}_M$ is also a belief function; and also that $m_M(A) > 0$ if and only if there exists some $k = 1, \ldots, m$ such that $m_k(A) > 0$. This implies that $\mathcal{F}_M = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_m$. ■

7.4. Pareto

It is easy to verify that the capacities in Miranda, Salamanca, and Montes (2025, Example 3), for which the Pareto rule does not produce a coherent capacity, are in particular belief functions. As a consequence, this very same counterexample allows us to establish that the family of belief functions is not closed under this rule.

7.5. Conjunction-disjunction and MCS

The same comments made in Section 5 allow us to establish that the family of belief functions it not closed under the conjunction-disjunction nor the maximal consistent subset rules. In this case, we do not even have the result in the case of cardinality three, since then a coherent capacity is always 2-monotone but it need not be a belief function. This can be seen in Example 7.1, where the maximum of two belief functions is a coherent capacity (and therefore coincides with the output of the conjunction, CD or MCS rules) but is not a belief function.

8. Minitive measures

Next, we consider that $\underline{P}_1, \ldots, \underline{P}_m$ are minitive measures or, equivalently, that their conjugate capacities $\overline{P}_1, \ldots, \overline{P}_m$ are maxitive. For the forthcoming results, note that the credal set of a minitive measure $\underline{P}$ can be equivalently expressed in terms of its conjugate maxitive

measure $\overline{P}$ as:

$$\mathcal{M}(\underline{P}) = \left\{P \in \mathbb{P}(\mathcal{X}) : P(A) \leq \overline{P}(A)\ \forall\, A \subseteq \mathcal{X}\right\},$$

and for this reason we shall also denote it $\mathcal{M}(\overline{P})$ in this section.

8.1. Conjunction

The study of the conjunction of maxitive measures was the subject of the work carried out in Miranda, Troffaes, and Destercke (2015). The conjunction of two maxitive measures $\overline{P}_1$ and $\overline{P}_2$ is the upper envelope of the intersection $\mathcal{M}(\overline{P}_1) \cap \mathcal{M}(\overline{P}_2)$:

$$\overline{E}(A) = \max\left\{P(A) : P \in \mathcal{M}(\overline{P}_1) \cap \mathcal{M}(\overline{P}_2)\right\}, \tag{15}$$

and it is the conjugate of the conjunction $\mathcal{A}_C$ of the minitive measures $\underline{P}_1$ and $\underline{P}_2$. In other words, $\overline{E} = \overline{\mathcal{A}_C}$. The following result was established in Miranda, Troffaes, and Destercke (2015)[8]:

Theorem 8.1 (Miranda, Troffaes, and Destercke 2015)**:** *Let $\overline{P}_1$ and $\overline{P}_2$ be two maxitive measures, define $\overline{P} = \min\{\overline{P}_1, \overline{P}_2\}$ and let $\overline{E}$ be the upper envelope of $\mathcal{M}(\overline{P}_1) \cap \mathcal{M}(\overline{P}_2)$ as in Equation (15). It holds that:*

(i) *$\overline{P}$ is maxitive if and only if*

$$\min_{i=1,2}\left(\max_{j=1,2}\overline{P}_i(\{x_j\})\right) = \max_{j=1,2}\left(\min_{i=1,2}\overline{P}_i(\{x_j\})\right) \quad \forall\, \{x_1, x_2\} \subseteq \mathcal{X}.$$

(ii) *If $\overline{P}$ is maxitive, then so is $\overline{E}$; however, the converse is not true.*
(iii) *If $\overline{P}_1$ and $\overline{P}_2$ are strictly positive on singletons, then $\overline{E}$ is maxitive if and only if $\overline{P}$ is.*

It was also established in Miranda, Troffaes, and Destercke (2015, Example 5) that the minimum of two maxitive measures may not be maxitive, even if it is coherent; in the notation of the theorem above, this means that $\overline{P} = \overline{E}$ does not imply that they are maxitive. Next we reformulate the example so as to show that the family of minitive measures is not closed under conjunction:

Example 8.1: Consider $\mathcal{X} = \{x_1, x_2, x_3\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be the minitive measures given in the following table:

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}_1(A)$	0.5	0	0	0.5	0.5	0
$\underline{P}_2(A)$	0	0.5	0	1	0	0.5
$\max\{\underline{P}_1(A), \underline{P}_2(A)\}$	0.5	0.5	0	1	0.5	0.5

We observe that $\max\{\underline{P}_1, \underline{P}_2\}$ is coherent because it is a probability measure, meaning that it coincides with $\mathcal{A}_C$, but it is not minitive since the disjoint events $\{x_1\}$ and $\{x_2\}$ are focal elements, meaning that the focal elements are not nested.

8.2. Disjunction

Concerning the disjunction, it is quite straightforward to see that the minimum of minitive measures is again minitive and it is also possible to establish some relationship between their focal elements:

Proposition 8.2: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be minitive measures with sets of focal elements $\mathcal{F}_1, \ldots, \mathcal{F}_m$, respectively. Then, the disjunction $\mathcal{A}_D$ is also minitive, and its set of focal elements $\mathcal{F}_D$ satisfies $\mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m \subseteq \mathcal{F}_D$.*

Proof: Consider two events A, B. The monotonicity due to coherence of $\mathcal{A}_D$ implies that $\mathcal{A}_D(A \cap B) \leq \min\{\mathcal{A}_D(A), \mathcal{A}_D(B)\}$. To see the converse, assume that $\mathcal{A}_D(A \cap B) = \underline{P}_i(A \cap B)$. Since $\underline{P}_i$ is minitive, we deduce that

$$\mathcal{A}_D(A \cap B) = \underline{P}_i(A \cap B) = \min\{\underline{P}_i(A), \underline{P}_i(B)\} \geq \min\{\mathcal{A}_D(A), \mathcal{A}_D(B)\},$$

whence $\mathcal{A}_D$ is minitive.

To see the second part, observe that if A is a focal element of all of $\underline{P}_1, \ldots, \underline{P}_m$ then it holds that

$$\underline{P}_i(A) > \max\{\underline{P}_i(B) : B \subset A\} \quad \forall\, i = 1, \ldots, m,$$

from which it follows that

$$\mathcal{A}_D(A) = \min\{\underline{P}_1(A), \ldots, \underline{P}_m(A)\} > \max\{\mathcal{A}_D(B) : B \subset A\}$$

and therefore that A is a focal element of $\mathcal{A}_D$. ■

Note that the inclusion between the focal elements in the above proposition may be strict: if for instance $\underline{P}_1$ and $\underline{P}_2$ are the minitive measures on $\mathcal{X} = \{x_1, x_2, x_3\}$ with respective focal elements $\mathcal{F}_1 = \{x_1\}$ and $\mathcal{F}_2 = \{x_2\}$, then $\mathcal{A}_D = \min\{\underline{P}_1, \underline{P}_2\}$ is the minitive measure whose only focal element is $\{x_1, x_2\}$.

8.3. Convex mixtures

While belief functions are preserved under convex mixtures, it is quite easy to establish that minitive measures are not; for this, observe that the convex mixture of two minitive measures will not be minitive as soon as the capacities assign strictly positive values to different singletons, since this prevents the focal elements from being nested. This said, following a similar reasoning as in Proposition 9.3, we can characterise in which cases the convex mixture of minitive measures is again minitive:

Corollary 8.3: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be minitive measures with sets of focal elements $\mathcal{F}_1, \ldots, \mathcal{F}_m$, respectively. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0, 1)$. $\mathcal{A}_M$ is a minitive measure if and only if the set $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_m$ is totally ordered by means of set inclusion.*

Proof: This is an immediate consequence of Proposition 7.4 and the characterisation of minitive measures in terms of their focal elements. ■

8.4. Pareto rule

That minitivity is not preserved by the Pareto rule can already be seen from Example 8.1, given that in that case the Pareto rule agrees with the conjunction. In our next example the Pareto rule does not agree with the conjunction, even if the latter produces a coherent capacity. This implies immediately that the output of the Pareto rule avoids sure loss but is not coherent, let alone minitive.

Example 8.2: Consider $\mathcal{X} = \{x_1, x_2, x_3\}$ and let $\underline{P}_1$ and $\underline{P}_2$ be the minitive measures given in the following table:

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\underline{P}_1(A)$	0.3	0	0	0.3	0.5	0	1
$\underline{P}_2(A)$	0	0.4	0	0.4	0	0.6	1
$\max\{\underline{P}_1(A), \underline{P}_2(A)\}$	0.3	0.4	0	0.4	0.5	0.6	1
$\min\{\overline{P}_1(A), \overline{P}_2(A)\}$	0.4	0.5	0.6	1	0.6	0.7	1

The Pareto rule gives $\mathcal{A}_P = \max\{\underline{P}_1, \underline{P}_2\}$, that is not a minitive measure; it is not even coherent, since it violates superadditivity: $\mathcal{A}_P(\{x_1, x_2\}) < \mathcal{A}_P(\{x_1\}) + \mathcal{A}_P(\{x_2\})$.

The example also serves as an illustration of Corollary 8.3: for any convex mixture $\alpha\underline{P}_1 + (1-\alpha)\underline{P}_2$ with $\alpha \in (0,1)$, the set of focal elements is

$$\{\{x_1\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}, \{x_2\}, \{x_2, x_3\}\}$$

and as a consequence it is not a minitive measure.

8.5. CD and MCS rules

Example 8.1 shows that the conjunction of two minitive measures need not be minitive. Since in that case the conjunction rule gives a coherent capacity, it coincides with the output of the conjunction-disjunction and the maximal consistent subsets rules.

9. *p*-boxes

The last model to be analysed in this paper are p-boxes. Assume thus that $\underline{P}_1, \ldots, \underline{P}_m$ are p-boxes defined on an ordered space $\mathcal{X} = \{x_1, \ldots, x_n\}$ and that they are determined by $(\underline{F}_1, \overline{F}_1), \ldots, (\underline{F}_m, \overline{F}_m)$, respectively.

9.1. Conjunction

We begin by analysing the conjunction. It is not difficult to establish a result akin to that stated in Proposition 6.1 for probability intervals.

Proposition 9.1: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be p-boxes determined by $(\underline{F}_1, \overline{F}_1), \ldots, (\underline{F}_m, \overline{F}_m)$, respectively. Their conjunction $\mathcal{A}_C$ is a p-box if and only if $\mathcal{M}(\underline{F}_1, \overline{F}_1) \cap \ldots \cap \mathcal{M}(\underline{F}_m, \overline{F}_m) \neq \emptyset$. In that case, it is determined by* $(\max_{i=1,\ldots,m} \underline{F}_i, \min_{i=1,\ldots,m} \overline{F}_i)$.

Proof: On the one hand, if the intersection of the credal sets is empty, the conjunction does not avoid sure loss, hence it cannot be a p-box.

On the other hand, if the intersection of the credal sets is non-empty, we get:

$$\begin{aligned}
&\mathcal{M}(\underline{F}_1, \overline{F}_1) \cap \ldots \cap \mathcal{M}(\underline{F}_n, \overline{F}_n) \\
&\quad = \left\{P \in \mathbb{P}(\mathcal{X}) : \underline{F}_i \leq F_P \leq \overline{F}_i \quad \forall\, i = 1, \ldots, m\right\} \\
&\quad = \left\{P \in \mathbb{P}(\mathcal{X}) : \max_{i=1,\ldots,m} \underline{F}_i \leq F_P \leq \min_{i=1,\ldots,m} \overline{F}_i\right\} = \mathcal{M}(\underline{F}, \overline{F}),
\end{aligned}$$

where $\underline{F} = \max_{i=1,\ldots,m} \underline{F}_i$ and $\overline{F} = \min_{i=1,\ldots,m} \overline{F}_i$. ■

9.2. Disjunction

Regarding the disjunction, the next example shows that it does not necessarily lead to a p-box, in the sense that the resulting model is not determined by its values on the cumulative events.

Example 9.1: Let us consider a three-element ordered possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the p-boxes $\underline{P}_1$ and $\underline{P}_2$ determined by $(\underline{F}_1, \overline{F}_1)$ and $(\underline{F}_2, \overline{F}_2)$, whose values are reported in the following table:

	x_1	x_2	x_3
$(\underline{F}_1, \overline{F}_1)$	$[0, 0]$	$[0, 0.5]$	$[1, 1]$
$(\underline{F}_2, \overline{F}_2)$	$[0.5, 1]$	$[1, 1]$	$[1, 1]$

Their credal sets can be expressed as:

$$\begin{aligned}
\mathcal{M}\left(\underline{F}_1, \overline{F}_1\right) &= \{P \in \mathbb{P}(\mathcal{X}) : \underline{F}_1 \leq F_P \leq \overline{F}_1\} \\
&= \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_1\}) = 0,\ P(\{x_2\}) \in [0, 0.5]\} \\
&= \{P = (0, \alpha, 1 - \alpha) : \alpha \in [0, 0.5]\}. \\
\mathcal{M}\left(\underline{F}_2, \overline{F}_2\right) &= \{P \in \mathbb{P}(\mathcal{X}) : \underline{F}_2 \leq F_P \leq \overline{F}_2\} \\
&= \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_3\}) = 0,\ P(\{x_1\}) \in [0.5, 1]\} \\
&= \{P = (1 - \beta, \beta, 0) : \beta \in [0, 0.5]\}.
\end{aligned}$$

The credal set of the disjunction can be expressed as:

$$\mathcal{M}(\mathcal{A}_D) = \mathcal{CH}\left(\mathcal{M}\left(\underline{F}_1, \overline{F}_1\right) \cup \mathcal{M}\left(\underline{F}_2, \overline{F}_2\right)\right) = \{P \in \mathbb{P}(\mathcal{X}) : P(\{x_2\}) \leq 0.5\}.$$

Restricted to cumulative events, $\mathcal{A}_D$ satisfies:

$$\mathcal{A}_D(\{x_1\}) = \mathcal{A}_D(\{x_1, x_2\}) = 0, \quad \overline{\mathcal{A}}_D(\{x_1\}) = \overline{\mathcal{A}}_D(\{x_1, x_2\}) = 1.$$

Hence, if $\mathcal{A}_D$ were a p-box, then it should be determined by:

	x_1	x_2	x_3
$(\underline{F}, \overline{F})$	$[0, 1]$	$[0, 1]$	$[1, 1]$

However, its associated credal set is $\mathcal{M}(\underline{F}, \overline{F}) = \mathbb{P}(\mathcal{X})$, which clearly differs from $\mathcal{M}(\mathcal{A}_D)$. We conclude that the disjunction is not a p-box in general.

Nevertheless, and similarly to the case of probability intervals, it is not difficult to determine the unique undominated outer approximation of the disjunction of a finite number of p-boxes:

Proposition 9.2: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be p-boxes determined by $(\underline{F}_1, \overline{F}_1), \ldots, (\underline{F}_m, \overline{F}_m)$, respectively. Consider $\underline{F}$ and $\overline{F}$ given by $\underline{F} = \min_{i=1,\ldots,m} \underline{F}_i$ and $\overline{F} = \max_{i=1,\ldots,m} \overline{F}_i$. Then, the capacity obtained as the lower envelope of $\mathcal{M}(\underline{F}, \overline{F})$ is the unique undominated outer approximation of the disjunction $\mathcal{A}_D$ in the family of p-boxes.*

Proof: To see that the lower probability $\underline{P}$ determined by $(\underline{F}, \overline{F})$ is an outer approximation of the disjunction, note that

$$P \geq \mathcal{A}_D \Leftrightarrow P \geq \min_{i=1,\ldots,m} \underline{P}_i$$
$$\Rightarrow \min_{i=1,\ldots,m} \underline{F}_i(x_j) \leq F_P(x_j) \leq \max_{i=1,\ldots,m} \overline{F}_i(x_j) \Leftrightarrow \underline{F} \leq F_P \leq \overline{F}.$$

This also shows that any other p-box $(\underline{F}^*, \overline{F}^*)$ that outer approximates $\mathcal{A}_D$ must satisfy $\underline{F}^* \leq \underline{F}$ and $\overline{F}^* \geq \overline{F}$, from which it follows that $(\underline{F}, \overline{F})$ is the unique undominated outer approximation. ■

9.3. Convex mixtures

Concerning convex mixtures, we can use a similar reasoning as in Proposition 7.4 to characterise in which cases p-boxes are preserved by convex mixtures.

Proposition 9.3: *Let $\underline{P}_1, \ldots, \underline{P}_m$ be p-boxes determined by $(\underline{F}_1, \overline{F}_1), \ldots, (\underline{F}_m, \overline{F}_m)$, respectively, and denote by $\mathcal{F}_1, \ldots, \mathcal{F}_m$ their set of focal elements. Consider their convex mixture $\mathcal{A}_M$ with weights $\alpha_1, \ldots, \alpha_m \in (0, 1)$. $\mathcal{A}_M$ is a p-box if and only if the set $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_m$ is formed by ordered intervals, in the sense of Equation (9).*

Proof: From Proposition 7.4 we know that the set of focal elements of the convex mixture $\mathcal{A}_M$ is $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_m$. It suffices then to apply that a belief function is a p-box if and only if the focal elements are ordered intervals in the sense of Equation (9). ■

9.4. Pareto

The Pareto rule applied to p-boxes may give rise to a non-coherent capacity, that is therefore not a p-box. This is illustrated in the following example.

Example 9.2: Consider the ordered possibility space $\mathcal{X} = \{x_1, x_2, x_3\}$ and the p-boxes $\underline{P}_1$ and $\underline{P}_2$ given by:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}_1$	0.3	0	0.3	0.3	0.6	0.4
$\overline{P}_1$	0.6	0.4	0.7	0.7	1	0.7
$\underline{P}_2$	0	0	0.5	0.1	0.5	0.8
$\overline{P}_2$	0.2	0.5	0.9	0.5	1	1

They are determined by $(\underline{F}_1, \overline{F}_1)$ and $(\underline{F}_2, \overline{F}_2)$ whose values are given in the following table:

	x_1	x_2	x_3
$(\underline{F}_1, \overline{F}_1)$	$[0.3, 0.6]$	$[0.3, 0.7]$	$[1, 1]$
$(\underline{F}_2, \overline{F}_2)$	$[0, 0.2]$	$[0.1, 0.5]$	$[1, 1]$

The Pareto rule takes the following values:

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\mathcal{A}_P$	0.2	0	0.5	0.3	0.6	0.7

It is not coherent since it violates super-additivity: $\mathcal{A}_P(\{x_1, x_3\}) = 0.6 < 0.7 = \mathcal{A}_P(\{x_1\}) + \mathcal{A}_P(\{x_3\})$. Consequently, it is not a p-box either.

In this example, the maximum of the capacities avoids sure loss and as a consequence coincides with the output by the Pareto rule, but is not coherent. When the maximum of the capacities is coherent, then by Miranda, Salamanca, and Montes (2025, Proposition 7) the Pareto rule agrees with the conjunction, and therefore produces a coherent capacity that is moreover a probability box, by Proposition 9.1. That would be then a sufficient condition for the Pareto rule to be closed within the family of p-boxes.

9.5. Conjunction-disjunction and MCS

Since the disjunction of p-boxes need not be a p-box, we immediately deduce that the conjunction-disjunction rule is not a closed operator in the family of probability boxes. On the other hand, when the models are compatible, i.e. when the intersection of all the credal sets is non-empty or $\max_{i=1,\dots,m} \underline{P}_i$ avoids sure loss, then the conjunction-disjunction rule coincides with the conjunction and, from Proposition 9.1, determines a p-box. Similar comments apply to the MCS rule.

10. Conclusions

In this paper, we address the problem of aggregating a finite number of imprecise probability models of a given family, and we analyse whether certain aggregation rules yield an aggregated model that belongs to the same family. While none of the common aggregation rules is closed for all models, in some cases we have found sufficient conditions that guarantee that the rule is closed. A summary of the results is provided in Table 1.

While all the aggregation rules save for Pareto preserve coherence as soon as there exists a probability measure that is compatible with all the models, they are not necessarily closed once we focus on some particular subfamilies of coherent lower previsions. Although in this paper we have established some sufficient conditions for the aggregated model to be in the same family as the sources, a much deeper study is needed in order to completely characterise these rules, similarly to what was done in Miranda, Troffaes, and Destercke (2015) for the conjunction of possibility measures.

When the aggregated model does not belong to the same family, two main avenues appear before us: we may on the one hand look for (possibly unique) inner (Miranda and Montes 2023) or outer (Montes, Miranda, and Vicig 2018, 2019) approximations of the

Table 1. Summary of the models preserved by each aggregation rule. A double tick means that the rule is closed, and a single tick that it is closed under some conditions.

	Conjunction	Disjunction	Convex Mixture	Pareto	CD	MCS
Comparative Probabilities	✓✓ (Miranda and Destercke 2015, Proposition 9)	✓ (Theorem 4.2)	✓ (Corollary 4.6)	✓ (Corollary 4.8)	✓✓ (Proposition 4.1)	✓✓ (Proposition 4.1)
2-monotone capacities	✓ (Proposition 5.1)	✓ (Corollary 5.3)	✓✓ (Proposition 5.4)	✗ (Miranda, Salamanca, and Montes 2025, Example 3)	✗ (Example 5.3)	✗ (Example 5.3)
Probability interval	✓ (de Campos, Huete, and Moral 1994, Proposition 7)	✗ (de Campos, Huete, and Moral 1994, Example 1)	✗ (Example 6.1)	✗ (Example 6.2)	✓ (de Campos, Huete, and Moral 1994, Proposition 7)	✓ (de Campos, Huete, and Moral 1994, Proposition 7)
Belief Functions	✗ (Example 7.1)	✓ (Corollary 7.3)	✓✓ (Proposition 7.4)	✗ (Miranda, Salamanca, and Montes 2025, Example 3)	✗ (Example 7.1)	✗ (Example 7.1)
Minitive measures	✓ (Miranda, Troffaes, and Destercke 2015)	✓✓ (Proposition 8.2)	✓ (Corollary 8.3)	✗ (Example 8.2)	✗ (Example 8.1)	✗ (Example 8.1)
Probability Box	✓ (Proposition 9.1)	✗ (Example 9.1)	✓ (Proposition 9.3)	✗ (Example 9.2)	✓ (Proposition 9.1)	✓ (Proposition 9.1)

aggregated model; or we may also propose other rules that are tailor-made for that specific family. As an illustration, in the case of belief functions we may look for a rule that distributes the mass among all possible focal elements, using aggregation rules for precise probabilities.

The finiteness of both the possibility space $\mathcal{X}$ and the number of models to be aggregated has been an essential assumption in this paper. The extension to infinite spaces is relatively simple, although in some cases, such as belief functions, we should take into account some continuity properties (Shafer 1979). A more delicate matter that should be addressed carefully would be the aggregation of an infinity of models. Here issues of compactness and continuity should be considered.

One topic we have not discussed in this already long paper are the other rationality criteria that were considered in Miranda, Salamanca, and Montes (2025). In this respect, we should investigate (a) whether some of the properties that a rule does not satisfy in general can be shown to hold when we restrict its domain to a particular subfamily of coherent lower previsions; and (b) if the rationality criteria may be suitably reformulated for such subfamilies taking into account their specific features. Another interesting study would be to consider other models such as comparative probabilities in a wider sense, where the comparisons are made between events and not only between singletons. For this purpose, the results in Erreygers and Miranda (2021) may be of interest. And finally, some suitable modifications of conjunction that may allow to achieve coherence (Quaeghebeur 2015), for instance based on distortions of credal sets (Nieto-Barba, Miranda, and Montes 2026; Nieto-Barba, Montes, and Miranda 2025) or second order models (Troffaes 2006), may be useful as a tool to address the conflict between imprecise probability models.

Notes

1. The general approach to comparative probabilities is based on considering a set $\mathcal{C}$ formed by pairs of events (A, B), understood as "the probability of A is at least as large as that of B". However, this general case is more involved: for starters, there may not exist a probability measure that satisfies all the assessments, and, when there is, the number of extreme points of the associated credal set may be much larger than when the comparisons are made on singletons only, as shown in Erreygers and Miranda (2021).
2. Note that there many other rules that have been proposed in the literature, particularly in the context of belief functions (Bronevich and Rozenberg 2015a; Denoeux 2021; Destercke, Dubois, and Chojnacki 2007; Lefévre, Colot, and Vannoorenberghe 2002); our choice in this paper has been of those that are applicable for arbitrary lower previsions, and those whose rationality does not hinge on the evidential interpretation of Dempster-Shafer theory.
3. When $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_m = \emptyset$, we obtain an empty set of comparisons, meaning that it determines a vacuous comparative probability on $\mathcal{X}$ whose credal set is $\mathbb{P}(\mathcal{X})$.
4. Following the terminology in Montes, Miranda, and Vicig (2018) and Montes, Miranda, and Vicig (2019), $\underline{Q}$ is an *outer approximation* of a lower prevision $\underline{P}$ if $\underline{Q} \leq \underline{P}$ or, equivalently, if $\mathcal{M}(\underline{Q}) \supseteq \mathcal{M}(\overline{P})$.
5. We say that $\underline{Q}$ is an undominated outer approximation of $\underline{P}$ in a family $\mathcal{H}$ if it is an outer approximation, $\underline{Q} \in \mathcal{H}$ and there is no other outer approximation $\underline{Q}' \in \mathcal{H}$ of $\underline{P}$ satisfying $\underline{Q}' \gtrdot \underline{Q}$.
6. See (Bronevich 2005) for a more general result in terms of aggregation functions.
7. Notice the difference with the situation depicted in Example 5.1: while the conjunction is not a closed rule in the class of 2-monotone capacities, it is closed if we focus on the subclass of probability intervals.

8. See also (Bronevich 2001) for an earlier study of point (i) of this theorem.

Acknowledgments

We would like to thank the helpful comments by Erik Quaeghebeur, Matthias Troffaes, Teddy Seidenfeld and the participants of the ISIPTA 2025 conference. Finally, we are grateful to the anonymous reviewers for their comments and suggestions.

Author contributions

CRediT: **Enrique Miranda:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Writing – original draft, Writing – review & editing; **Ignacio Montes:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This contribution is part of grant PID2022-140585NB-I00 funded by MICIU/AEI/10.13039/501100011033 and "FEDER/UE".

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